NOTE

Simple Non-trivial Designs with an Arbitrary Automorphism Group

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We give a new recursive construction of simple non-trivial designs. Using this construction, we show that given a natural number \( t \) and a finite group \( G \), a simple non-trivial \( t \)-design admitting an automorphism group isomorphic to \( G \) exists. Further, we apply our construction to get a recursive construction of large sets. © 2002 Elsevier Science (USA)

1. INTRODUCTION

For positive integers \( t, v, k \) and \( \lambda \) satisfying \( k > t \) and \( v > k \), a \( t \)-design \( \mathcal{D} \) with parameters \( (v, k, \lambda) \) is a set \( X \) of \( v \) points, together with a collection \( \mathcal{B} \) of subsets, called blocks, such that each block contains exactly \( k \) points and each \( t \)-tuple of points is contained in exactly \( \lambda \) blocks. A \( t - (v, k, \lambda) \) design is simple if repeated blocks are not allowed and is non-trivial if not all \( k \)-sets of points are blocks. In this paper, all the designs are supposed simple and non-trivial.

Let \( \mathcal{D} = (X, \mathcal{B}) \) be a \( t - (v, k, \lambda) \) design. It is well known that for all \( 1 \leq i \leq t \), \( \mathcal{D} \) is also a \( i - (v, k, \lambda_{(i)}) \) design with \( \lambda_{(i)} = \lambda \binom{v-1}{t-i}/\binom{k-1}{t-i} \).

Let \( Y \) be a set of \( m \) points \((0 \leq m \leq t)\), we define

\[ \mathcal{B}^Y = \{ B \mid Y \subseteq B \text{ and } Y \cap B = \emptyset \} , \]

\[ \mathcal{B}_Y = \{ B \mid B \subseteq \mathcal{B} \text{ and } Y \cap B = \emptyset \} . \]

Then, \( (X \setminus Y, \mathcal{B}^Y) \) is a \( (t - m) - (v - m, k - m, \lambda) \) design called an \( m \)th derived design of \( (X, \mathcal{B}) \) and \( (X \setminus Y, \mathcal{B}_Y) \) is a \( (t - m) - (v - m, k, \lambda \binom{v-t}{k-t+m}/\binom{v-t}{k-t}) \) design called an \( m \)th residual design of \( (X, \mathcal{B}) \).
For a design $\mathcal{D} = (X, B)$, the automorphism group $Aut(\mathcal{D})$ consists of all permutations of $X$ that leave $B$ invariant. An automorphism group of $\mathcal{D}$ is a subgroup of $Aut(\mathcal{D})$.

Although in 1987, Teirlinck [6, 7] showed the existence of $t$-designs for all $t \geq 2$, the construction of $t$-designs with large $t$ is still a difficult and interesting problem. In order to construct a design, there are two well-known methods. A recursive construction is to construct a $t$-design from some $t'$-designs with $t \leq t'$, for example, [4, 8]. An extension construction is to construct a $t$-design from some $t'$-designs with $t > t'$, see for example [1, 2]. In Section 2, we give a new recursive construction of designs.

**Theorem 1.1.** Let $j_0, j_1, \ldots, j_n$ and $m$ be natural numbers such that $0 = j_0 < j_1 < \cdots < j_n = m + 1$. Suppose that there exist $(t + j_{l+1} - j_l - 1)/(v + j_{l+1} - j_l - 1, k + j_{l+1} - 1, \lambda^{(l)})$ designs such that for $l = 0, \ldots, m$,

$$
\lambda^{(j_l)} = \lambda^{(0)} \begin{pmatrix}
(v - t) \\
(k - t + j_l) \\
(v - t) \\
(k - t)
\end{pmatrix}.
$$

Then there exists a $t - (v + m, k + m, \lambda^{(0)}_{(t-m)})$ having $S_m$ as an automorphism group.

In recent years, many new $t$-designs have been constructed with the help of DISCRETA, a program written by Betten et al. [3]. This program constructs $t$-designs having a given automorphism group. Hence, it is natural to ask the following question.

**Problem.** For which couples $(t, G)$ where $t$ is a natural number and $G$ is a finite group, there exists a $t$-design having an automorphism group isomorphic to $G$.

The first result in this direction was due to Sebille [5]. He proved

**Theorem 1.2 (Sebille [5]).** For every pair $(t, n)$ of natural numbers, there exists a simple non-trivial $t$-design with an automorphism group isomorphic to $\mathbb{Z}_2^n$.

In this paper, applying Theorem 1.1, we give an affirmative answer to the previous problem and so generalize the result of Sebille.
Theorem 1.3. Let $t$ be a natural number and $G$ be a finite group. Then there exists a simple non-trivial $t$-design with an automorphism group isomorphic to $G$.

An application of Theorem 1.1 to large sets will be discussed in Section 4.

2. RECURSIVE CONSTRUCTION

We recall the following theorem of Tran van Trung.

Theorem 2.1 (van Trung [8]). Let $(X, \mathcal{B}(j))$ be designs with parameters $t-(v, k^{(j)}, \lambda^{(j)})$ for $0 \leq j \leq m$ ($m \leq t$). Suppose that we have:

(i) $k^{(j)} = k^{(j-1)} + 1$ for all $1 \leq j \leq m$,

(ii) $\sum_{l=0}^{j} \binom{j}{l} \lambda_{(t-m+j)}^{(l)} = \lambda_{(t-m)}^{(0)}$ for all $0 \leq j \leq m$.

Then there exists a $t-(v+m, k^{(m)}, \lambda^{(0)}_{(t-m)})$ design admitting the symmetric group $S_m$ as an automorphism group.

Proof of Theorem 1.1. Let $l, j$ be two integers such that $0 \leq l < n$ and $j_l \leq j < j_{l+1}$. Now, we denote with $\mathcal{D}_{l}$ the $(t + j_{l+1} - j_l - 1) - (v + j_{l+1} - j_l - 1, k + j_{l+1} - 1, \lambda^{(l)}_{(j_{l+1}-j_l-1)})$ design given in the hypothesis. We construct $\mathcal{D}_{l}'$ as the $(j_{l+1} - j_l - 1)$th derived design of $\mathcal{D}_{l}$ and then construct $(X, \mathcal{B}(j))$ the $(j - j_l)$th residual design of $\mathcal{D}_{l}'$. This is a $t-(v, k^{(j)}, \lambda^{(j)})$ design with

$$k^{(j)} = k + j,$$

$$\lambda^{(j)} = \binom{j}{l} = \frac{v - t}{k - t + j} = \frac{v - t}{k - t}.$$

To conclude, from Theorem 2.1, it is sufficient to verify that for all $0 \leq j \leq m$, we have

$$\sum_{l=0}^{j} \binom{j}{l} \lambda_{(t-m+j)}^{(l)} = \lambda_{(t-m)}^{(0)}.$$
Now we write
\[
\sum_{l=0}^{j} \binom{j}{l} \lambda^{(l)}_{(t-m+j)}
\]
\[
= \sum_{l=0}^{j} \binom{j}{l} \lambda^{(l)} \frac{(v - (t - m + j))}{(t - (t - m + j))} \frac{(k + l) - (t - m + j)}{t - (t - m + j)}
\]
\[
= \sum_{l=0}^{j} \binom{j}{l} \lambda^{(0)} \frac{(v - t)}{k - t} \frac{(k + l) - (t - m + j)}{t - (t - m + j)}
\]
\[
= \lambda^{(0)} \left( \frac{v - t}{k - t} \right)^{-1} \sum_{l=0}^{j} \binom{j}{l} \frac{(v - t + m - j)}{v - k - l}
\]
\[
= \lambda^{(0)} \left( \frac{v - t}{k - t} \right)^{-1} \left( \frac{v - t + m}{v - k} \right)
\]
\[
= \lambda^{(0)} \left( \frac{k - (t - m)}{t - (t - m)} \right)^{-1} \left( \frac{v - (t - m)}{t - (t - m)} \right)
\]
\[
= \lambda^{(0)}_{(t-m)}.
\]

Thus we obtain Theorem 1.1. \(\square\)

If we take \(m = 1\), \(j_0, j_1, j_2 = (0, 1, 2)\), then we get the recursive construction considered by van Leijenhorst [9].

**Corollary 2.1.** (van Leijenhorst [9]). Suppose that there exist a \(t - (v, k + 1, \lambda')\) design and a \(t - (v, k, \lambda)\) design such that \(\lambda' = \lambda \frac{v - k}{k - t + 1}\). Then there exists a \(t - (v + 1, k + 1, \lambda_{(t-1)})\) design.

If we take \(m = 2\), \(j_0, j_1, j_2 = (0, 1, 3)\), then we get the recursive construction considered by Sebille [5].

**Corollary 2.2.** (Sebille [5]). Suppose that there exist a \(t + 1 - (v + 1, k + 2, \lambda')\) design and a \(t - (v, k, \lambda)\) design such that \(\lambda' = \lambda \frac{v - k}{k - t + 1}\). Then there exists a \(t - (v + 2, k + 2, \lambda_{(t-2)})\) design.
3. APPLICATION TO DESIGNS

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. Teirlinck [6, 7] proved the existence of \( t \)-design for every \( t \geq 1 \). Therefore, for every pair \( (t, m) \) of natural integers, there exists a \( (t + m) - (v + m, k + m, \lambda) \) for some \( v, k \) and \( \lambda \). Applying Theorem 1.1 for \( j_0 = 0 \), \( j_1 = m + 1 \), we imply that there exists a simple non-trivial \( t - (v + m, k + m, \lambda(t)) \) design with an automorphism group isomorphic to \( S_m \). Since every finite group is a subgroup of a symmetric group \( S_m \) for some \( m \), we have Theorem 1.3.

Remark. It is well known that a \( (t + m) - (v + m, k + m, \lambda) \) design is also a \( t - (v + m, k + m, \lambda(t)) \) design with the same parameters as above. However, this construction does not give any information on automorphism groups.

4. APPLICATION TO LARGE SETS

Recall that a large set \( LS[N](t, k, v) \) is a partition of the complete \( k \)-uniform hypergraph with \( v \) vertices into \( N \) designs with parameters \( t - (v, k, \lambda) \). A counting argument gives \( \lambda = \binom{v - t}{k - t} / N \).

Theorem 4.1. Let \( j_0, j_1, \ldots, j_n \), \( m \) and \( N \) be natural numbers such that \( 0 = j_0 < j_1 < \cdots < j_n = m + 1 \). Suppose that there exist large sets \( LS[N] \) \( (t + j_{i+1} - j_i - 1, k + j_{i+1} - 1, v + j_{i+1} - j_i - 1) \). Then \( LS[N](t, k + m, v + m) \) exists.

Proof. The previous discussion implies that each partition of \( LS[N] \) \( (t + j_{i+1} - j_i - 1, k + j_{i+1} - 1, v + j_{i+1} - j_i - 1) \) is a \( (t + j_{i+1} - j_i - 1) - (v + j_{i+1} - j_i - 1, k + j_{i+1} - 1), \lambda^{(j)}(t, m) \) design with

\[
\lambda^{(j)}(t) = \frac{1}{N} \binom{v - t}{k - t + j_i}.
\]

In particular, \( \lambda(t) = \lambda^{(0)}(v - t) / (v - t) \). Thus we obtain Theorem 4.1 by a proof similar to that of Theorem 1.1.

If we take \( m = 2 \), \( (j_0, j_1, j_2) = (0, 1, 3) \), then we get a result of Sebille [5].

Theorem 4.2 (Sebille [5]). Suppose that there exist large sets \( LS[N] \) \( (t + 1, k + 2, v + 1) \) and \( LS[N](t, k, v) \). Then \( LS[N](t, k + 2, v + 2) \) exists.
As remarked by Sebille [5], this construction gives rise to some new large sets, for example $LS[3](2, 5, 13)$, $LS[3](2, 6, 14)$, $LS[3](3, 7, 15)$, $LS[3](3, 6, 14)$, $LS[3](4, 7, 15)$ and $LS[7](2, 6, 18)$.

**REFERENCES**

3. A. Betten, R. Laue, and A. Wassermann, “DISCRETA, a program system for the construction of $t$-designs with a prescribed automorphism group,” University of Bayreuth, http://www.mathe2.uni-bayreuth.de/betten/DISCRETA/Index.html