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On intertwining operators and finite automorphism groups of vertex operator algebras

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Abstract

Let V be a simple vertex operator algebra and G a finite automorphism group. We give a construction of intertwining operators for irreducible V^G-modules which occur as submodules of irreducible V-modules by using intertwining operators for V.

We also determine some fusion rules for a vertex operator algebra as an application.

1. Introduction

Let V be a simple vertex operator algebra (cf. [1,8,9]), and G a finite automorphism group. It is an important problem to understand the module category for the vertex operator algebra V^G of G-invariants. In [2], this question was asked and several ideas are proposed.

For a simple vertex operator algebra V, it is shown in [6] that every irreducible V-module is a completely reducible V^G-module as a natural consequence of a duality theorem of Schur–Weyl type. In this paper we give a construction of intertwining operators for irreducible V^G-modules which occur as submodules of irreducible V-modules by using intertwining operators for V.

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Let us state our results more explicitly. Firstly, we need to recall the results of Dong–Yamskulna [6]. For an irreducible V-module (L, Y_L) and $a \in G$ we define a new irreducible V-module $(L \circ a, Y_{L\circ a})$. Here $L \circ a$ is equal to L and $Y_{L\circ a}(u, z) = Y_L(au, z)$. Let $\mathcal{S}$ be a finite set of inequivalent irreducible V-modules which is closed under this right action of G. In [6] they define a finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G, \mathcal{S})$ over $\mathbb{C}$ and show a duality theorem of Schur–Weyl type for the actions of V^G and $\mathcal{A}_\alpha(G, \mathcal{S})$ on the direct sum of V-modules in $\mathcal{S}$ which is denoted by $\mathcal{L}$. That is, as a $\mathcal{A}_\alpha(G, \mathcal{S}) \otimes V^G$-module,

$$\mathcal{L} = \bigoplus_{(j,\lambda)\in\Gamma} W_{(j,\lambda)} \otimes M_{(j,\lambda)},$$

where $\{W_{(j,\lambda)}\}_{(j,\lambda)\in\Gamma}$ is the set of all inequivalent irreducible $\mathcal{A}_\alpha(G, \mathcal{S})$-modules and $M_{(j,\lambda)}$ is the multiplicity spaces of $W_{(j,\lambda)}$ in $\mathcal{L}$. Each $M_{(j,\lambda)}$ is a nonzero irreducible V^G-module and the different multiplicity spaces are inequivalent V^G-modules. In this paper we consider intertwining operators for irreducible V^G-modules constructed from irreducible V-modules in this way.

For each $i = 1, 2, 3$ let $\mathcal{S}_i$ be a finite set of inequivalent irreducible V-modules which is closed under the action of G and let $\mathcal{L}_i$ be the direct sum of V-modules in $\mathcal{S}_i$. We have the decomposition $\mathcal{L}_i = \bigoplus_{(j_i,\lambda_i)\in\Gamma_i} W^i_{(j_i,\lambda_i)} \otimes M^i_{(j_i,\lambda_i)}$ as a $\mathcal{A}_{\alpha_i}(G, \mathcal{S}_i) \otimes V^G$-module. Set

$$\mathcal{I} = \bigoplus_{(L^1,L^2,L^3)\in\mathcal{S}_1\times\mathcal{S}_2\times\mathcal{S}_3} I_V\begin{pmatrix} L^3 \\ L^1\ L^2 \end{pmatrix} \otimes L^1 \otimes L^2,$$

where $I_V\binom{L^3}{L^1\ L^2}$ is the set of all intertwining operators of type $\binom{L^3}{L^1\ L^2}$. $\mathcal{I}$ has a natural $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-module structure. For each $i = 1, 2, 3$, fix $(j_i, \lambda_i) \in \Gamma_i$ and nonzero $v^{10} \in M^1_{(j_1,\lambda_1)}$, $v^{20} \in M^1_{(j_2,\lambda_2)}$. Set

$$\mathcal{I}_{(j_1,\lambda_1),(j_2,\lambda_2)} = \mathrm{Span}_{\mathbb{C}}\bigl\{f \otimes \bigl(w^1 \otimes v^{10}\bigr) \otimes \bigl(w^2 \otimes v^{20}\bigr) \in \mathcal{I} \bigm| w^1 \in W^1_{(j_1,\lambda_1)},\ w^2 \in W^2_{(j_2,\lambda_2)}\bigr\}.$$

$\mathcal{I}_{(j_1,\lambda_1),(j_2,\lambda_2)}$ is a $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-submodule of $\mathcal{I}$. We will construct an injective linear map from the multiplicity space of $W^3_{(j_3,\lambda_3)}$ in $\mathcal{I}_{(j_1,\lambda_1),(j_2,\lambda_2)}$ to the set of all intertwining operators for V^G of type

$$\begin{pmatrix} M^3_{(j_3,\lambda_3)} \\ M^1_{(j_1,\lambda_1)}\ M^2_{(j_2,\lambda_2)} \end{pmatrix}.$$

Therefore, the fusion rule for V^G of type

$$\begin{pmatrix} M^3_{(j_3,\lambda_3)} \\ M^1_{(j_1,\lambda_1)}\ M^2_{(j_2,\lambda_2)} \end{pmatrix}$$

is greater than or equal to the multiplicity of $W^3_{(j_3,\lambda_3)}$ in $\mathcal{I}_{(j_1,\lambda_1),(j_2,\lambda_2)}$.

This paper is organized as follows. In Section 2 we first recall a construction of irreducible V^G-modules from irreducible V-modules in [6]. We also recall the definitions of intertwining operators and fusion rules. In Section 3 we give a construction of intertwining operators for irreducible V^G-modules which occur as submodules of irreducible V-modules. In Section 4 we apply the main result to determine some fusion rules for a vertex operator algebra $\mathcal{W}$ studied in [3]. In Appendix, we give some singular vectors in $\mathcal{W}$-modules used in Section 4.

2. Preliminary

We assume that the reader is familiar with the basic knowledge on vertex operator algebras as presented in [1,8,9].

The following notation will be in force throughout the paper: $V = (V, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra and G a finite automorphism group of V. For any V-module L, we always arrange the grading on $L = \bigoplus_{n=0}^{\infty} L(n)$ so that $L(0) \neq 0$ if $L \neq 0$ by using a grading shift.

2.1. *Irreducible V^G-modules constructed from irreducible V-modules*

In this subsection we review the results of Dong–Yamskulna [6]. For a simple vertex operator algebra V, they showed that every irreducible V-module is a completely reducible V^G-module as a natural consequence of a duality theorem of Schur–Weyl type.

Let (L, Y_L) be an irreducible V-module and $a \in G$. We define a new irreducible V-module $(L \circ a, Y_{L\circ a})$. Here $L \circ a$ is equal to L and $Y_{L\circ a}(u, z) = Y_L(au, z)$. Note that $L \circ a$ is also an irreducible V-module. A set $\mathcal{S}$ of irreducible V-modules is called *stable* if for any $L \in \mathcal{S}$ and $a \in G$ there exists $M \in \mathcal{S}$ such that $L \circ a \simeq M$.

Now we take a finite G-stable set $\mathcal{S}$ consisting of inequivalent irreducible V-modules. Let $L \in \mathcal{S}$ and $a \in G$. Then there exists $M \in \mathcal{S}$ such that $M \simeq L \circ a^{-1}$. That is, there is a linear map $\phi(a, L) : L \to M$ satisfying the condition:

$$\phi(a, L)Y_L(v, z) = Y_M(av, z)\phi(a, L).$$

By simplicity of L, there exists $\alpha_L(b, a) \in \mathbb{C}^\times$ such that

$$\phi(b, M)\phi(a, L) = \alpha_L(b, a)\phi(ba, L).$$

Moreover, for $a, b, c \in G$ we have

$$\alpha_L(c, ba)\alpha_L(b, a) = \alpha_M(c, b)\alpha_L(cb, a).$$

For $L \in \mathcal{S}$ and $a \in G$, we denote $M \in \mathcal{S}$ such that $L \circ a \simeq M$ by $L \cdot a$.

Define a vector space $\mathbb{C}\mathcal{S} = \bigoplus_{L\in\mathcal{S}} \mathbb{C}e(L)$ with a basis $e(L)$ for $L \in \mathcal{S}$. The space $\mathbb{C}\mathcal{S}$ is an associative algebra under the product $e(L)e(M) = \delta_{L,M}e(L)$. Let $\mathcal{U}(\mathbb{C}\mathcal{S}) =$

$\{\sum_{L\in\mathcal{S}}\lambda_L e(L) \mid \lambda_L \in \mathbb{C}^\times\}$ be the set of unit elements on $\mathbb{C}\mathcal{S}$. $\mathcal{U}(\mathbb{C}\mathcal{S})$ is a multiplicative right G-set by the action

$$\left(\sum_{L\in\mathcal{S}}\lambda_L e(L)\right)\cdot a = \sum_{L\in\mathcal{S}}\lambda_L e(L\cdot a) \quad \text{for } a\in G.$$

Set $\alpha(a,b) = \sum_{L\in\mathcal{S}}\alpha_L(a,b)e(L)$. Then $(\alpha(a,b)\cdot c)\alpha(ab,c) = \alpha(a,bc)\alpha(b,c)$ hold for all $a,b,c\in G$. So $\alpha: G\times G\to\mathcal{U}(\mathbb{C}\mathcal{S})$ is a 2-cocycle.

Define the vector space $\mathcal{A}_\alpha(G,\mathcal{S}) = \mathbb{C}[G]\otimes\mathbb{C}\mathcal{S}$ with a basis $a\otimes e(L)$ for $a\in G$ and $L\in\mathcal{S}$ and a multiplication on it:

$$a\otimes e(L)\cdot b\otimes e(M) = \alpha_M(a,b)ab\otimes e(L\cdot b)e(M).$$

Then $\mathcal{A}_\alpha(G,\mathcal{S})$ is an associative algebra with the identity element $\sum_{L\in\mathcal{S}} 1\otimes e(L)$.

We define an action of $\mathcal{A}_\alpha(G,\mathcal{S})$ on $\mathcal{L} = \bigoplus_{L\in\mathcal{S}} L$ as follows: For $L, M\in\mathcal{S},\ w\in M$ and $a\in G$ we set

$$a\otimes e(L)\cdot w = \delta_{L,M}\phi(a,L)w.$$

Note that the actions of $\mathcal{A}_\alpha(G,\mathcal{S})$ and V^G on $\mathcal{L}$ commute with each other.

For each $L\in\mathcal{S}$ set $G_L = \{a\in G \mid L\circ a\simeq L \text{ as } V\text{-modules}\}$. Let $\mathcal{O}_L$ be the orbit of L under the action of G and let $G = \bigsqcup_{j=1}^k G_L g_j$ be a right coset decomposition with $g_1 = 1$. Then $\mathcal{O}_L = \{L\cdot g_j \mid j = 1,\ldots,k\}$ and $G_{L\cdot g_j} = g_j^{-1}G_L g_j$. We define several subspaces of $\mathcal{A}_\alpha(G,\mathcal{S})$ by:

$$\begin{aligned} S(L) &= \mathrm{Span}_{\mathbb{C}}\{a\otimes e(L) \mid a\in G_L\},\\ D(L) &= \mathrm{Span}_{\mathbb{C}}\{a\otimes e(L) \mid a\in G\} \quad \text{and}\\ D(\mathcal{O}_L) &= \mathrm{Span}_{\mathbb{C}}\{a\otimes e(L\cdot g_j) \mid j = 1,\ldots,k,\ a\in G\}. \end{aligned}$$

Decompose $\mathcal{S}$ into a disjoint union of orbits $\mathcal{S} = \bigsqcup_{j\in J}\mathcal{O}_j$. Let $L^{(j)}$ be a representative element of $\mathcal{O}_j$. Then

$$\mathcal{O}_j = \{L^{(j)}\cdot a \mid a\in G\} \quad \text{and} \quad \mathcal{A}_\alpha(G,\mathcal{S}) = \bigoplus_{j\in J} D(\mathcal{O}_{L^{(j)}}).$$

We recall the following properties of $\mathcal{A}_\alpha(G,\mathcal{S})$.

Lemma 1 [6, Lemma 3.4]. *Let $L\in\mathcal{S}$ and $G = \bigsqcup_{j=1}^k G_L g_j$. Then*

(1) *$S(L)$ is a subalgebra of $\mathcal{A}_\alpha(G,\mathcal{S})$ isomorphic to $\mathbb{C}^{\alpha_L}[G_L]$ where $\mathbb{C}^{\alpha_L}[G_L]$ is the twisted group algebra with 2-cocycle α_L.*
(2) *$D(\mathcal{O}_L) = \bigoplus_{j=1}^k D(L\cdot g_j)$ is a direct sum of left ideals.*

(3) *Each* $D(\mathcal{O}_L)$ *is a two sided ideal of* $\mathcal{A}_\alpha(G,\mathcal{S})$ *and* $\mathcal{A}_\alpha(G,\mathcal{S}) = \bigoplus_{j\in J} D(\mathcal{O}_{L^{(j)}})$. *Moreover,* $D(\mathcal{O}_L)$ *has the identity element* $\sum_{M\in\mathcal{O}_L} 1\otimes e(M)$.

Lemma 2 [6, Theorem 3.6].

(1) $D(\mathcal{O}_L)$ *is semisimple for all* $L \in \mathcal{S}$ *and the simple* $D(\mathcal{O}_L)$*-modules are precisely equal to*

$$\mathrm{Ind}_{S(L)}^{D(L)} U = D(L) \otimes_{S(L)} U$$

where U *ranges over the simple* $\mathbb{C}^{\alpha_L}[G_L]$*-modules.*

(2) $\mathcal{A}_\alpha(G,\mathcal{S})$ *is semisimple and simple* $\mathcal{A}_\alpha(G,\mathcal{S})$*-modules are precisely* $\mathrm{Ind}_{S(L^{(j)})}^{D(L^{(j)})} U$ *where* U *ranges over the simple* $\mathbb{C}^{\alpha_{L^{(j)}}}[G_{L^{(j)}}]$*-modules and* $j \in J$.

For $L \in \mathcal{S}$ let Λ_{G_L,α_L} be the set of all irreducible characters λ of $\mathbb{C}^{\alpha_L}[G_L]$. We denote the corresponding simple module by $U(L,\lambda)$. Note that L is a semisimple $\mathbb{C}^{\alpha_L}[G_L]$-module. Let L^λ be the sum of simple $\mathbb{C}^{\alpha_L}[G_L]$-submodules of L isomorphic to $U(L,\lambda)$. Then

$$L = \bigoplus_{\lambda\in\Lambda_{G_L,\alpha_L}} L^\lambda.$$

Moreover $L^\lambda = U(L,\lambda) \otimes L_\lambda$ where $L_\lambda = \mathrm{Hom}_{\mathbb{C}^{\alpha_L}[G_L]}(U(L,\lambda), L)$ is the multiplicity space of $U(L,\lambda)$ in L. We can realize L_λ as a subspace of L in the following way: Let $w \in U(L,\lambda)$ be a fixed nonzero vector. Then we can identify $\mathrm{Hom}_{\mathbb{C}^{\alpha_L}[G_L]}(U(L,\lambda), L)$ with the subspace

$$\bigl\{ f(w) \mid f \in \mathrm{Hom}_{\mathbb{C}^{\alpha_L}[G_L]}\bigl(U(L,\lambda), L\bigr)\bigr\}$$

of L^λ. Note that the actions of $\mathbb{C}^{\alpha_L}[G_L]$ and V^{G_L} on L commute with each other. So L^λ and L_λ are ordinary V^{G_L}-modules. Furthermore, L^λ and L_λ are ordinary V^G-modules.

For convenience, we set

$$G_j = G_{L^{(j)}}, \quad \Lambda_j = \Lambda_{L^{(j)},\alpha_{L^{(j)}}} \quad \text{and} \quad U_{(j,\lambda)} = U\bigl(L^{(j)}, \lambda\bigr)$$

for $j \in J$ and $\lambda \in \Lambda_j$. We denote by Γ the set $\{(j,\lambda) \mid j \in J, \lambda \in \Lambda_j\}$. We have a decomposition

$$L^{(j)} = \bigoplus_{\lambda\in\Lambda_j} U_{(j,\lambda)} \otimes M_{(j,\lambda)}$$

as a $\mathbb{C}^{\alpha_{L^{(j)}}}[G_j] \otimes V^{G_j}$-module. We also have

$$\mathcal{L} = \bigoplus_{(j,\lambda)\in\Gamma} \mathrm{Ind}_{S(L^{(j)})}^{D(L^{(j)})} U_{(j,\lambda)} \otimes M_{(j,\lambda)}$$

as a $\mathcal{A}_\alpha(G,\mathcal{S}) \otimes V^G$-module.

For $(j,\lambda) \in \Gamma$ we set

$$W_{(j,\lambda)} = \mathrm{Ind}_{S(L^{(j)})}^{D(L^{(j)})} U_{(j,\lambda)}.$$

Then $W_{(j,\lambda)}$ forms a complete list of simple $\mathcal{A}_\alpha(G,\mathcal{S})$-modules by Lemma 2.

A duality theorem of Schur–Weyl type holds.

Theorem 1 [6, Theorem 6.14]. *As a $\mathcal{A}_\alpha(G,\mathcal{S}) \otimes V^G$-module,*

$$\mathcal{L} = \bigoplus_{(j,\lambda)\in\Gamma} W_{(j,\lambda)} \otimes M_{(j,\lambda)}.$$

Moreover,

(1) *Each $M_{(j,\lambda)}$ is a nonzero irreducible V^G-module.*
(2) *$M_{(j_1,\lambda_1)}$ and $M_{(j_2,\lambda_2)}$ are isomorphic V^G-modules if and only if $(j_1,\lambda_1) = (j_2,\lambda_2)$.*

2.2. Intertwining operators and fusion rules

We recall the definition of intertwining operators for V-modules which is introduced in [8].

Definition 1. Let L^i $(i = 1,2,3)$ be V-modules. An intertwining operator of type $\binom{L^3}{L^1\ L^2}$ is a linear map

$$I(\cdot,z) : L^1 \to \mathrm{Hom}_{\mathbb{C}}\big(L^2, L^3\big)\{z\},$$
$$v \mapsto I(v,z) = \sum_{\gamma\in\mathbb{C}} v_\gamma z^{-\gamma-1},$$

which satisfies the following conditions: Let $u \in V$, $v \in L^1$, and $w \in L^2$.

(1) For any fixed $\gamma \in \mathbb{C}$, $v_{\gamma+n}w = 0$ for $n \in \mathbb{Z}$ sufficiently large.
(2) $I(L(-1)v,z) = \frac{d}{dz} I(v,z)$.
(3) $z_0^{-1}\delta(\frac{z_1-z_2}{z_0}) Y_{L^3}(u,z_1) I(v,z_2)w - z_0^{-1}\delta(\frac{-z_2+z_1}{z_0}) I(v,z_2) Y_{L^2}(u,z_1)w$
$= z_2^{-1}\delta(\frac{z_1-z_0}{z_2}) I(Y_{L^1}(u,z_0)v,z_2)w.$

We denote by $I_V\binom{L^3}{L^1\ L^2}$ the set of all intertwining operators of type $\binom{L^3}{L^1\ L^2}$. The dimension of $I_V\binom{L^3}{L^1\ L^2}$ is called the *fusion rule* of type $\binom{L^3}{L^1\ L^2}$.

For a V-module $L = \bigoplus_{n=0}^\infty L(n)$, it is shown in [8, Theorem 5.2.1] that the graded vector space $L' = \bigoplus_{n=0}^\infty L(n)^*$ carries the structure of a V-module, where $L(n)^* = \mathrm{Hom}_{\mathbb{C}}(L(n),\mathbb{C})$. L' is called the *contragredient module* of L.

The fusion rules have some symmetries.

Lemma 3 [8, Propositions 5.4.7 and 5.5.2]. *Let L^i $(i = 1, 2, 3)$ be V-modules. Then*

$$\dim_{\mathbb{C}} I_V\binom{L^3}{L^1\ L^2} = \dim_{\mathbb{C}} I_V\binom{L^3}{L^2\ L^1} = \dim_{\mathbb{C}} I_V\binom{(L^2)'}{L^1\ (L^3)'}.$$

Let L^1 and L^2 be irreducible V-modules. We use a notation

$$L^1 \times L^2 = \sum_{L^3} \dim_{\mathbb{C}} I_V\binom{L^3}{L^1\ L^2} L^3$$

to represent the fusion rules, where L^3 ranges over the irreducible V-modules. Note that $L^1 \times L^2 = L^2 \times L^1$ by Lemma 3.

3. Intertwining operators for irreducible V^G-modules which occur as submodules of irreducible V-modules

In this section we give a construction of intertwining operators for irreducible V^G-modules which occur as submodules of irreducible V-modules by using intertwining operators for V.

Let $\mathcal{S}_i$ $(i = 1, 2, 3)$ be finite G-stable sets consisting of inequivalent irreducible V-modules. Set $\mathcal{L}_i = \bigoplus_{L \in \mathcal{S}_i} L$. For $L^i \in \mathcal{S}_i$ and $a \in G$, $\phi_i(a, L^i) : L^i \to L^i \cdot a^{-1}$ denote the fixed V-module isomorphisms. For $L^i \in \mathcal{S}_i$ and $a, b \in G$, $\alpha_{i,L^i} \in \mathbb{C}^\times$ denote nonzero complex numbers such that

$$\phi_i\big(b, L^i \cdot a^{-1}\big)\phi_i\big(a, L^i\big) = \alpha_{i,L^i}(b, a)\phi\big(ba, L^i\big).$$

Set $\alpha_i(a, b) = \sum_{L^i \in \mathcal{S}_i} \alpha_{i,L^i}(a, b)e(L^i)$. Let $\mathcal{S}_i = \bigsqcup_{j \in J_i} \mathcal{O}_j$ be the orbit decompositions under the action of G and set $\Gamma_i = \{(j_i, \lambda_i) \mid j_i \in J_i, \lambda_i \in \Lambda_{j_i}\}$.

For $f \in I_V\binom{L^3}{L^1\ L^2}$ and $a \in G$, we define ${}_af \in I_V\binom{L^3 \cdot a^{-1}}{L^1 \cdot a^{-1}\ L^2 \cdot a^{-1}}$ as follows: For $v \in L^1$ we set

$${}_af(v, z) = \phi_3\big(a, L^3\big) f\big(\phi_1\big(a, L^1\big)^{-1} v, z\big)\phi_2\big(a, L^2\big)^{-1}.$$

Set

$$\mathcal{I} = \bigoplus_{(L^1, L^2, L^3) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3} I_V\binom{L^3}{L^1\ L^2} \otimes_{\mathbb{C}} L^1 \otimes_{\mathbb{C}} L^2.$$

We define an action of $\mathcal{A}_{\alpha_3}(G, \mathcal{S})$ on $\mathcal{I}$ as follows: Let $L^i \in \mathcal{S}_i$ $(i = 1, 2, 3)$. For $a \otimes e(M) \in \mathcal{A}_{\alpha_3}(G, \mathcal{S})$, $v \in L^1$, $w \in L^2$, and $f \in I_V\binom{L^3}{L^1\ L^2}$, we set

$$\big(a \otimes e(M)\big) \cdot (f \otimes v \otimes w) = \delta_{M,L^3 \cdot a} f \otimes \phi_1\big(a, L^1\big)v \otimes \phi_2\big(a, L^2\big)w \in I_V\begin{pmatrix} L^3 \cdot a^{-1} \\ L^1 \cdot a^{-1}\ L^2 \cdot a^{-1} \end{pmatrix} \otimes L^1 \cdot a^{-1} \otimes L^2 \cdot a^{-1}.$$

We define a map $\Psi : \mathcal{I} \to \mathcal{L}_3\{z\}$ by

$$\Psi\big(f \otimes v^1 \otimes v^2\big) = f\big(v^1, z\big)v^2$$

for $v^1 \in L^1$, $v^2 \in L^2$, and $f \in I_V\binom{L^3}{L^1\ L^2}$, where $L^i \in \mathcal{S}_i$ $(i = 1, 2, 3)$. Note that Ψ is a $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-module homomorphism.

Lemma 4. *The map $\Psi : \mathcal{I} \to \mathcal{L}_3\{z\}$ is injective.*

Proof. We use the same method that was used in the proof of Lemma 3.1 of [5]. Assume false. Then there is a nonzero $X \in \operatorname{Ker} \Psi$. Since $\mathcal{L}_3 = \bigoplus_{L \in \mathcal{S}_3} L$, we may assume

$$X = \sum_{i,j} f^{ij} \otimes v^{1i} \otimes v^{2j},$$

where $v^{1i} \in L^{1i} (i = 1, \ldots, l_1)$ are linearly independent homogeneous vectors in $\mathcal{L}_1$, $v^{2j} \in L^{2j}$ $(j = 1, \ldots, l_2)$ are linearly independent homogeneous vectors in $\mathcal{L}_2$, $f^{ij} \in I_V\binom{L^3}{L^{1i}\ L^{2j}}$, $L^{1i} \in \mathcal{S}_1$, $L^{2j} \in \mathcal{S}_2$, and $L^3 \in \mathcal{S}_3$. We may also assume $f^{11} \otimes v^{11} \otimes v^{21}$ is nonzero. Since $\sum_{i,j} f^{ij}(v^{1i}, z)v^{2j} = 0$, for all $u \in V$ we have

$$\sum_{i,j} Y_{\mathcal{L}_3}(u, z_1) f^{ij}\big(v^{1i}, z\big)v^{2j} = 0.$$

Using the associativity of intertwining operators [4, Proposition 11.5],

$$\sum_{i,j} f^{ij}\big(Y_{\mathcal{L}_1}(u, z_1)v^{1i}, z\big)v^{2j} = 0. \tag{1}$$

We denote $Y_{\mathcal{L}_1}(u, z_1) = \sum_{n \in \mathbb{Z}} u_n^{\mathcal{L}_1} z_1^{-n-1}$. Fix $N \in \mathbb{Z}$ such that $v^{1i} \in \bigoplus_{n=0}^{N} L^{1i}(n)$ for all i. Since $\mathcal{S}_1$ consists of inequivalent irreducible V-modules, the linear map $\sigma_N : V \to \bigoplus_{L \in S_1} \bigoplus_{m=0}^{N} \operatorname{End}_{\mathbb{C}} L(m)$ defined by

$$\sigma_N(u) = u^{\mathcal{L}_1}_{\mathrm{wt}\,u-1}$$

for homogeneous $u \in V$ is an epimorphism by Lemma 6.13 of [6]. So there exists $u^1 \in V$ such that $\sigma_N(u^1)v^{1i} = \delta_{1,i} v^{11}$. From formula (1), we have

$$0 = \sum_{i,j} f^{ij}\big(\sigma_N\big(u^1\big)v^{1i}, z\big)v^{2j} = \sum_j f^{1j}\big(v^{11}, z\big)v^{2j}. \tag{2}$$

Therefore, for all $u \in V$ we have

$$\sum_j Y_{\mathcal{L}_3}(u, z_1) f^{1j}\big(v^{11}, z\big) v^{2j} = 0.$$

Using the commutativity of intertwining operators [4, Proposition 11.4],

$$\sum_j f^{1j}\big(v^{11}, z\big) Y_{\mathcal{L}_2}(u, z_1) v^{2j} = 0.$$

Since $\mathcal{S}_2$ consists of inequivalent irreducible V-modules, we have

$$f^{11}\big(v^{11}, z\big) v^{21} = 0$$

for the same reason to obtain formula (2). So $f^{11} \otimes v^{11} \otimes v^{21} \in \operatorname{Ker} \Psi$. Since L^{11} and L^{21} are irreducible V-modules and $f^{11} \otimes v^{11} \otimes v^{21}$ is nonzero, this contradicts Proposition 11.9 of [4]. $\square$

We have the decomposition of each $\mathcal{L}_i$ as a $\mathcal{A}_{\alpha_i}(G, \mathcal{S}_i)$-module in Theorem 1:

$$\mathcal{L}_i = \bigoplus_{(j_i, \lambda_i) \in \Gamma_i} W^i_{(j_i, \lambda_i)} \otimes M^i_{(j_i, \lambda_i)}.$$

For $i = 1, 2$ let $(j_i, \lambda_i) \in \Gamma_i$ and let $v^{i0} \in M^i_{(j_i, \lambda_i)}$. Set

$$\begin{aligned}
&\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}\big(v^{10}, v^{20}\big) \\
&\quad = \operatorname{Span}_{\mathbb{C}}\big\{ f \otimes \big(w^1 \otimes v^{10}\big) \otimes \big(w^2 \otimes v^{20}\big) \in \mathcal{I} \bigm| w^1 \in W^1_{(j_1, \lambda_1)},\ w^2 \in W^2_{(j_2, \lambda_2)} \big\}.
\end{aligned}$$

$\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})$ is a $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-submodule of $\mathcal{I}$. For any nonzero $v^{10}, v^1 \in M^1_{(j_1, \lambda_1)}$ and nonzero $v^{20}, v^2 \in M^2_{(j_2, \lambda_2)}$, $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})$ and $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^1, v^2)$ are isomorphic $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-modules.

Theorem 2. *Fix a nonzero $v^{10} \in M^1_{(j_1, \lambda_1)}$ and a nonzero $v^{20} \in M^2_{(j_2, \lambda_2)}$.*

For any $((j_1, \lambda_1), (j_2, \lambda_2), (j_3, \lambda_3)) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3$, there exists an injective linear map

$$\operatorname{Hom}_{\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)}\big(W^3_{(j_3, \lambda_3)}, \mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}\big(v^{10}, v^{20}\big)\big) \to I_{V^G}\begin{pmatrix} & M^3_{(j_3, \lambda_3)} & \\ M^1_{(j_1, \lambda_1)} & & M^2_{(j_2, \lambda_2)} \end{pmatrix}.$$

In particular,

$$\dim_{\mathbb{C}} I_{V^G}\begin{pmatrix} & M^3_{(j_3, \lambda_3)} & \\ M^1_{(j_1, \lambda_1)} & & M^2_{(j_2, \lambda_2)} \end{pmatrix} \geqslant \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)}\big(W^3_{(j_3, \lambda_3)}, \mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}\big(v^{10}, v^{20}\big)\big).$$

Proof. For convenience, we set

$$W^i = W^i_{(j_i,\lambda_i)}, \quad M^i = M^i_{(j_i,\lambda_i)} \quad (i = 1, 2, 3),$$
$$\mathcal{A}_3 = \mathcal{A}_{\alpha_3}(G, \mathcal{S}_3), \quad \text{and} \quad \mathcal{I}_{12}\big(v^{10}, v^{20}\big) = \mathcal{I}_{(j_1,\lambda_1),(j_2,\lambda_2)}\big(v^{10}, v^{20}\big).$$

Fix a nonzero $w^{30} \in W^3$. Let $F \in \mathrm{Hom}_{\mathcal{A}_3}(W^3, \mathcal{I}_{12}(v^{10}, v^{20}))$. We shall define $\Phi(F) \in I_{V^G}\binom{M^3}{M^1\ M^2}$.

For $v^1 \in M^1$, $v^2 \in M^2$, we set

$$H\big(v^1, v^2\big) \in \mathrm{Hom}_{\mathcal{A}_3}\big(W^3, \mathcal{I}_{12}\big(v^1, v^2\big)\big)$$

as follows: For $w^3 \in W^3$, let

$$F\big(w^3\big) = \sum_i f^i \otimes \big(w^{1i} \otimes v^{10}\big) \otimes \big(w^{2i} \otimes v^{20}\big) \in \mathcal{I}_{12}\big(v^{10}, v^{20}\big),$$

where $w^{1i} \otimes v^{10} \in L^{1i}$, $w^{2i} \otimes v^{20} \in L^{2i}$, $f^i \in I_V\binom{L^{3i}}{L^{1i}\ L^{2i}}$ and $L^{ji} \in \mathcal{S}_j$ $(j = 1, 2, 3)$. Note that $w^{ji} \otimes v^j \in L^{ji}$ by the definition of M^j for $j = 1, 2$. We define

$$H\big(v^1, v^2\big)\big(w^3\big) = \sum_i f^i \otimes \big(w^{1i} \otimes v^1\big) \otimes \big(w^{2i} \otimes v^2\big).$$

It is clear that $H(v^1, v^2) \in \mathrm{Hom}_{\mathcal{A}_3}(W^3, \mathcal{I}_{12}(v^1, v^2))$. Since the map $\Psi : \mathcal{I} \to \mathcal{L}_3\{z\}$ is a $\mathcal{A}_3$-module homomorphism, the map $w^3 \mapsto \Psi(H(v^1, v^2)(w^3))$ is a $\mathcal{A}_3$-module homomorphism from W^3 to $\mathcal{L}_3\{z\}$. So $\Psi(H(v^1, v^2)(W^3))$ is a $\mathcal{A}_3$-submodule of $(W^3 \otimes M^3)\{z\}$. Let $w^{3,1}, w^{3,2}, \ldots, w^{3,\dim_{\mathbb{C}} W^3}$ be a basis of W^3. Since W^3 is an irreducible $\mathcal{A}_3$-module, there exists $a \in \mathcal{A}_3$ such that $aw^{3,i} = \delta_{1,i} w^{3,1}$. Let $\Psi(H(v^1, v^2)(w^{3,1})) = \sum_i w^{3,i} \otimes p^i$, where $p_i \in M^3\{z\}$. Then

$$\begin{aligned}\Psi\big(H\big(v^1, v^2\big)\big(w^{3,1}\big)\big) &= \Psi\big(H\big(v^1, v^2\big)\big(aw^{3,1}\big)\big) = a\Psi\big(H\big(v^1, v^2\big)\big(w^{3,1}\big)\big) \\ &= a\sum_i w^{3,i} \otimes p^i = \sum_i \big(aw^{3,i}\big) \otimes p^i = w^{3,1} \otimes p^1.\end{aligned}$$

So $\Psi(H(v^1, v^2)(w^3)) \in (w^3 \otimes M^3)\{z\}$ for all $w^3 \in W^3$. We hence have an unique $\Phi(F)(v^1, z)v^2 \in M^3\{z\}$ such that

$$w^{30} \otimes \Phi(F)\big(v^1, z\big)v^2 = \Psi\big(H\big(v^1, v^2\big)\big(w^{30}\big)\big). \tag{3}$$

Since f^i are intertwining operators, we have $\Phi(F) \in I_{V^G}\binom{M^3}{M^1\ M^2}$ from formula (3).

We will show that Φ is injective. Suppose $\Phi(F) = 0$. Then

$$0 = w^{30} \otimes \Phi(F)\big(v^{10}, z\big)v^{20} = \Psi\big(F\big(w^{30}\big)\big).$$

Since Ψ is injective by Lemma 4, $F(w^{30}) = 0$. Since W^3 is an irreducible $\mathcal{A}_3$-module and $F \in \mathrm{Hom}_{\mathcal{A}_3}(W^3, \mathcal{I}_{12}(v^{10}, v^{20}))$, $F = 0$. □

Let $\mathbb{C}G$ be the group algebra of G and $\mathrm{Irr}\, G$ the set of all irreducible characters of G. We set $\mathcal{S}_i = \{V\}$ $(i = 1, 2, 3)$ in Theorem 2. Then $\mathcal{A}_{\alpha_i}(G, \mathcal{S}_i) = \mathbb{C}G$ and $\Gamma_i = \mathrm{Irr}\, G$. Note that $\dim_{\mathbb{C}} I_V\binom{V}{V\ V} = 1$ since V is simple. In this case we have the following result:

Corollary 3. *Let* $\chi_i \in \mathrm{Irr}\, G$ $(i = 1, 2, 3)$. *Then*

$$\dim_{\mathbb{C}} I_{V^G}\begin{pmatrix} V_{\chi_3} \\ V_{\chi_1}\ V_{\chi_2} \end{pmatrix} \geqslant \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(W_{\chi_3}, W_{\chi_1} \otimes_{\mathbb{C}} W_{\chi_2}).$$

In [2, Section 3], it is conjectured that if V is rational then for all $\chi_1, \chi_2 \in \mathrm{Irr}\, G$,

$$V_{\chi_1} \times V_{\chi_2} = \sum_{\chi_3 \in \mathrm{Irr}\, G} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(W_{\chi_3}, W_{\chi_1} \otimes_{\mathbb{C}} W_{\chi_2}) V_{\chi_3}.$$

The conjecture implies that if V is rational then the representation algebra of the finite group G is always realized as a subalgebra of the fusion algebra of V^G.

4. An application

In [3] we studied a vertex operator algebra $\mathcal{W}$ which is a realization of an algebra denoted by $[Z_3^{(5)}]$ in [7]. $\mathcal{W}$ is a fixed point subalgebra of a vertex operator algebra M_k^0. It is expected that the $\mathbb{Z}_3$ symmetry of $\mathcal{W}$ affords $3B$ elements of the Monster simple group [11]. For M_k^0, the irreducible modules are classified and the fusion rules are determined in [12]. Let L^i $(i = 1, 2, 3)$ be irreducible $\mathcal{W}$-modules such that L^1 and L^2 occur as submodules of irreducible M_k^0-modules. In this section we determine the fusion rule of type $\binom{L^3}{L^1\ L^2}$ by using Theorem 2.

4.1. Subalgebra M_k^0 of $V_{\sqrt{2}A_2}$

In this subsection we review some properties of M_k^0 in [12]. Let A_2 be the ordinary root lattice of type A_2 and $V_{\sqrt{2}A_2}$ the lattice vertex operator algebra associated with $\sqrt{2}A_2$. Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -\alpha_1 - \alpha_2$.

For basic definitions concerning lattice vertex operator algebras we refer to [4] and [9]. Our notation for the lattice vertex operator algebra is standard [9]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \sqrt{2}A_2$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n) \mid \alpha \in \mathfrak{h},\ n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$, $n > 0$, and $c = 1$. As a vector space $V_{\sqrt{2}A_2} = M(1) \otimes \mathbb{C}[\sqrt{2}A_2]$ and for each $v \in V_{\sqrt{2}A_2}$, a vertex operator

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V_{\sqrt{2}A_2})[[z, z^{-1}]]$$

is defined. The vector $\mathbf{1} = 1 \otimes 1$ is called the vacuum vector. We use the symbol $e^{\alpha}, \alpha \in \sqrt{2}A_2$ to denote a basis of $\mathbb{C}[\sqrt{2}A_2]$.

There exists an isometry τ of $\sqrt{2}A_2$ such that $\tau(\sqrt{2}\alpha_1) = \sqrt{2}\alpha_2$ and $\tau(\sqrt{2}\alpha_2) = \sqrt{2}\alpha_0$. The isometry τ lifts naturally to an automorphism of $V_{\sqrt{2}A_2}$:

$$\alpha^1(-n_1)\cdots\alpha^k(-n_k)e^{\beta} \mapsto \bigl(\tau\alpha^1\bigr)(-n_1)\cdots\bigl(\tau\alpha^k\bigr)(-n_k)e^{\tau\beta}.$$

By abuse of notation, we denote it by τ. Let G be the cyclic group generated by τ. Set

$$\begin{aligned}\omega^3 = {}& \frac{1}{15}\bigl(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2\bigr) \\ & + \frac{1}{10}\bigl(e^{\sqrt{2}\alpha_1} + e^{-\sqrt{2}\alpha_1} + e^{\sqrt{2}\alpha_2} + e^{-\sqrt{2}\alpha_2} + e^{\sqrt{2}\alpha_0} + e^{-\sqrt{2}\alpha_0}\bigr)\end{aligned}$$

and $M_k^0 = \{v \in V_{\sqrt{2}A_2} \mid (\omega^3)_1 v = 0\}$. Since $\tau\omega^3 = \omega^3$, M_k^0 is invariant under the action of τ. τ is an automorphism group of M_k^0 of order 3 by [3, Theorem 2.1]. Set

$$L_0 = L, \qquad L_a = \frac{\sqrt{2}\alpha_2}{2} + L, \qquad L_b = \frac{\sqrt{2}\alpha_0}{2} + L, \qquad L_c = \frac{\sqrt{2}\alpha_1}{2} + L$$

and

$$\begin{aligned} M_k^i &= \bigl\{v \in V_{L_i} \bigm| \bigl(\omega^3\bigr)_1 v = 0\bigr\}, \\ W_k^i &= \bigl\{v \in V_{L_i} \bigm| \bigl(\omega^3\bigr)_1 v = \tfrac{2}{5}v\bigr\}, \quad \text{for } i = 0, a, b, c. \end{aligned}$$

It is shown in [12] that $\{M_k^i, W_k^i \mid i = 0, a, b, c\}$ is the set of all irreducible M_k^0-modules and the fusion rules are determined.

4.2. Subalgebra $\mathcal{W}$ of M_k^0

We denote by $\mathcal{W}$ the subalgebra $(M_k^0)^{\tau}$ of fixed points of τ in M_k^0. We recall some properties of $\mathcal{W}$ in [3]. $\mathcal{W}$ is generated by the Virasoro element ω and an element J of weight 3. Let $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ and $Y(J, z) = \sum_{n\in\mathbb{Z}} J(n)z^{-n-3}$. They satisfy the following commutation relations:

$$\begin{aligned} \bigl[L(m), L(n)\bigr] &= (m-n)L(m+n) + \frac{m^3 - m}{12}\cdot\frac{6}{5}\cdot\delta_{m+n,0}, \\ \bigl[L(m), J(n)\bigr] &= (2m-n)J(m+n), \\ \bigl[J(m), J(n)\bigr] &= (m-n)\bigl(22(m+n+2)(m+n+3) + 35(m+2)(n+2)\bigr)L(m+n) \\ &\quad - 120(m-n)\biggl(\sum_{k\leqslant -2} L(k)L(m+n-k) + \sum_{k\geqslant -1} L(m+n-k)L(k)\biggr) \\ &\quad - \frac{7}{10}m\bigl(m^2-1\bigr)\bigl(m^2-4\bigr)\delta_{m+n,0}. \end{aligned} \tag{4}$$

$\mathcal{W}$ has exactly 20 irreducible modules. 8 irreducible $\mathcal{W}$-modules occur as submodules of irreducible M_k^0-modules. We introduce those 8 $\mathcal{W}$-modules. τ acts on the irreducible M_k^0-modules as follows:

$$M_k^0 \circ \tau \simeq M_k^0, \qquad W_k^0 \circ \tau \simeq W_k^0,$$
$$M_k^a \circ \tau \simeq M_k^c, \qquad M_k^c \circ \tau \simeq M_k^b, \qquad M_k^b \circ \tau \simeq M_k^a,$$
$$W_k^a \circ \tau \simeq W_k^c, \qquad W_k^c \circ \tau \simeq W_k^b, \qquad W_k^b \circ \tau \simeq W_k^a.$$

So $\{M_k^0\}$, $\{W_k^0\}$, $\{M_k^a, M_k^b, M_k^c\}$, and $\{W_k^a, W_k^b, W_k^c\}$ are G-stable sets. The automorphism τ of $V_{\sqrt{2}A_2}$ fixes ω^3 and so W_k^0 is invariant under τ. Hence we can take τ as $\phi(\tau, W_k^0)$ in Section 2.1. For these G-stable sets, we can take the 2-cocycles α in Section 2.1 to be trivial. Let $\xi = e^{2\pi\sqrt{-1}/3}$. We set

$$M_k^{0(i)} = \{v \in M_k^0 \mid \tau v = \xi^i v\} \quad \text{and} \quad W_k^{0(i)} = \{v \in W_k^0 \mid \tau v = \xi^i v\} \quad \text{for } i \in \mathbb{Z}.$$

Note that $\mathcal{W} = M_k^{0(0)}$. By Theorem 1, we have $M_k^{0(i)}$, $W_k^{0(i)}$ $(i = 0, 1, 2)$, M_k^a, and W_k^a are inequivalent irreducible $\mathcal{W}$-modules. Moreover, $M_k^a \simeq M_k^b \simeq M_k^c$ and $W_k^a \simeq W_k^b \simeq W_k^c$ as $\mathcal{W}$-modules. The contragredient modules of these $\mathcal{W}$-modules are

$$(M_k^{0(i)})' \simeq M_k^{0(-i)}, \quad (W_k^{0(i)})' \simeq W_k^{0(-i)} \quad (i = 0, 1, 2),$$
$$(M_k^a)' \simeq M_k^a \quad \text{and} \quad (W_k^a)' \simeq W_k^a.$$

All the other irreducible $\mathcal{W}$-modules occur as submodules of irreducible τ^i-twisted M_k^0-modules for $i = 1, 2$.

4.3. *An upper bound for the fusion rule*

We review some notations and formulas for the Zhu algebra $A(V)$ of an arbitrary vertex operator algebra V in [14] and the $A(V)$-bimodule $A(L)$ of an arbitrary V-module L in [10] and [13]. For $u, v \in V$ with u being homogeneous, define two bilinear operations

$$u * v = \operatorname{Res}_z\left(Y(u, z)v\frac{(1+z)^{\operatorname{wt} u}}{z}\right),$$
$$u \circ v = \operatorname{Res}_z\left(Y(u, z)v\frac{(1+z)^{\operatorname{wt} u}}{z^2}\right).$$

We extend $*$ and $\circ$ for arbitrary $u, v \in V$ by linearity. Let $O(V)$ be the subspace of V spanned by all $u \circ v$ for $u \in V, v \in L$. Set $A(V) = V/O(V)$. By [14, Theorem 2.1.1.], $O(V)$ is a two-sided ideal with respect to the operation $*$ and $(A(V), *)$ is an associative algebra with identity $1 + O(V)$. For every V-module N, $N(0)$ is a left $A(V)$-module.

Let L be a V-module. For $u \in V$, $v \in L$ with u being homogeneous, define three bilinear operations

$$u * v = \mathrm{Res}_z\left(Y(u,z)v\frac{(1+z)^{\mathrm{wt}\,u}}{z}\right),$$
$$v * u = \mathrm{Res}_z\left(Y(u,z)v\frac{(1+z)^{\mathrm{wt}\,u-1}}{z}\right),$$
$$u \circ v = \mathrm{Res}_z\left(Y(u,z)v\frac{(1+z)^{\mathrm{wt}\,u}}{z^2}\right).$$

We extend $*$ and $\circ$ for arbitrary $u \in V$, $v \in L$ by linearity. Let $O(L)$ be the subspace of L spanned by all $u \circ v$ for $u \in V$, $v \in L$. By [10, Theorem 1.5.1], $O(L)$ is a two-sided ideal with respect to the operation $*$. Thus it induces an operation on $A(L) = L/O(L)$. Denote by $[v]$ the image of $v \in L$ in $A(L)$. $A(L)$ is a $A(V)$-bimodule under the operation $*$. Using $A(L)$, we have an upper bound for every fusion rule.

Lemma 5 [13, Proposition 2.10]. *Let $L^i = \bigoplus_{n=0}^{\infty} L^i(n)$ $(i = 1, 2, 3)$ be irreducible V-modules. Then*

$$\dim_{\mathbb{C}} I_V\binom{L^3}{L^1\ L^2} \leqslant \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}\big(L^3(0)^* \otimes_{A(V)} A(L^1) \otimes_{A(V)} L^2(0)\big).$$

4.4. The fusion rules for irreducible $\mathcal{W}$-modules which occur as submodules of irreducible M_k^0-modules

Let L^i $(i = 1, 2, 3)$ be irreducible $\mathcal{W}$-modules such that L^1 and L^2 occur as submodules of irreducible M_k^0-modules. In this subsection we determine the fusion rule of type $\binom{L^3}{L^1\ L^2}$. As a first step, we give a lower bound for every fusion rule by using Theorem 2.

Lemma 6.

(1) *The fusion rule of following types is greater than or equal to* 1: *Let $i, j \in \{0, 1, 2\}$.*

$$\binom{M_k^{0(i+j)}}{M_k^{0(i)}\ M_k^{0(j)}}, \quad \binom{W_k^{0(i+j)}}{W_k^{0(i)}\ W_k^{0(j)}}, \quad \binom{W_k^{0(i+j)}}{M_k^{0(i)}\ W_k^{0(j)}}, \quad \binom{M_k^a}{M_k^{0(i)}\ M_k^a},$$
$$\binom{W_k^a}{M_k^{0(i)}\ W_k^a}, \quad \binom{M_k^a}{W_k^{0(i)}\ W_k^a}, \quad \binom{W_k^a}{W_k^{0(i)}\ W_k^a}.$$

(2) *The fusion rule of following types is greater than or equal to* 2:

$$\binom{M_k^a}{M_k^a\ M_k^a}, \quad \binom{W_k^a}{M_k^a\ W_k^a}, \quad \binom{W_k^a}{W_k^a\ W_k^a}.$$

Proof. We use notations in Section 3. The fusion rules for M_k^0 are obtained in [12].

(1) We consider the case that $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \{W_k^0\}$. We have

$$\dim_{\mathbb{C}} I_{M_k^0}\begin{pmatrix} W_k^0 \\ W_k^0\ W_k^0 \end{pmatrix} = 1.$$

Let $\mathbb{C}x_i$ $(i = 0, 1, 2)$ be the one dimensional G-modules such that $\tau \cdot x_i = \xi^i x_i$. Fix nonzero $v^i \in \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}x_i, W_k^{0(i)})$. Then

$$W_k^0 = \bigoplus_{i=0}^{2} x_i \otimes \mathrm{Hom}_{\mathbb{C}G}\big(\mathbb{C}x_i, W_k^{0(i)}\big),$$

$$\mathcal{I} = I_{M_k^0}\begin{pmatrix} W_k^0 \\ W_k^0\ W_k^0 \end{pmatrix} \otimes_{\mathbb{C}} W_k^0 \otimes_{\mathbb{C}} W_k^0, \quad \text{and}$$

$$\mathcal{I}_{W_k^{0(i)}, W_k^{0(j)}}\big(v^i, v^j\big) = I_{M_k^0}\begin{pmatrix} W_k^0 \\ W_k^0\ W_k^0 \end{pmatrix} \otimes \big(x_i \otimes v^i\big) \otimes \big(x_j \otimes v^j\big).$$

Let $f \in I_{M_k^0}\binom{W_k^0}{W_k^0\ W_k^0}$ be a nonzero intertwining operator. Then

$${}_\tau f = f$$

by the construction of intertwining operators in [12]. So

$$\big(\tau \otimes e\big(W_k^0\big)\big) \cdot f \otimes \big(x_i \otimes v^i\big) \otimes \big(x_j \otimes v^j\big) = \xi^{i+j} f \otimes \big(x_i \otimes v^i\big) \otimes \big(x_j \otimes v^j\big)$$

and

$$\dim_{\mathbb{C}} I_{\mathcal{W}}\begin{pmatrix} W_k^{0(i+j)} \\ W_k^{0(i)}\ W_k^{0(j)} \end{pmatrix} \geqslant 1.$$

We can compute the other cases in the same way.

(2) We consider the case that $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = \{M_k^a, M_k^b, M_k^c\}$. For $i_1, i_2, i_3 \in \{a, b, c\}$, we have

$$\dim_{\mathbb{C}} I_{M_k^0}\begin{pmatrix} M_k^{i_3} \\ M_k^{i_1}\ M_k^{i_2} \end{pmatrix} = \begin{cases} 1, & \text{if } \{i_1, i_2, i_3\} = \{a, b, c\}, \\ 0, & \text{otherwise.} \end{cases}$$

We define an action of τ on $\{a, b, c\}$ by $\tau(a) = b$, $\tau(b) = c$ and $\tau(c) = a$. It is possible to take $\phi(\tau, M_k^i)$ $(i = a, b, c)$ such that

$$\phi\big(\tau, M_k^c\big)\phi\big(\tau, M_k^b\big)\phi\big(\tau, M_k^a\big) = \mathrm{id}_{M_k^a}.$$

Fix a nonzero $v^a \in M_k^a$ and set

$$v^b = \phi\bigl(\tau, M_k^a\bigr)v^a \in M_k^b, \qquad v^c = \phi\bigl(\tau, M_k^b\bigr)v^b \in M_k^c.$$

Fix nonzero $f_{a,b,c} \in I_{M_k^0}\binom{M_k^c}{M_k^a\ M_k^b}$, $f_{b,a,c} \in I_{M_k^0}\binom{M_k^c}{M_k^b\ M_k^a}$ and set

$$f_{b,c,a} = {}_\tau f_{a,b,c} \in I_{M_k^0}\begin{pmatrix} M_k^a \\ M_k^b\ M_k^c \end{pmatrix},$$

$$f_{c,a,b} = {}_\tau f_{b,c,a} \in I_{M_k^0}\begin{pmatrix} M_k^b \\ M_k^c\ M_k^a \end{pmatrix},$$

$$f_{c,b,a} = {}_\tau f_{b,a,c} \in I_{M_k^0}\begin{pmatrix} M_k^a \\ M_k^c\ M_k^b \end{pmatrix},$$

$$f_{a,c,b} = {}_\tau f_{c,b,a} \in I_{M_k^0}\begin{pmatrix} M_k^b \\ M_k^a\ M_k^c \end{pmatrix}.$$

We have

$$\mathcal{I} = \bigoplus_{\substack{i_1,i_2,i_3\in\{a,b,c\} \\ \{i_1,i_2,i_3\}=\{a,b,c\}}} I_{M_k^0}\begin{pmatrix} M_k^{i_3} \\ M_k^{i_1}\ M_k^{i_2} \end{pmatrix} \otimes_{\mathbb{C}} M_k^{i_1} \otimes_{\mathbb{C}} M_k^{i_2}$$

and

$$\mathcal{I}_{M_k^a,M_k^a} = \bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(a),\tau^i(b),\tau^i(c)} \otimes v^{\tau^i(a)} \otimes v^{\tau^i(b)} \oplus \bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(b),\tau^i(a),\tau^i(c)} \otimes v^{\tau^i(b)} \otimes v^{\tau^i(a)}.$$

Since $\bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(a),\tau^i(b),\tau^i(c)} \otimes v^{\tau^i(a)} \otimes v^{\tau^i(b)}$ and $\bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(b),\tau^i(a),\tau^i(c)} \otimes v^{\tau^i(b)} \otimes v^{\tau^i(a)}$ are isomorphic irreducible $\mathcal{A}_{\alpha_3}(G, \mathcal{S}_3)$-modules, we have

$$\dim_{\mathbb{C}} I_{\mathcal{W}}\begin{pmatrix} M_k^a \\ M_k^a\ M_k^a \end{pmatrix} \geqslant 2.$$

We can compute the other cases in the same way. □

We will show that every fusion rule meets the lower bound obtained in Lemma 6. We use Lemma 5. For an irreducible $\mathcal{W}$-module $N = \bigoplus_{n=0}^{\infty} N(n)$, we fix a nonzero vector in $N(0)$ and denote it by w_N. By the same argument as in [3, Lemma 5.2], N is spanned by the vectors of the form

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)w_N \tag{5}$$

with $m_1 \geqslant \cdots \geqslant m_p \geqslant 1$, $n_1 \geqslant \cdots \geqslant n_q \geqslant 1$, $p = 0, 1, \ldots,$ and $q = 0, 1, \ldots.$ By the same argument as in [3, Section 5.4], we have for $n \leqslant -1$ and $u \in N$,

$$\begin{aligned} L(n)u &= (-1)^{n-1}[\omega] * [u] + (-1)^{n-1} n[u] * [\omega] + (-1)^n \operatorname{wt} u[u], \\ J(n)u &= (-1)^n \biggl(n\bigl[J(-1)u\bigr] + (n+1)\bigl[J[0]u\bigr] - (n+1)[J] * [u] \\ &\quad - \frac{n(n+1)}{2}[u] * [J] \biggr) \end{aligned} \tag{6}$$

in $A(N)$. Using formula (6) and commutation relations (4) repeatedly, it is shown that $A(N)$ is generated by $\{J(-1)^i w_N\}_{i=0}^{\infty}$ as a $A(\mathcal{W})$-bimodule.

A *singular vector* w of weight h for N is by definition a vector w which satisfies $L(0)w = hw$ and $L(n)w = J(n)w = 0$ for $n \geqslant 1$. By commutation relations (4), it is easy to show that w is a singular vector of weight h if and only if $L(0)w = hw$ and $L(1)w = L(2)w = J(1)w = 0$. If $h - \operatorname{wt} N(0) > 0$, then the submodule of N generated by w does not contain $N(0)$. We hence have $w = 0$.

Theorem 4. *The fusion rule $L^1 \times L^2$ is given by the following list, where $\{L^1, L^2\}$ is an arbitrary pair of irreducible $\mathcal{W}$-modules which occur as submodules of irreducible M_k^0-modules*: *Let $i, j \in \{0, 1, 2\}$.*

$$\begin{aligned} M_k^{0(i)} \times M_k^{0(j)} &= M_k^{0(i+j)}, \\ M_k^{0(i)} \times W_k^{0(j)} &= W_k^{0(i+j)}, \\ W_k^{0(i)} \times W_k^{0(j)} &= M_k^{0(i+j)} + W_k^{0(i+j)}, \\ M_k^{0(i)} \times M_k^a &= M_k^a, \\ M_k^{0(i)} \times W_k^a &= W_k^a, \\ W_k^{0(i)} \times M_k^a &= W_k^a, \\ W_k^{0(i)} \times W_k^a &= M_k^a + W_k^a, \\ M_k^a \times M_k^a &= \sum_{i=0}^{2} M_k^{0(i)} + 2M_k^a, \\ M_k^a \times W_k^a &= \sum_{i=0}^{2} W_k^{0(i)} + 2W_k^a, \\ W_k^a \times W_k^a &= \sum_{i=0}^{2} M_k^{0(i)} + \sum_{i=0}^{2} W_k^{0(i)} + 2M_k^a + 2W_k^a. \end{aligned}$$

Proof. For an irreducible $\mathcal{W}$-module N, h_N denotes the eigenvalue for $L(0)$ on $N(0)$ and k_N denotes the eigenvalue for $J(0)$ on $N(0)$. For all irreducible $\mathcal{W}$-modules those eigenvalues are computed in [3]. For irreducible $\mathcal{W}$-modules L^i $(i = 1, 2, 3)$, set

$$F\left(L^1, L^2, L^3\right) = \dim_{\mathbb{C}}\left(L^3(0)^* \otimes_{A(\mathcal{W})} A\left(L^1\right) \otimes_{A(\mathcal{W})} L^2(0)\right).$$

Note that $F(L^1, L^2, L^3)$ is lower than or equal to the number of generators of $A(L^1)$ as a $A(\mathcal{W})$-bimodule since the dimension of top level of every irreducible $\mathcal{W}$-module is 1.

The simplicity of $\mathcal{W}$ implies that

$$\dim_{\mathbb{C}} I_{\mathcal{W}}\begin{pmatrix} L^3 \\ \mathcal{W}\ L^2 \end{pmatrix} = \begin{cases} 1, & \text{if } L^2 \simeq L^3, \\ 0, & \text{otherwise} \end{cases}$$

for all irreducible $\mathcal{W}$-modules L^2 and L^3. We consider the case $L^1 \neq \mathcal{W}$. We will show that $F(L^1, L^2, L^3)$ is lower than or equal to the lower bound for $\dim_{\mathbb{C}} I_{\mathcal{W}}\binom{L^3}{L^1\ L^2}$ given in Lemma 6. Then, by Lemmas 3 and 5 we get the desired results. We use a computer algebra system Risa/Asir to find singular vectors used in the following argument.The explicit forms of those singular vectors are given in Appendix.

(1) We consider the case $L^1 = W_k^{0(1)}$. Since $(5\sqrt{-3}L(-1) + J(-1))w_{W_k^{0(1)}}$ is a singular vector and $A(W_k^{0(1)})$ is generated by

$$\left\{J(-1)^i w_{W_k^{0(1)}}\right\}_{i=0}^{\infty}$$

as a $A(\mathcal{W})$-bimodule, $A(W_k^{0(1)})$ is generated by $w_{W_k^{0(1)}}$ as a $A(\mathcal{W})$-bimodule. So

$$F\left(W_k^{0(1)}, L^2, L^3\right) \leqslant 1.$$

Set

$$\begin{aligned} w_1 = \big(&-30\sqrt{-3}L(-1)J(-1) + 39\sqrt{-3}J(-2) \\ &+ 5J(-1)^2 + 336L(-2) + 405L(-1)^2\big)w_{W_k^{0(1)}}. \end{aligned}$$

Since w_1 is a singular vector, we have a relation

$$\begin{aligned} 0 = {}& 50\sqrt{-3}\left([\omega]^2 * [w_{W_k^{0(1)}}] + [w_{W_k^{0(1)}}] * [\omega]^2\right) \\ & - 20\sqrt{-3}\left([\omega] * [w_{W_k^{0(1)}}] + [w_{W_k^{0(1)}}] * [\omega]\right) \\ & + 4\sqrt{-3}[w_{W_k^{0(1)}}] - 100\sqrt{-3}[\omega] * [w_{W_k^{0(1)}}] * [\omega] \\ & - 5[w_{W_k^{0(1)}}] * [J] + 5[J] * [w_{W_k^{0(1)}}] \end{aligned}$$

in $A(W_0^k(1))$ by using formulas (6). Therefore,

$$0 = \big(50\sqrt{-3}(h_{L^2}^2 + h_{L^3}^2) - 20\sqrt{-3}(h_{L^2} + h_{L^3}) \\ + 4\sqrt{-3} - 100\sqrt{-3}h_{L^2}h_{L^3} - 5k_{L^2} + 5k_{L^3}\big) w_{(L^3)'} \otimes [w_{W_k^{0(1)}}] \otimes w_{L^2}$$

in $L^3(0)^* \otimes_{A(\mathcal{W})} A(W_k^{0(1)}) \otimes_{A(\mathcal{W})} L^2(0)$. Set

$$\psi(h_{L^2}, k_{L^2}, h_{L^2}, k_{L^3}) = 50\sqrt{-3}\big(h_{L^2}^2 + h_{L^2}^2\big) - 20\sqrt{-3}(h_{L^2} + h_{L^3}) \\ + 4\sqrt{-3} - 100\sqrt{-3}h_{L^2}h_{L^3} - 5k_{L^2} + 5k_{L^3}.$$

If $\psi(h_{L^2}, k_{L^2}, h_{L^2}, k_{L^3}) \neq 0$, then

$$w_{(L^3)'} \otimes [w_{W_k^{0(1)}}] \otimes w_{L^2} = 0 \quad \text{and} \quad F\big(W_k^{0(1)}, L^2, L^3\big) = 0.$$

By computing $\psi(h_{L^2}, k_{L^2}, h_{L^3}, k_{L^3})$ for all pairs (L^2, L^3) of 20 irreducible $\mathcal{W}$-modules, we have $F(W_k^{0(1)}, L^2, L^3) \leqslant 1$ if the pair (L^2, L^3) is one of

$$\big(M_k^{0(i)}, W_k^{0(1+i)}\big), \quad \big(W_k^{0(i)}, M_k^{0(1+i)}\big), \quad \big(W_k^{0(i)}, W_k^{0(1+i)}\big) \quad (i = 0, 1, 2), \\ \big(M_k^a, W_k^a\big), \qquad \big(W_k^a, M_k^a\big), \qquad \big(W_k^a, W_k^a\big)$$

and $F(W_k^{0(1)}, L^2, L^3) = 0$ otherwise. Combining these results and Lemma 6, we have

$$\begin{aligned} W_k^{0(1)} \times M_k^{0(i)} &= W_k^{0(1+i)}, \\ W_k^{0(1)} \times W_k^{0(i)} &= M_k^{0(1+i)} + W_k^{0(1+i)}, \\ W_k^{0(1)} \times M_k^a &= W_k^a, \\ W_k^{0(1)} \times W_k^a &= M_k^a + W_k^a. \end{aligned}$$

In the case $L^1 = W_k^{0(2)}$, we can compute the fusion rules in the same way. In the case $L^1 = M_k^{0(i)}$ $(i = 1, 2)$, it is shown that $A(M_k^{0(i)})$ is generated by $[w_{M_k^{0(i)}}]$ in the same way. But in these cases we need two singular vectors

$$v_{31} \in M_k^{0(i)}(3) \quad \text{and} \quad v_{41} \in M_k^{0(i)}(4).$$

We can compute the fusion rules by using two relations $v_{41} = 0$ and $J(-1)v_{31} = 0$.

(2) We consider the case $L^1 = W_k^{0(0)}$. There are two singular vectors $u_{2i} \in W_k^{0(0)}(2)$ $(i = 1, 2)$ and there is one singular vector $u_{41} \in W_k^{0(0)}(4)$. Since $u_{21} = 0$, $A(W_0^k(0))$ is generated by $\{w_{W_k^{0(0)}}, J(-1)w_{W_k^{0(0)}}\}$ as a $A(\mathcal{W})$-bimodule and for $X \in W_k^{0(0)}$ we have $a_1(L^2, L^3; X), a_2(L^2, L^3; X) \in \mathbb{C}$ such that

$$w_{(L^3)'} \otimes [X] \otimes w_{L^2} = a_1\big(L^2, L^3; X\big) w_{(L^3)'} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2}$$
$$+ a_2\big(L^2, L^3; X\big) w_{(L^3)'} \otimes \big[J(-1) w_{W_k^{0(0)}}\big] \otimes w_{L^2}.$$

Therefore, $F(W_k^{0(0)}, L^2, L^3) \leqslant 2$. Set a matrix

$$A = \begin{pmatrix} a_1(L^2, L^3; u_{22}) & a_2(L^2, L^3; u_{22}) \\ a_1(L^2, L^3; J(-1)u_{22}) & a_2(L^2, L^3; J(-1)u_{22}) \\ a_1(L^2, L^3; v_{41}) & a_2(L^2, L^3; u_{41}) \\ a_1(L^2, L^3; J(-1)u_{41}) & a_2(L^2, L^3; J(-1)u_{41}) \end{pmatrix}.$$

We have

$$A \begin{pmatrix} (w_{(L^3)'} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2} \\ w_{(L^3)'} \otimes [J(-1) w_{W_k^{0(0)}}] \otimes w_{L^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If rank $A = 2$, then

$$w_{(L^3)'} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2} = w_{(L^3)'} \otimes \big[J(-1) w_{W_k^{0(0)}}\big] \otimes w_{L^2} = 0$$

in $L^3(0)^* \otimes_{A(\mathcal{W})} A(W_k^{0(0)}) \otimes_{A(\mathcal{W})} L^2(0)$. If rank $A = 1$, then $w_{(L^3)'} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2}$ and $w_{(L^3)'} \otimes [J(-1) w_{W_k^{0(0)}}] \otimes w_{L^2}$ are linearly dependent. We hence have

$$F\big(W_k^{0(0)}, L^2, L^3\big) \leqslant \begin{cases} 0, & \text{if rank } A = 2, \\ 1, & \text{if rank } A = 1, \\ 2, & \text{if rank } A = 0. \end{cases}$$

By computing the rank of A for all pairs (L^2, L^3) of 20 irreducible $\mathcal{W}$-modules, we have for $i = 0, 1, 2$

$$W_k^{0(0)} \times M_k^{0(i)} = W_k^{0(i)},$$
$$W_k^{0(0)} \times W_k^{0(i)} = M_k^{0(i)} + W_k^{0(i)},$$
$$W_k^{0(0)} \times M_k^{a} = W_k^{a},$$
$$W_k^{0(0)} \times W_k^{a} = M_k^{a} + W_k^{a}.$$

In the case that L^1 is one of M_k^a and W_k^a, we can compute the fusion rules is the same way. We roughly explain each case. In the case $L^1 = M_k^a$, there are two singular vectors v_{21}, v_{22} in $M_k^a(2)$ and three singular vectors v_{61}, v_{62}, v_{63} in $M_k^a(6)$. Set the same matrix for $v_{22}, v_{61}, v_{62}, v_{63}, J(-1)v_{61}, J(-1)v_{62}$ and $J(-1)v_{63}$ as in the case of $W_k^{0(0)}$. By computing the rank of the matrix, we can determine the fusion rules. In the case $L^1 = W_k^a$, there is a singular vector v_{21} in $W_k^a(2)$ and there are two singular vectors v_{41}, v_{42} in $W_k^a(4)$. Set the same matrix for $v_{41}, v_{42}, J(-1)v_{41}$ and $J(-1)v_{42}$ as in the case of $W_k^{0(0)}$. By computing the rank of the matrix, we can determine the fusion rules. □

Appendix

We give some singular vectors in irreducible $\mathcal{W}$-modules used in Theorem 4. We omit $w_N \in N(0)$ from the explicit form of every singular vector in a $\mathcal{W}$-module N.

- $M_k^{0(1)}$.

$$
\begin{aligned}
&(1)\quad 9\sqrt{-3}L(-1) + J(-1),\\
&(2)\quad 6255L(-1)J(-2) - 375\sqrt{-3}L(-1)J(-1)^2 + 36960L(-2)J(-1)\\
&\qquad + 5175L(-1)^2J(-1) - 26208J(-3) + 1425\sqrt{-3}J(-2)J(-1)\\
&\qquad + 25J(-1)^3 + 23040\sqrt{-3}L(-3) + 147600\sqrt{-3}L(-2)L(-1)\\
&\qquad - 44625\sqrt{-3}L(-1)^3,\\
&(3)\quad -36720\sqrt{-3}L(-1)J(-3) + 324L(-1)J(-2)J(-1)\\
&\qquad - 16\sqrt{-3}L(-1)J(-1)^3 + 9360\sqrt{-3}L(-2)J(-2)\\
&\qquad + 12852\sqrt{-3}L(-1)^2J(-2) + 2400L(-2)J(-1)^2\\
&\qquad + 462L(-1)^2J(-1)^2 - 5040\sqrt{-3}L(-3)J(-1)\\
&\qquad + 8640\sqrt{-3}L(-2)L(-1)J(-1) - 5232\sqrt{-3}L(-1)^3J(-1)\\
&\qquad + 35280\sqrt{-3}J(-4) - 819J(-2)^2\\
&\qquad + 76\sqrt{-3}J(-2)J(-1)^2 + J(-1)^4\\
&\qquad - 751680)L(-4) - 1028160)L(-3)L(-1)\\
&\qquad + 254880L(-2)L(-1)^2 + 16929L(-1)^4.
\end{aligned}
$$

We obtain singular vectors in $M_k^{0(2)}$ by replacing $J(n)$ with $-J(n)$ in the above vectors.

- $W_k^{0(0)}$.

$$
\begin{aligned}
&(1)\quad -70\sqrt{-3}L(-1)J(-1) + 91\sqrt{-3}J(-2) - 5J(-1)^2 - 2496L(-2) + 195L(-1)^2,\\
&(2)\quad -70\sqrt{-3}L(-1)J(-1) + 91\sqrt{-3}J(-2) + 5J(-1)^2 + 2496L(-2) - 195L(-1)^2,\\
&(3)\quad -1500L(-1)J(-2)J(-1) + 1200L(-2)J(-1)^2 + 750L(-1)^2J(-1)^2\\
&\qquad + 3600J(-3)J(-1) + 825J(-2)^2 + J(-1)^4\\
&\qquad - 633600L(-4) + 46800L(-3)L(-1) + 230400L(-2)^2\\
&\qquad - 126000L(-2)L(-1)^2 + 50625L(-1)^4,
\end{aligned}
$$

• $W_k^{0(1)}$.

(1) $5\sqrt{-3}L(-1)+J(-1)$,

(2) $-30\sqrt{-3}L(-1)J(-1)+39\sqrt{-3}J(-2)+5J(-1)^2+336L(-2)+405L(-1)^2$.

We obtain singular vectors in $W_k^{0(2)}$ by replacing $J(n)$ with $-J(n)$ in the above vectors.

• W_k^a.

(1) $J(-1)^2-30L(-2)+75L(-1)^2$,

(2)
$$\begin{aligned}&-5040\sqrt{-3}L(-1)J(-3)-4980L(-1)J(-2)J(-1)\\&+200\sqrt{-3}L(-1)J(-1)^3+7668\sqrt{-3}L(-2)J(-2)\\&+990\sqrt{-3}L(-1)^2J(-2)+9660L(-2)J(-1)^2\\&+7350L(-1)^2J(-1)^2+35760\sqrt{-3}L(-3)J(-1)\\&-78960\sqrt{-3}L(-2)L(-1)J(-1)+34200\sqrt{-3}L(-1)^3J(-1)\\&+5208\sqrt{-3}J(-4)+12780J(-3)J(-1)\\&+2622J(-2)J(-2)-310\sqrt{-3}J(-2)J(-1)^2\\&+25J(-1)^4-9000L(-4)\\&-255780L(-3)L(-1)-28044L(-2)^2\\&+557460L(-2)L(-1)^2-78975L(-1)^4,\end{aligned}$$

(3)
$$\begin{aligned}&5040\sqrt{-3}L(-1)J(-3)-4980L(-1)J(-2)J(-1)\\&-200\sqrt{-3}L(-1)J(-1)^3-7668\sqrt{-3}L(-2)J(-2)\\&-990\sqrt{-3}L(-1)^2J(-2)+9660L(-2)J(-1)^2\\&+7350L(-1)^2J(-1)^2-35760\sqrt{-3}L(-3)J(-1)\\&+78960\sqrt{-3}L(-2)L(-1)J(-1)-34200\sqrt{-3}L(-1)^3J(-1)\\&-5208\sqrt{-3}J(-4)+12780J(-3)J(-1)\\&+2622J(-2)J(-2)+310\sqrt{-3}J(-2)J(-1)^2\\&+25J(-1)^4-9000L(-4)\\&-255780L(-3)L(-1)-28044L(-2)^2\\&+557460L(-2)L(-1)^2-78975L(-1)^4.\end{aligned}$$

• M_k^a.

(1) $8\sqrt{-3}L(-1)J(-1)-6\sqrt{-3}J(-2)+J(-1)^2+90L(-2)+27L(-1)^2$,

(2) $-8\sqrt{-3}L(-1)J(-1)+6\sqrt{-3}J(-2)+J(-1)^2+90L(-2)+27L(-1)^2$.

There are another three singular vectors v_{6i} $(i=1,2,3)$ in M_k^a,

$$
\begin{aligned}
v_{61} = {} & 63987840\sqrt{-3}L(-1)J(-5) - 1059480L(-1)J(-4)J(-1) \\
& + 1587600L(-1)J(-3)J(-2) + 14400\sqrt{-3}L(-1)J(-3)J(-1)^2 \\
& - 26208\sqrt{-3}L(-1)J(-2)^2J(-1) + 4260L(-1)J(-2)J(-1)^3 \\
& + 32\sqrt{-3}L(-1)J(-1)^5 + 23284800\sqrt{-3}L(-2)J(-4) \\
& - 19867680\sqrt{-3}L(-1)^2J(-4) + 145800L(-2)J(-3)J(-1) \\
& - 1414260L(-1)^2J(-3)J(-1) - 427140L(-2)J(-2)^2 \\
& - 155358L(-1)^2J(-2)^2 - 15840\sqrt{-3}L(-2)J(-2)J(-1)^2 \\
& + 10224\sqrt{-3}L(-1)^2J(-2)J(-1)^2 + 3990L(-2)J(-1)^4 \\
& - 447L(-1)^2J(-1)^4 - 25401600\sqrt{-3}L(-3)J(-3) \\
& - 5443200\sqrt{-3}L(-2)L(-1)J(-3) + 1995840\sqrt{-3}L(-1)^3J(-3) \\
& - 659880L(-3)J(-2)J(-1) + 1666440L(-2)L(-1)J(-2)J(-1) \\
& + 341388L(-1)^3J(-2)J(-1) - 41280\sqrt{-3}L(-3)J(-1)^3 \\
& + 32640\sqrt{-3}L(-2)L(-1)J(-1)^3 + 7872\sqrt{-3}L(-1)^3J(-1)^3 \\
& + 4294080\sqrt{-3}L(-4)J(-2) + 16420320\sqrt{-3}L(-3)L(-1)J(-2) \\
& + 4989600\sqrt{-3}L(-2)^2J(-2) - 1360800\sqrt{-3}L(-2)L(-1)^2J(-2) \\
& + 122472\sqrt{-3}L(-1)^4J(-2) - 3483720L(-4)J(-1)^2 \\
& - 1742220L(-3)L(-1)J(-1)^2 + 38700L(-2)^2J(-1)^2 \\
& + 588420L(-2)L(-1)^2J(-1)^2 - 169749L(-1)^4J(-1)^2 \\
& - 42370560\sqrt{-3}L(-5)J(-1) - 45861120\sqrt{-3}L(-4)L(-1)J(-1) \\
& - 2332800\sqrt{-3}L(-3)L(-2)J(-1) - 31164480\sqrt{-3}L(-3)L(-1)^2J(-1) \\
& - 3542400\sqrt{-3}L(-2)^2L(-1)J(-1) + 4429440\sqrt{-3}L(-2)L(-1)^3J(-1) \\
& + 23328\sqrt{-3}L(-1)^5J(-1) - 101787840\sqrt{-3}J(-6) \\
& - 546480J(-5)J(-1) - 1186920J(-4)J(-2) \\
& + 5280\sqrt{-3}J(-4)J(-1)^2 - 2381400J(-3)^2 \\
& - 47520\sqrt{-3}J(-3)J(-2)J(-1) - 3420J(-3)J(-1)^3 \\
& + 11088\sqrt{-3}J(-2)^3 + 870J(-2)^2J(-1)^2 \\
& - 88\sqrt{-3}J(-2)J(-1)^4 + J(-1)^6 \\
& + 879076800L(-6) + 990072720L(-5)L(-1)
\end{aligned}
$$

$$+65091600L(-4)L(-2)+323666280L(-4)L(-1)^2$$
$$+102173400L(-3)^2+73823400L(-3)L(-2)L(-1)$$
$$-3027780L(-3)L(-1)^3-152523000L(-2)^3$$
$$+95652900L(-2)^2L(-1)^2-15326010L(-2)L(-1)^4$$
$$-505197L(-1)^6.$$

The vector v_{62} is obtained by replacing $J(n)$ with $-J(n)$ in v_{61}.

$$\begin{aligned}
v_{63} = {} & -1478136600L(-1)J(-4)J(-1)+423979920L(-1)J(-3)J(-2) \\
& -1273500L(-1)J(-2)J(-1)^3-29005560L(-2)J(-3)J(-1) \\
& +58538700L(-1)^2J(-3)J(-1)+134322300L(-2)J(-2)^2 \\
& -133840350L(-1)^2J(-2)^2+1341750L(-2)J(-1)^4 \\
& +680625L(-1)^2J(-1)^4-778588200L(-3)J(-2)J(-1) \\
& +143310600L(-2)L(-1)J(-2)J(-1)+37975500L(-1)^3J(-2)J(-1) \\
& -2232505800L(-4)J(-1)^2-311863500L(-3)L(-1)J(-1)^2 \\
& +586133100L(-2)^2J(-1)^2-239341500L(-2)L(-1)^2J(-1)^2 \\
& +127186875L(-1)^4J(-1)^2-393666480J(-5)J(-1) \\
& -1236672360J(-4)J(-2)-104786136J(-3)^2 \\
& +3968100J(-3)J(-1)^3+2139750J(-2)^2J(-1)^2 \\
& +625J(-1)^6+275225065920L(-6) \\
& +450006898320L(-5)L(-1)+221829042960L(-4)L(-2) \\
& -60223224600L(-4)L(-1)^2+121440173400L(-3)^2 \\
& +145205865000L(-3)L(-2)L(-1)-22975690500L(-3)L(-1)^3 \\
& -5826151800L(-2)^3+65516566500L(-2)^2L(-1)^2 \\
& -55664516250L(-2)L(-1)^4+5974171875L(-1)^6.
\end{aligned}$$

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