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A note on a generalized Joukowski transformation ☆

H.R. Malonek [a], R. De Almeida [b,*]

[a] Departamento de Matemática, Universidade de Aveiro, P-3810-193 Aveiro, Portugal

[b] Departamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, P-5001-801 Vila Real, Portugal

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ABSTRACT

It is well known that the Joukowski transformation plays an important role in physical applications of conformal mappings, in particular in the study of flows around airfoils. We present, for $n \geq 2$, an n-dimensional hypercomplex analogue of the Joukowski transformation and describe in some detail the 3D case. A generalized 3D Joukowski profile, produced with Maple, is included.

1. Introduction

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with a non-commutative product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \ldots, m$, where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, m\}\}$ with $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \leq h_1 \leq \cdots \leq h_r \leq m$, $e_\emptyset = e_0 = 1$, forms a basis of the 2^m-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$. Let $\mathbb{R}^{m+1}$ be embedded in $Cl_{0,m}$ by identifying $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1 x_1 + \cdots + e_m x_m$. The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_m^2$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e., $Cl_{0,m}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{m+1}$. If f is a solution of the Cauchy–Riemann system $Df = 0$ ($fD = 0$) where $D = \frac{\partial}{\partial x_0} + \partial_{\underline{x}}$ for $\partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_m \frac{\partial}{\partial x_m}$, then f is called a left (right) monogenic function or left (right) holomorphic in the sense of Clifford analysis; see [1]. We suppose that f is hypercomplex differentiable in Ω in the sense of [2], i.e., f has a uniquely defined areolar derivative f' in each point of Ω (see [3]). Then f is real differentiable and f' can be expressed in terms of the real partial derivatives as $f' = (1/2)(\frac{\partial}{\partial x_0} - \partial_{\underline{x}})f$. Since a hypercomplex differentiable function belongs to the kernel of D, then it follows that in fact $f' = \frac{\partial}{\partial x_0} f$ like in the complex case.

We will concentrate on the case $m = 2$, but the applied methods allow a generalization to higher dimensions. Using the generalized Joukowski transformation we describe in some detail the corresponding Joukowski profile in $\mathbb{R}^{m+1}$.

2. The classical Joukowski transformation revisited

The Joukowski transformation

$$w = w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z = x_0 + ix_1 \in \mathbb{C} \setminus \{0\} \tag{1}$$

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* Corresponding author. Tel.: +351 914557603.
E-mail address: ralmeida@utad.pt (R. De Almeida).

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is a special conformal mapping which maps circles $|z| = \rho \neq 1$ onto confocal ellipses with foci $w = 1$ and $w = -1$. The unit circle is mapped into the interval $[-1, 1]$ of the real axis in the w-plane traced twice (see [4,5]). It is evident that there is no essential difference between transforming the unit circle into the real interval $[-1, 1]$ and transforming it into the imaginary interval $[-\mathrm{i}, \mathrm{i}]$. Of course, the simplest way to transform one into the other would be the multiplication of w by i. But, to embed the complex case in the framework of a generalized Joukowski transformation $w : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, $m \geq 1$, we consider instead of (1) the modified Joukowski transformation

$$\widetilde{w} = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{2}$$

Taking $\widetilde{w} = w_0 + \mathrm{i}w_1$ and changing also the ordinary polar coordinates by setting $z = \rho \sin\varphi + \mathrm{i}\rho\cos\varphi$, where $\rho > 0$ and $\varphi \in [0, 2\pi]$, this leads to

$$w_0 = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi, \qquad w_1 = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi. \tag{3}$$

We can directly see that $\rho = 1$ results in $w_0 = 0$ and $w_1 = \cos\varphi$ as desired. Furthermore, circles of radius $\rho \neq 1$ are mapped onto ellipses with semi-axis $a = \frac{1}{2}\left|\rho - \frac{1}{\rho}\right|$, $b = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$ and foci changed to $\widetilde{w} = -i$ and $\widetilde{w} = i$ on the imaginary axis.

Finally, we note that the complex case corresponds to the choice of $m = 1$ and the imaginary unit i is replaced by e_1 in the hypercomplex theory.

3. A generalized Joukowski transformation

On the basis of the reformulation of the complex Joukowski function in the previous section, we now introduce a higher dimensional analogue of the Joukowski transformation as follows:

Definition 1. Let $m \geq 1$, $\mathcal{P}^m(x) = x_0 + \frac{1}{m}\underline{x}$, a left and right monogenic linear function with hypercomplex derivative $(\mathcal{P}^m(x))' \equiv 1$, and $E^m(x) = \frac{\bar{x}}{|x|^{m+1}}$, the fundamental solution of the corresponding Cauchy–Riemann system. Then

$$w(x_0 + \underline{x}) = \frac{m}{m+1}\left(\mathcal{P}^m(x) - E^m(x)\right), \tag{4}$$

is called the generalized Joukowski transformation in $\mathbb{R}^{m+1}$.

The polynomial $\mathcal{P}^m(x)$ is equal to the first-degree polynomial in a set of hypercomplex Appell polynomials considered in [6].

Notice that for $m = 1$ the function (4) coincides exactly with the complex Joukowski transformation in the form (2). Moreover, it is clear that for any $m > 1$ the function (4) satisfies the following properties:

(i) $w(x)$ is monogenic outside the origin and is of the form $w = w_0 + \underline{w}$;
(ii) the unit sphere S^m is twofold mapped to S^{m-1} (including its interior) in the hyperplane $w_0 = 0$;
(iii) spheres in $\mathbb{R}^{m+1}$ with radius $\rho \neq 1$ are mapped onto ellipsoids in $\mathbb{R}^{m+1}$.

The properties (i)–(iii) are analogous to those for the complex case $m = 1$ and justify the designation of (4) as a generalized Joukowski transformation. As far as we know, no other hypercomplex generalization has been proposed until now. In addition, the generalized Joukowski transformation is – like in the complex case – an important example for the study of the mapping properties of non-linear monogenic functions. To investigate in more detail some similarities as well as differences between (4) and the complex Joukowski transformation we will continue with the simplest case $m = 2$,

$$w(x) = w(x_0 + \underline{x}) = \frac{2}{3}\left(\mathcal{P}^2(x) - E^2(x)\right). \tag{5}$$

We use spherical coordinates (ρ, θ, φ) with $\rho = |x|$, θ as the longitude and φ as the latitude (not, like usually, the polar angle $\alpha = \frac{\pi}{2} - \varphi$ or co-latitude; cf. Section 2). This means that

$$x_1 = \rho\cos\varphi\cos\theta, \qquad x_2 = \rho\cos\varphi\sin\theta, \qquad x_0 = \rho\sin\varphi,$$

where $\rho > 0, -\pi < \theta \leq \pi$ and $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. Let $w = w_0 + w_1e_1 + w_2e_2$; then we get from (5) the Joukowski transformation in $\mathbb{R}^3$, in terms of spherical coordinates, where

$$\begin{aligned} w_0 &= \frac{2}{3}\left(\rho - \frac{1}{\rho^2}\right)\sin\varphi, \\ w_1 &= \frac{2}{3}\left(\frac{\rho}{2} + \frac{1}{\rho^2}\right)\cos\varphi\cos\theta, \\ w_2 &= \frac{2}{3}\left(\frac{\rho}{2} + \frac{1}{\rho^2}\right)\cos\varphi\sin\theta. \end{aligned} \tag{6}$$

We immediately see that spheres in $\mathbb{R}^3$ are transformed into spheroids with $a = \frac{2}{3}\left|\rho - \frac{1}{\rho^2}\right|$ as the polar radius and $b = \frac{2}{3}\left(\frac{\rho}{2} + \frac{1}{\rho^2}\right)$ as the equatorial radius (speaking again in geographical terms).

As was proved in detail in [7], the Jacobian determinant of the mapping (5) is non-zero almost everywhere. Furthermore, the different geometrical configurations of the image of spheres with radius $|x| = \rho \geq 1$ are given in:

Proposition 1. *Consider the Joukowski transformation (5). Then the following hold:*

1. *The unit sphere $|x| = 1$ is mapped into $w = \cos\varphi\,(\cos\theta e_1 + \sin\theta e_2)$, i.e., the twofold-mapped S^1, including its interior, in the hyperplane $w_0 = 0$.*
2. *A sphere of radius $|x| = \rho$ and $1 < \rho < \sqrt[3]{4}$ is mapped onto an oblate spheroid.*
3. *A sphere of radius $|x| = \sqrt[3]{4}$ is mapped onto a sphere of radius $\frac{1}{\sqrt[3]{2}}$.*
4. *A sphere of radius $|x| = \rho > \sqrt[3]{4}$ is mapped onto a prolate spheroid with*
   - 4.1. *equatorial radius $b < 1$, for spheres of radius $\sqrt[3]{4} < \rho < 1 + \sqrt{3}$;*
   - 4.2. *equatorial radius $b = 1$, for the sphere of radius $|x| = 1 + \sqrt{3}$;*
   - 4.3. *equatorial radius $b > 1$, for spheres of radius $|x| = \rho > 1 + \sqrt{3}$.*

The proofs of the statements 1.–4. follow immediately by direct inspection of (6). Notice that in the case of an oblate spheroid one has $a < b$ and in the case of a prolate spheroid $a > b$.

Although for higher dimensions the generalized Joukowski transformation (4) and the complex Joukowski transformation (2) being analogous is evident, the case $m = 2$ already reveals some new properties, for example the change of the images from oblate to prolate spheroids and the intermediate result of a sphere, here for the value of the radius $\rho = \sqrt[3]{4}$ (see the cases 2.–4. of Proposition 1). Another difference is the fact that only the value of the radius $\rho > 1 + \sqrt{3}$ results in a mapping of the sphere onto a prolate spheroid which contains the unit disc (see the cases 4.2. and 4.3. of Proposition 1). This happens in the complex case always. We expect that a detailed study of the mapping properties of the generalized Joukowski transformation (4) for arbitrary $m > 1$ will reveal further interesting insights into the use of monogenic functions as tools for $(m + 1)$D mappings. Furthermore, we use the following definition of quasiconformality given in [8].

Definition 2. Let D and D' be open subsets of $\mathbb{R}^{m+1}$ and $f : D \to D'$ a diffeomorphism. Then the inner dilatation and outer dilatation are defined respectively by

$$K_I(f) = \sup_{x\in D} H_I(J_f(x)), \qquad K_O(f) = \sup_{x\in D} H_O(J_f(x)),$$

where

$$H_I(J_f(x)) = \frac{|J(x,f)|}{m(J_f(x))^{m+1}}, \qquad H_O(J_f(x)) = \frac{M(J_f(x))^{m+1}}{|J(x,f)|}$$

defining $J(x, f) = \det J_f(x)$ as the Jacobian determinant of f, $m(J_f(x)) = \min_{|h|=1} |J_f(x)h|$ and $M(J_f(x)) = \max_{|h|=1} |J_f(x)h|$.

The maximum dilatation of f is defined by $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K < \infty$, then f is K-quasiconformal.

Remark 1. Notice that the image of the unit ball B^{m+1} under f is an ellipsoid E. Denoting as B_I and B_O the inscribed and circumscribed balls of E, respectively, then

$$H_I = \frac{m(E)}{m(B_I)}, \qquad H_O = \frac{m(B_O)}{m(E)}$$

where $m(\cdot)$ stands for the volume measure.

We obtain:

Theorem 1. *Let $\rho > \sqrt[3]{4}$; then the generalized Joukowski transformation in $\mathbb{R}^3$ (5) is orientation preserving and 4-quasi-conformal.*

Proof. By the generalized Joukowski transformation (5), spheres of radii ρ in $\mathbb{R}^3$ are transformed onto spheroids with $a = \frac{2}{3}\left|\rho - \frac{1}{\rho^2}\right|$ as the polar radius and $b = \frac{2}{3}\left(\frac{\rho}{2} + \frac{1}{\rho^2}\right)$ as the equatorial radius. Since $\rho > \sqrt[3]{4}$, the Jacobian determinant of the generalized Joukowski transformation is positive for (ρ, ϕ) in

$$\left\{(\rho,\phi) \in \mathbb{R}_0^+ \setminus [1, \sqrt[3]{4}] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\} \bigcup \left\{(\rho,\phi) \in [1, \sqrt[3]{4}] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] : \cos\phi > \frac{\rho^3 + 2}{3\rho^{\frac{3}{2}}}\right\}$$

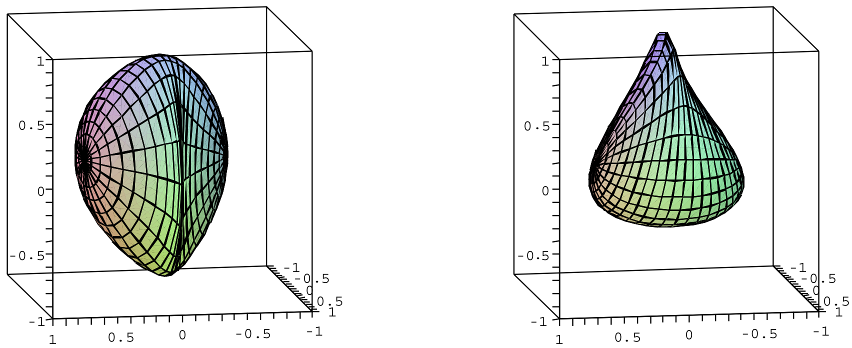

Fig. 1. Generalized Joukowski profiles associated with (8) and (10).

and $a > b$. Therefore, using Definition 2, the outer and inner dilatation are given, respectively, by

$$K_O(\rho) = \sup_{\rho \geq \sqrt[3]{4}} \frac{2(\rho^3 - 1)}{(\rho^3 + 2)} \quad \text{and} \quad K_I(\rho) = \sup_{\rho \geq \sqrt[3]{4}} \frac{4(\rho^3 - 1)^2}{(\rho^3 + 2)^2}.$$

Since $K_O(\rho)$ and $K_I(\rho)$ are increasing functions then

$$K_O(\rho) = \lim_{\rho \to +\infty} \frac{2(\rho^3 - 1)}{(\rho^3 + 2)} = 2 \quad \text{and} \quad K_I(\rho) = \lim_{\rho \to +\infty} \frac{4(\rho^3 - 1)^2}{(\rho^3 + 2)^2} = 4.$$

Consequently, $K = \max\{K_O, K_I\} = 4$; therefore the generalized Joukowski transformation is 4-quasiconformal for $\rho > \sqrt[3]{4}$. $\square$

4. The Joukowski profile in $\mathbb{R}^3$

In the complex case, the mapping (1) transforms suitable circles into airfoils denoted as *Joukowski airfoils.* On the basis of the classical Joukowski profile we introduce a 3D analogue for the Joukowski profile.

Consider the parametrization of the sphere that passes through the point $z = e_1$ with center at $z = -\rho e_1$,

$$x_0 = (1+\rho)\sin\varphi, \qquad x_1 = -\rho + (1+\rho)\cos\varphi\cos\theta, \qquad x_2 = (1+\rho)\cos\varphi\sin\theta, \tag{7}$$

for $\rho > 0$ fixed, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ and $-\pi < \theta \leq \pi$. Substituting (7) into the generalized Joukowski transformation (5) leads to

$$\begin{aligned} w = {} & \frac{2}{3}(1+\rho)\sin\varphi\left(1 - \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\cos\theta\right)^{-\frac{3}{2}}\right)e_0 \\ & + \frac{2}{3}\left(-\rho + (1+\rho)\cos\varphi\cos\theta\right)\left(\frac{1}{2} + \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\cos\theta\right)^{-\frac{3}{2}}\right)e_1 \\ & + \frac{2}{3}(1+\rho)\cos\varphi\sin\theta\left(\frac{1}{2} + \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\cos\theta\right)^{-\frac{3}{2}}\right)e_2. \end{aligned} \tag{8}$$

Analogously, we obtain a similar result considering the sphere that passes through the point $z = e_2$ with center at $z = -\rho e_2$. Therefore, consider the following parameterization:

$$x_0 = (1+\rho)\sin\varphi, \qquad x_1 = (1+\rho)\cos\varphi\cos\theta, \qquad x_2 = -\rho + (1+\rho)\cos\varphi\sin\theta. \tag{9}$$

Substituting (9) into the generalized Joukowski transformation (5) leads to

$$\begin{aligned} w = {} & \frac{2}{3}(1+\rho)\sin\varphi\left(1 - \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\sin\theta\right)^{-\frac{3}{2}}\right)e_0 \\ & + \frac{2}{3}(1+\rho)\cos\varphi\cos\theta\left(\frac{1}{2} + \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\sin\theta\right)^{-\frac{3}{2}}\right)e_1 \\ & + \frac{2}{3}\left(-\rho + (1+\rho)\cos\varphi\sin\theta\right)\left(\frac{1}{2} + \left(\rho^2 + (1+\rho)^2 - 2\rho(1+\rho)\cos\varphi\sin\theta\right)^{-\frac{3}{2}}\right)e_2. \end{aligned} \tag{10}$$

Fig. 1 visualizes the Joukowski transformation of the sphere that passes through the point $z = e_1$ with center $z = -\rho e_1$, and that which passes through the point $z = e_2$ with center $z = -\rho e_2$. The figures show the similarity of the classical 2D and the generalized 3D Joukowski profiles.

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