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# Some old and new problems in various branches of combinatorics

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**1.** A very old problem of mine is as follows: Is it true that every graph of $10n$ vertices every subgraph of $5n$ vertices of which contains (or induces) more than $2n^2$ edges contains a triangle? This problem has been open for several decades, and I offer 250 dollars for a proof or disproof. First observe that the best possible result is the statement being true. To see this consider a blown up pentagon, each vertex of the pentagon is replaced by an independent set of $2n$ vertices. This gives a $G\,(10n;\,20n^2)$ which is of course triangle free and every set of $5n$ vertices induces at least $2n^2$ edges ($G(n;\,e)$ denotes a graph of $n$ vertices and $e$ edges). Several years ago, Simonovits observed that if we blow up the Petersen graph, each vertex of which is replaced by an independent set of $n$ vertices, we obtain a triangle-free $G\,(10n;\,15n^2)$ every subgraph of $5n$ vertices induces at least $2n^2$ edges. It is possible that these two graphs are the only ones which show that the conjecture is best possible. In other words, every $G\,(10n)$ every subgraph of $5n$ vertices induces $\geqslant 2n^2$ edges, contains a triangle unless it is isomorphic with one of our two graphs. At the moment we have no counterexample but a trivial counterexample may exist which I overlooked.

Recently, Ervin Györi and I considered the following rather annoying problem: what is the smallest $e_n$ for which there is a $G\,(10n;\,e_n)$ every subgraph of $5n$ vertices of which induces at least $2n^2$ edges. It is easy to see that $e_n = (1 + o(1))\,8n^2$ but we would not determine the exact value of $e_n$ (perhaps we overlook a trivial point). It is easy to see that every graph of 10 vertices every subgraph of 5 vertices of which has 2 edges has 12 edges. The extreme graph (probably the only one) is the vertex-disjoint union of a $k(4)$ and two $k(3)$'s. For 20 vertices the extreme graph is probably the vertex disjoint union of $4k(5)$'s and has 40 edges — we did not prove this since too many cases would have to be considered. We proved without much difficulty that every $G\,(4n)$ every set of $2n$ vertices of which induces at least $n(n-1)$ edges has $4n^2 - 2n$ edges. The extremal graph is given by two vertex disjoint $k(2n)$'s. Our "Success" is, of course, caused by the fact that it was easy to guess the extremal graph.

Several related and challenging and perhaps interesting problems can be asked.

[†] Sadly, the author passed away on September 20, 1996.


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**2.** Very recently, Gyárfás and I considered several problems which are related to Ramsey's theorem. Here I only state one of them. Denote by $f_k^{(r)}(n)$ the largest integer for which one can color the edges of a complete graph of $f_k^{(r)}(n)$ vertices by $r$ colours so that every set of $n$ vertices contains a complete subgraph of $k$ vertices in each of the $r$ colors. For $r = k = 2$ we obtain the ordinary Ramsey problem. We investigated the use of $r = 2$, $k = 3$. We proved by the probability method that

$$f_3^{(r)}(n) > \exp(c_1 n^{1/2}). \tag{1}$$

We conjectured but could not prove

$$f_3^{(r)}(n) < \exp(c_2 n^{1/2}). \tag{2}$$

Eq. (2) should be proved by Ramsey's theoretical methods but so far we had no success. Very likely,

$$\exp(c_1^{(r)} n^{1/k-1}) < f_k^{(r)}(n) < \exp(c_2^{(r)} n^{1/k-1}). \tag{3}$$

The probabilistic proof will probably prove the lower bound in (3). Gyárfás and I considered several related problems but we will discuss these in a separate paper.

**3.** Does there exist a sequence $a_1 < a_2 < \cdots$ of density 0 for which there is an absolute constant $c$ so that every $G(n; cn)$ contains a cycle of length $a_i$ for some $i$. Probably, $a_i = 2^i$ does not have this property, what about $a_i = i^2$ or $p_i \pm 1$ ($p_i$ is the $i$th prime)? I conjectured and Bollobás proved that if the $a$'s form an arithmetic progression which contains even numbers then $c$ exists, but as far as I know the best value of $c$ is not known.

In fact, is there a sequence of density 0 for which every graph of infinite chromatic number contains cycles of length $a_i$ for infinitely many $i$?. A very old conjecture of Hajnal and myself asks: let $G$ have infinite chromatic number and let $b_1 < b_2 < \cdots$ be the cycles of odd length which occur in $G$. Is it true that $\sum 1/b_i = \infty$? and perhaps the $b$'s have positive upper density? It is well known that the lower density can be 0.

We got nowhere with this conjecture and more than 10 years ago Mihok and I conjectured that if $G$ has infinite chromatic number then for infinitely many $k$, $G$ must contain cycles of length $2^k$. Again, of course, $2^k$ could be replaced by any other much slower growing sequence e.g. $a_k = k^2$, but again we got nowhere.

About three years ago, Gyárfás and I thought at first that if $G(n)$ is any graph every vertex of which has degree $\geqslant 3$ then our $G(n)$ has a cycle of length $2^k$ for some $k$. We are convinced now that this is false and no doubt these are graphs for every $r$ every vertex of which has degree $\geqslant r$ and which contains no cycle of length $2^k$, but we never found a counterexample even for $r = 3$. These problems lead eventually to the problems formulated at the beginning of **3** and I offer 1000 dollars for a satisfactory answer.

**4.** Here is a very nice old problem of Hajnal, Szemerédi and myself which unfortunately has been forgotten and neglected: Let $f(n) \to \infty$ arbitrarily slowly. Is it true that there is a $G$ of infinite chromatic number every induced subgraph of $n$ vertices of which can be made bipartite by the omission of fewer than $f(n)$ edges? As far as I know the problem is open even for $f(n) = n^{1/2}$. I offer 500 dollars for a proof of the existence of such a graph $G$ but only 250 dollars for a counterexample.

**5.** Andrásfai and I some years ago asked: Let $\mathscr{S}$ be an infinite set of points in the plane in general position, i.e. no three on a line and no four on a circle. Join two of our points if their distance is an integer. Can the chromatic number of this graph be infinite? If the answer is negative how large can the chromatic number be? What is the largest complete graph that this graph can contain? Perhaps, I overlook a trivial point but at the moment I do not see whether our graph can contain an infinite complete graph.

**6.** Kohayakava, Gyárfás and I asked: Let $f(e)$ be the largest integer for which every graph of $e$ edges must contain a bipartite graph of $f(e)$ edges. It follows from a result of Edwards that

$$f(e) \geqslant \frac{e}{2} + \frac{(8e)^{1/2}}{8} \tag{4}$$

and (4) is best possible if $e = \binom{r}{2}$ and our graph is complete; we could not decide if for infinitely many $e$

$$f(e) - \frac{e}{2} - \frac{(8e)^{1/2}}{8} \to \infty . \tag{5}$$

Noga Alon recently proved that for $e = \binom{r}{2} + \frac{r}{2}$,

$$f(e) > \binom{r}{2} + \frac{(8e)^{1/2}}{8} + ce^{1/4} \tag{6}$$

but the largest possible value of $c$ is not yet known.

**7.** Let $f(n)$ be the smallest integer for which there is a triangle-free graph $G(n)$ of $n$ vertices, diameter two the maximum degree of which is $f(n)$. Pach and I conjectured some time ago that $f(n)^{1/2}_{/n} \to \infty$. I forgot our conjecture and did not see how to find such a graph with $f(n)_{/n} \to 0$. Simonovits found a nice Kneser graph with $f(n) < n^{1-\varepsilon}$. Let $S = 3m - 1$ the vertices of our graph are the subsets of size $m$ of $S$. $A_i \subset S$, $|A_i| = m$, $1 \leqslant i \leqslant \binom{3m-1}{m}$ are the vertices of our graph, $A_i$ and $A_j$ are joined if $A_i \cap A_j = \emptyset$. This graph is regular of degree $\binom{2m-1}{m}$ and the graph is easily seen to be triangle free and of diameter two. Also a simple computation shows that $f(n) < n^{1-\varepsilon}$ $(n = \binom{3m-1}{m})$. It is not impossible that this graph of Simonovits gives the smallest possible value of $f(n)$. Perhaps this conjecture is too optimistic and the interested reader should perhaps first try to find a counterexample.

A few weeks ago, Gyárfás and I considered the following problem. Let $G'(n)$ be a triangle-free graph of $n$ vertices we want to add to our $G(n)$ edges (no new vertices) so that our new graph $G'(n)$ should be triangle free and have diameter two. It is easy to see that such a $G(n)$ always exist. Let $h(G)$ be the smallest number of edges for which we can obtain our $G(n)$ which is triangle free and has diameter two. First of all we proved that max $h_{2n}(G) = (n-1)^2$. The proof is not quite trivial and we hope that it will be written up with all details soon.

Denote by $d_n(G)$ the maximum degree of $G(n)$; we prove that if $d_n(G)$ is less than $c \log n/\log\log n$ then

$$h_n(G) = o(n^2). \tag{7}$$

We hope that (7) holds already if

$$d_n(G) < n^{1/2-\varepsilon}, \tag{8}$$

perhaps $d_n(G) < cn^{1/2}$ already implies (7) for sufficiently small $c$, perhaps this conjecture is too optimistic, a simple construction of Simonovits showed that (8) fails for large $c$.

**8.** To end these types of problems, let me restate two of my old problems on triangles in graphs. Is it true that a triangle-free graph of $5n$ vertices can be made bipartite by the omission of at most $n^2$ edges? The blown up pentagon shows that if true this result is the best possible.

Is it true that a triangle-free graph of $5n$ vertices has at most $n^5$ pentagons? E. Györi showed that the number of pentagons is at most $1.03n^5$ and this result was further improved by Füredi but at this moment $n^5$ has not yet been proved. To end our paper I state some problems in combinatorial number theory and combinatorial geometry.

**9.** Sárközy and I recently considered the following problem: Let $a_1 < a_2 < \cdots$ be an infinite sequence no $a_i$ divides the sum of two other $a_n'$. We will denote this property by P. Is there a sequence of property P satisfying $a_n > n^2$ for all $n > n_0$. If $a_n = (p_n^2, p_n \equiv 3 \pmod 4)$ then property P is satisfied but $P_n$ increases just a little too fast. Perhaps every sequence with property P satisfies $a_n > n^{1+c}$ for infinitely many $n$ and sufficiently small $c$. Probably, $\sum 1/a_m < \infty$ holds for every sequence with property P.

An old problem of Sárközy and myself stated: Let $a_1 < a_2 < \cdots < a_k \leqslant m$ be such that no $a_i$ divides the sum of two larger $a$'s. Is it true that max $k \leqslant m/3 + 0(1)$? The integers $2m/3 < a_i \leqslant m$ show that our conjecture is the best possible if it is true.

This problem is discussed in our paper: "On the divisibility properties of sequences of integers", Proc. London Math. Soc. 9 (1970) 97–101. This paper is dedicated to the memory of Littlewood.

**10.** Another very recent problem of Sárközy and myself states: Let $a_1 < a_2 < \cdots < a_k \leqslant m$ be a sequence of integers and no $a_i$ divides any distinct sum of other $a_n'$. Put max $h = f(n)$. $f(n) < cn^{1/2}$ is easy. Sándor Csaba a young student at the

University of Budapest showed $f(n) > cn^{1/5}$. Is it true that $f(n) > n^{1/2-\varepsilon}$? Many related questions can be asked.

Now some problems in geometry.

**11.** Here is a problem of Pach and myself: Let $x_1, x_2, \ldots, x_m$ be $m$ points in the plane. Denote by $d_1 > d_2 > \cdots \geqslant d_k$ the distinct distances determined by our points. A very old problem of mine states $k > cn/_{(\log n)}{}^{1/2}$ (Amer. Math. Monthly, 1946). I offer 500 dollars for a proof or disproof. Erica Pannowitz proved that the diameter $d_1$ can occur at most $n$ times. Pach and I now ask: can it happen that for $n > 4$ every other distance occurs more than $n$ times? We believe that the answer is no! Also can it happen that there are $c_1 n$ distances which occur more than $n$ times? Many related questions can be asked.

**12.** To end the paper I state another problem. Let $x_1, x_2, \ldots, x_m$ be $m$ points in the plane. Assume that any four of our points determine at least five distinct distances. I conjectured several years ago that our set then determines at least $c_1 m^2$ distinct distances. I could get no where with this problem. All I could do is to state an even stronger conjecture: our set contains $c_2 m$ points for which all the distances are distinct. I offer 250 dollars for a proof or disproof.

Assume now that our points are on a line and every set of four points determines at least five distinct distances. V. T. S's and I easily proved that there is then a subset of $n/2$ points with all distances distinct. In a very nice paper of Gyárfás and Lehel it is proved that for a fixed but small $\varepsilon > 0$ there are always $(\frac{1}{2} + \varepsilon)\, n$ points for which all the distances are distinct. The best value of $\varepsilon$ is not known but they prove that $\varepsilon$ cannot be greater than $\frac{1}{10}$.

Very recently, Gyárfás and I considered the following somewhat related problem: Let $f(n)$ be the smallest integer for which one can color the edges of a $k(n)$ by $f(n)$ colors so that every $k(4)$ gets at least 5 colors. We observed that

$$\tfrac{2}{3}n < f(n) < n \tag{9}$$

and we prove $f(9) = 8$. I believed that the upper bound in (9) is closer to the truth (Gyárfás believed in the lower bound). Neither of us had much evidence. Many (we believe) interesting generalisations are possible, but this has to be left for another occasion.

Once again let $k(n)$ be a complete graph of $n$ vertices. Color the edges so that every $k(4)$ gets at least 5 colors. How large a complete graph must be which is totally multicolored, i.e. the coloring is rainbow and any two edges get different colors?