JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 38, 619–632 (1972)

Nonlinear Programming in Complex Space: Sufficient Conditions and Duality*

ROBERT A. ABRAMS

Department of Industrial Engineering and Management Sciences, The Technological Institute, Northwestern University, Evanston, Illinois 60201

Submitted by Norman Levinson

1. INTRODUCTION

The development of nonlinear programming in finite-dimensional complex space of [1] is continued here. In [1], necessary conditions for optimal points of two classes of problems are obtained. In this paper sufficient conditions for optimal points of problems of the form

$$\begin{aligned} &\text{minimize} && \operatorname{Re} f(z, \bar{z}) \\ &\text{subject to} && g(z, \bar{z}) \in S, \end{aligned} \tag{1}$$

where S is a polyhedral cone, are obtained. A dual theorem, which in the case of linear constraints reduces to that of [6], and a converse dual theorem are given.

For the case in which $f(z, \bar{z})$ is quadratic and $g(z, \bar{z})$ is linear, the results given here reduce to those of [3]. When both functions are linear the duality results of [4] and [9], where programming in complex space was first studied, are obtained.

The notation and definitions of [1] are used here and are given in Section 2. Results given in the preliminary section of [1] and identities from the appendix of [1] are not repeated here. In Section 3 of this paper, convexity of a complex-valued function with respect to a cone is defined in two equivalent ways. It is then shown that a nonlinear analytic function of n complex variables cannot have convex real part with respect to R_+. Therefore, problems of the first class considered in [1], i.e., problems of the form

$$\begin{aligned} &\text{minimize} && \operatorname{Re} f(z) \\ &\text{subject to} && g(z) \in S, \end{aligned}$$

* Research partly supported by the National Science Foundation Project GP 13546.

© 1972 by Academic Press, Inc.

where $f(z)$ and $g(z)$ are analytic cannot satisfy the convexity requirements needed here. Thus sufficiency and duality theorems are given only for problems of the form (1). In Sections 4 and 5, the sufficiency and saddle point theorems of the Kuhn and Tucker [8] type are obtained. Dual convex programs (in the sense of Wolfe [11] and Hanson [5]) are considered in Section 6 and examples are given in the final section.

2. Preliminaries

2.1. *Notation*

$C^n[R^n]$ — *n-dimensional complex [real] vector space*
$C^{m\times n}[R^{m\times n}]$ — *$m \times n$ complex [real] matrices*
$R_+^n \equiv \{x \in R^n : x_i \geqslant 0\ (i = 1,..., n)\}$ — *nonnegative orthant of R^n*
$x \geqslant y$ denotes $x - y \in R_+^n$, for $x, y \in R^n$.

For $A = (a_{ij}) \in C^{m\times n}$,

$$\bar{A} \equiv (\bar{a}_{ij}) \text{ — } \textit{conjugate}$$
$$A^T \equiv (a_{ji}) \text{ — } \textit{transpose}$$
$$A^H \equiv \bar{A}^T \text{ — } \textit{conjugate transpose.}$$

For $x = (x_i) \in C^n$, $y \in C^n$,

$$(x, y) \equiv y^H x \text{ — } \textit{inner product} \text{ of } x \text{ and } y$$
$$\bar{x} \equiv (\bar{x}_i) \text{ — } \textit{conjugate}$$
$$\operatorname{Re} x \equiv (\operatorname{Re} x_i) \in R^n \text{ — } \textit{real part}$$
$$\operatorname{Im} x \equiv (\operatorname{Im} x_i) \in R^n \text{ — } \textit{imaginary part}$$
$$\arg x \equiv (\arg x_i) \text{ — } \textit{argument} \text{ of } x.$$

For a subspace $L \subset C^n$,

$$L^\perp \equiv \{y \in C^n : l \in L \Rightarrow (y, l) = 0\} \text{ — } \textit{orthogonal complement} \text{ of } L.$$

For a nonempty set $S \subset C^n$,

$$S^* \equiv \{y \in C^n : x \in S \Rightarrow \operatorname{Re}(y, x) \geqslant 0\} \text{ — } \textit{dual} \text{ (also } \textit{polar}\text{) of } S.$$

For a nonempty set $S \subset R^n$,

$$S^* \equiv \{y \in R^n : x \in S \Rightarrow (x, y) \geqslant 0\}$$

$\tilde{S} = \left\{\begin{pmatrix}\operatorname{Re} w \\ \operatorname{Im} w\end{pmatrix} : w \in S\right\} \subset R^n \times R^n$ is a polyhedral cone (see Definition 2.2(c)) in $R^n \times R^n$ when S is a polyhedral cone in C^n.

For an analytic function $f: C^n \to C$ and a point $z^0 \in C^n$

$$\nabla_z f(z^0) \equiv \left(\frac{\partial f}{\partial z_i}(z^0)\right), \qquad i = 1,\dots, n \text{ — } \textit{gradient} \text{ of } f \text{ at } z^0.$$

For a complex function $f(w^1, w^2)$ analytic in the $2n$ variables (w^1, w^2) at the point $(z^0, \bar{z}^0) \in C^n \times C^n$,

$$\nabla_z f(z^0, \bar{z}^0) \equiv \left(\frac{\partial f}{\partial w_i^1}(z^0, \bar{z}^0)\right), \qquad i = 1,\dots, n$$

and

$$\nabla_{\bar{z}} f(z^0, \bar{z}^0) \equiv \left(\frac{\partial f}{\partial w_i^2}(z^0, \bar{z}^0)\right), \qquad i = 1,\dots, n.$$

For an analytic function $f: C^n \to C^m$,

$$D_z g(z^0) \equiv \left(\frac{\partial g_i}{\partial z_j}(z^0)\right), \qquad i = 1,\dots, m; \quad j = 1,\dots, n.$$

Similarly, for a function $g: C^n \times C^n \to C^m$ analytic in the $2n$ variables (w^1, w^2) at $(z^0, \bar{z}^0) \in C^n \times C^n$,

$$D_z g(z^0, \bar{z}^0) = \left(\frac{\partial g_i}{\partial w_j^1}(z^0, \bar{z}^0)\right), \qquad i = 1,\dots, m; \quad j = 1,\dots, n$$

and

$$D_{\bar{z}} g(z^0, \bar{z}^0) = \left(\frac{\partial g_i}{\partial w_j^2}(z^0, \bar{z}^0)\right), \qquad i = 1,\dots, m; \quad j = 1,\dots, n.$$

Also

$$D_z^T g(z^0, \bar{z}^0) = (D_z g(z^0, \bar{z}^0))^T,$$

$$D_z^H g(z^0, \bar{z}^0) = (D_z g(z^0, \bar{z}^0))^H.$$

2.2. *Definitions*

A nonempty set $S \subset C^n$ is

(a) *convex* if $0 \leqslant \lambda \leqslant 1 \Rightarrow \lambda S + (1 - \lambda) S \subset S$,

(b) a *cone* if $0 \leqslant \lambda \Rightarrow \lambda S \subset S$,

(c) a *polyhedral cone* if, for some positive integer k and $A \in C^{n \times k}$,

$$S = AR_+^k = \{Ax : x \in R_+^k\},$$

i.e., S is generated by finitely many vectors (the columns of A).

The following results are needed in the sequel:

2.3 A polyhedral cone in C^n is a closed convex cone.

2.4 A nonempty set $S \subset C^n$ is a closed convex cone if and only if $S = S^{**}$ (for proof, see, e.g., [4, Theorem 1.5]).

2.5 If S, T are polyhedral cones, then $S \times T$ is a polyhedral cone.

2.6 For any nonempty sets $S, T : (S \times T)^* = S^* \times T^*$.

2.7 Let $A \in C^{m\times n}$, $b \in C^m$ and $S \subset C^n$ a polyhedral cone. Then the following are equivalent:

(a) $Ax = b$, $x \in S$ is consistent.

(b) $A^H y \in S^* \Rightarrow \text{Re}(b, y) \geqslant 0$ [4, Theorem 3.5].

2.8 The nonnegative orthant R_+^n is a self-dual set in $R^n : (R_+^n)^* = R_+^n$.

2.9 Let S be a polyhedral cone in C^n. Then S is the intersection of finitely many closed halfspaces, each including the origin in its boundary:

$$S = \bigcap_{k=1}^{p} H_{u_k},$$

where

$$H_{u_k} = \{z \in C^n : \text{Re}(z, u_k) \geqslant 0\}.$$

(proved similarly to the real case, e.g., [12]).

2.10 Let

$$S = \bigcap_{k=1}^{p} H_{u_k}$$

be a polyhedral cone in C^n or R^n and let $z \in S$. Then $S(z^0)$ is defined to be the intersection of those closed half spaces H_{u_k} which include z^0 in their boundaries, i.e.,

$$S(z^0) = \bigcap_{k\in B(z^0)} H_{u_k}$$

where

$$B(z^0) \equiv \{k : \text{Re}(z^0, u_k) = 0\}.$$

If z^0 is in the interior of S, then $S(z^0) = C^n$.

2.11 Let $\phi \neq S \subset T \subset C^n$. Then $T^* \subset S^*$.

2.12 Let $\{S_i : i = 1,..., p\}$ be closed convex cones in C^n. Then

$$\left(\bigcap_{i=1}^{p} S_i\right)^* = cl \sum_{i=1}^{p} S_i^* \text{ (follows from [4, Corollary 1.7]).}$$

3. Convexity

To obtain the sufficiency, duality and saddlepoint theorems of Sections 4, 5 and 6, convexity of a complex-valued function is defined with respect to a closed convex cone.

Definition. The function $g : C^{2n} \to C^m$ is *convex with respect to the closed convex cone* S *on the manifold* $Q \equiv \{(w^1, w^2) \in C^{2n} : w^2 = \overline{w^1}\}$ if for any z^1 and z^2 and $0 \leqslant \lambda \leqslant 1$,

$$\lambda g(z^1, \bar{z}^1) + (1-\lambda) g(z^2, \bar{z}^2) - g(\lambda z^1 + (1-\lambda) z^2, \lambda \bar{z}^1 + (1-\lambda) \bar{z}^2) \in S. \tag{2}$$

When $g(w^1, w^2)$ is analytic, a condition equivalent to (2) is

$$g(z^1, \bar{z}^1) - g(z^2, \bar{z}^2) - D_z g(z^2, \bar{z}^2)(z^1 - z^2) - D_{\bar{z}} g(z^2, \bar{z}^2)(\bar{z}^1 - \bar{z}^2) \in S. \tag{3}$$

Similarly a function $f : C^n \to C^m$ is convex with respect to S if for any z^1 and z^2 and $0 \leqslant \lambda \leqslant 1$,

$$\lambda f(z^1) + (1-\lambda) f(z^2) - f(\lambda z^1 + (1-\lambda) z^2) \in S. \tag{2'}$$

If $f : C^n \to C^m$ is analytic, (2′) is equivalent to

$$f(z^1) - f(z^2) - D_z f(z^2)(z^1 - z^2) \in S. \tag{3'}$$

When referring to the objective function of a programming problem, convexity of the real part will be of interest. Thus if T is a closed convex cone in R^m, the real part of $g : C^{2n} \to C^m$ is convex with respect to T on the manifold $Q = \{(w^1, w^2) \in C^{2n} : w^2 = w^T\}$ if, for any z^1 and z^2 and $0 \leqslant \lambda = 1$,

$$\begin{aligned} &\lambda \operatorname{Re} g(z, \bar{z}) + (1-\lambda) \operatorname{Re} g(z^2, \bar{z}^2) \\ &\qquad - \operatorname{Re} g(\lambda z^1 + (1-\lambda) z^2, \lambda \bar{z}^1 + (1-\lambda) \bar{z}^2) \in T. \end{aligned} \tag{4}$$

When $g(w^1, w^2)$ is analytic, a condition equivalent to (4) is

$$\operatorname{Re}[g(z^1, \bar{z}^1) - g(z^2, \bar{z}^2) - D_z g(z^2, \bar{z}^2)(z^1 - z^2) - D_{\bar{z}} g(z^2, \bar{z}^2)(\bar{z}^1, \bar{z}^2)] \in T. \tag{5}$$

With $T = R_+{}^m$, (5) is the definition of convexity of a complex-valued function given by Hanson and Mond in [6]. Definitions of convexity of the real part of a function $f : C^n \to C^m$ are obtained from (4) and (5) by replacing $g(z, \bar{z})$ with $f(z)$ and noting that $D_{\bar{z}} f(z) = 0$.

A function will be called concave with respect to a closed convex cone S if it is convex with respect to $-S = \{z : -z \in S\}$.

For the sufficiency and duality theorems of the next sections convexity of the real part of the objective functions with respect to R_+ is required. It will now be shown that if the real part of $f : C^n \to C$ is convex in the ordinary sense, i.e., with respect to R_+, then $f(z) = az + b$ where a and b are constants. Thus in Sections 4, 5 and 6 only problems of the second class discussed in [1], i.e., problems of the form (1) will be considered.

Let $r(z)$ be analytic and have convex real part with respect to R_+ in a convex neighborhood of z^0. Then (5) implies

$$\operatorname{Re}[r(z) - r(z^0) - \nabla_z^T r(z^0)(z - z^0)] \geq 0. \tag{6}$$

Therefore, $\nabla_z r(z^0) = 0$ implies that z^0 is a local minimum of $\operatorname{Re} r(z)$.

Suppose $f(z)$ is analytic and has convex real part with respect to R_+ in a neighborhood of z_0. Define $h(z) = f(z) - \nabla_z^T f(z^0)\, z$. Then $h(z)$ is analytic and has convex real part with respect to R_+ in a neighborhood of z^0. Therefore $\operatorname{Re} h(z)$ has a local minimum at z^0. Hence $e^{-h(z)}$ has maximum modulus at z^0, which implies by the maximum modulus theorem that $e^{-h(z)}$ is constant. Therefore, $h(z)$ is constant, i.e.,

$$f(z) - \nabla_z^T f(z^0)\, z = b \qquad \text{and} \qquad f(z) = \nabla_z^T f(z^0)\, z + b.$$

As previously mentioned, this does not rule out convexity of "functions" of z and $\bar{z}$, i.e., of functions defined on the manifold $\{(w^1, w^2) \in C^{2n} : w^2 = \overline{w^1}\}$. For example, if $g(z, \bar{z}) = z\bar{z}$, then for $0 \leq \lambda \leq 1$

$$\begin{aligned}\lambda g(z^1, \bar{z}^1) + (1 - \lambda)\, g(z^2, \bar{z}^2) - g(\lambda z^1 + (1 - \lambda)\, z^2, \lambda \bar{z}^1 + (1 - \lambda)\, \bar{z}^2)\\ = \lambda(1 - \lambda) \mid z^2 - z^1 \mid^2 \geq 0\end{aligned}$$

and, therefore, $g(z, \bar{z}) = z\bar{z}$ is convex with respect to R_+.

4. Sufficiency

THEOREM 1. *Let $f : C^{2n} \to C$ be analytic and have convex real part with respect to R_+ on the manifold $Q = \{(w^1, w^2) \in C^{2n} : w^2 = \overline{w^1}\}$. Let $g : C^{2n} \to C^m$ be analytic and be concave with respect to S on the manifold Q where S is a closed convex cone in C^m. A sufficient condition for $(z^0, \bar{z}^0)$ to be an optimal point of (1) is the existence of a $u^0 \in S^*$ such that*

$$\operatorname{Re}(u^0, g(z^0, \bar{z}^0)) = 0 \tag{7}$$

and

$$\nabla_z f(z^0, \bar{z}^0) + \overline{\nabla_{\bar{z}} f(z^0, \bar{z}^0)} = D_z^T g(z^0, \bar{z}^0)\, \bar{u}^0 + D_{\bar{z}}^H g(z^0, \bar{z}^0)\, u^0. \tag{8}$$

Proof. Let z be any other feasible point, i.e., let $g(z, \bar{z}) \in S$. Since $u^0 \in S^*$, $\operatorname{Re}(u^0, g(z, \bar{z})) \geqslant 0$. Therefore, the definition (5) of convexity of $f(z, \bar{z})$ implies

$$\begin{aligned}\operatorname{Re} f(z, \bar{z}) \geqslant \operatorname{Re} f(z^0, \bar{z}^0) &+ \operatorname{Re}(\nabla_z f(z^0, \bar{z}^0)\, \bar{z} - \bar{z}^0) \\ &+ \operatorname{Re}(\nabla_{\bar{z}} f(z^0, \bar{z}^0), z - z^0) - \operatorname{Re}(u^0, g(z, \bar{z})).\end{aligned} \tag{9}$$

Concavity of $g(z, \bar{z})$ with respect to S and $u^0 \in S^*$ imply

$$\begin{aligned}\operatorname{Re}(u^0, g(z, \bar{z})) \leqslant \operatorname{Re}(u^0, g(z^0, \bar{z}^0)) &+ \operatorname{Re}(u^0, D_z g(z^0, \bar{z}^0)\,(z - z^0)) \\ &+ \operatorname{Re}(u^0, D_{\bar{z}} g(z^0, \bar{z}^0)\,(\bar{z} - \bar{z}^0)).\end{aligned} \tag{10}$$

Combining (9), (10) and (7) yields

$$\begin{aligned}\operatorname{Re} f(z, \bar{z}) \geqslant \operatorname{Re} f(z^0, \bar{z}^0) &+ \operatorname{Re}(\nabla_z f(z^0, \bar{z}^0) + \overline{\nabla_{\bar{z}} f(z^0, \bar{z}^0)}, \bar{z} - \bar{z}^0) \\ &- \operatorname{Re}(D_z{}^T g(z^0, \bar{z}^0)\, \bar{u}^0 + D_{\bar{z}}{}^H g(z^0, \bar{z}^0)\, u^0, \bar{z} - \bar{z}^0)\end{aligned} \tag{11}$$

which with (8) gives

$$\operatorname{Re} f(z^0, \bar{z}^0) \leqslant \operatorname{Re} f(z, \bar{z}).$$

By identity (A34) of [1], the left hand side of (8) may be replaced by $2\nabla_z f^R(z^0, \bar{z}^0)$ which is sometimes a more convenient form.

5. Saddlepoint Equivalence

Define the Lagrangian of (1) by

$$L(z, \bar{z}, u) = \operatorname{Re} f(z, \bar{z}) - \operatorname{Re}(u, g(z, \bar{z})). \tag{12}$$

Theorem 2. *Let $F : C^{2n} \to C$ and $g : C^{2n} \to C^m$ in* (1) *satisfy the hypothesis of Theorem* 1. *Let S be a polyhedral convex cone in C^m. Assume the constraint qualification as defined in* [1] *holds at the optimal point of* (1). *Then $(z^0, \bar{z}^0)$ is an optimal point of* (1) *if and only if there exists a $u^0 \in S^*$ such that*

$$L(z, \bar{z}, u^0) \geqslant L(z^0, \bar{z}^0, u^0) \geqslant L(z^0, \bar{z}^0, u) \qquad \text{for all } z \in C^n \text{ and all } u \in S^*. \tag{13}$$

Proof. Only if: If $(z^0, \bar{z}^0)$ is an optimal point of (1), Theorem 3 of [1] implies the existence of a $u^0 \in S^*$ such that $\operatorname{Re}(u^0, g(z^0, \bar{z}^0)) = 0$. Hence

$$L(z^0, \bar{z}^0, u^0) \geqslant L(z^0, \bar{z}^0, u) \qquad \text{for all } u \in S^*. \tag{14}$$

Using the definitions of convexity of both $\operatorname{Re} f(z, \bar{z})$ and $g(z, \bar{z})$ gives for any $(z, \bar{z}) \in C^{2n}$

$$\begin{aligned} L(z, \bar{z}, u^0) &- L(z^0, \bar{z}^0, u^0) \\ &\geqslant \operatorname{Re}(\nabla_z f(z^0, \bar{z}^0) + \overline{\nabla_{\bar{z}} f(z^0, \bar{z}^0)}, z - z^0) \\ &\quad - \operatorname{Re}(D_z{}^H g(z^0, \bar{z}^0)\, u^0 + D_{\bar{z}}{}^T g(z^0, \bar{z}^0)\, \bar{u}^0, z - z^0). \end{aligned} \tag{15}$$

Thus by Theorem 3 of [1],

$$L(z, \bar{z}, u^0) \geqslant L(z^0, \bar{z}^0, u^0).$$

If: Now assume (13) holds for all $z \in C^n$ and all $u \in S^*$. The second inequality implies

$$\operatorname{Re}(u - u^0, g(z^0, \bar{z}^0)) \geqslant 0, \qquad \text{for all } u \in S^*. \tag{16}$$

Letting $u = w + u_0$, (16) yields

$$\operatorname{Re}(w, g(z^0, \bar{z}^0)) \geqslant 0, \qquad \text{for all } w \in S^*. \tag{17}$$

Therefore, $g(z^0, \bar{z}^0) \in S^{**}$ and since $S^{**} = S$ (1.5 of [4]), $(z^0, \bar{z}^0)$ is feasible point of (1). Putting $u = 0$ in (16) and noting that $u^0 \in S^*$ and $g(z^0, \bar{z}^0) \in S$ gives

$$\operatorname{Re}(u^0, g(z^0, \bar{z}^0)) = 0. \tag{17}$$

Thus the first inequality of (13) yields for any feasible point $(z, \bar{z})$

$$\operatorname{Re} f(z, \bar{z}) \geqslant \operatorname{Re} f(z, \bar{z}) - \operatorname{Re}(u^0, g(z, \bar{z})) \geqslant \operatorname{Re} f(z^0, \bar{z}^0).$$

6. Duality

Assume $f(z, \bar{z})$ and $g(z, \bar{z})$ in (1) satisfy the hypothesis of Theorem 1. A dual of (1) in the sense of Wolfe [11] is

maximize
$$\operatorname{Re} f(w, w) - \operatorname{Re}(u, g(w, \bar{w})) \tag{18}$$
subject to

$$\nabla_z f(z, \bar{z}) + \overline{\nabla_{\bar{z}} f(z, \bar{z})} = D_z{}^T g(w, \bar{w})\, \bar{u} + D_{\bar{z}}{}^H g(w, \bar{w})\, u, \qquad u \in S^*.$$

THEOREM 3. *Let $f(z, \bar{z})$ and $g(z, \bar{z})$ in (1) satisfy the hypothesis of Theorem 1. Let z be a feasible point of (1) and let (w, u) be a feasible point of (18). Then*

$$\operatorname{Re} f(z, \bar{z}) \geqslant \operatorname{Re} f(w, \bar{w}) - \operatorname{Re}(u, g(w, \bar{w})). \tag{19}$$

If z^0 is an optimal point of (1), *then there is a $u^0 \in S^*$ such that (z^0, u^0) is optimal for* (18).

Proof. Since $g(z, \bar{z}) \in S$ and $u \in S^*$,

$$\begin{aligned} \operatorname{Re} f(z, \bar{z}) - \operatorname{Re} f(w, \bar{w}) + \operatorname{Re}(u, g(w, \bar{w}) \\ \geqslant \operatorname{Re} f(z, \bar{z}) - \operatorname{Re} f(w, \bar{w}) + \operatorname{Re}(u, g(w, \bar{w}) - g(z, \bar{z})). \end{aligned} \tag{20}$$

Applying convexity of the $\operatorname{Re} f(z, \bar{z})$ and $g(z, \bar{z})$, the right hand side of (20) is greater than

$$\operatorname{Re}(\nabla_z f(w, \bar{w}) + \nabla_{\bar{z}} f(w, \bar{w}), \bar{z} - \bar{w}) - \operatorname{Re}(D_z{}^T g(w, \bar{w})\, \bar{u} + D_{\bar{z}}{}^H g(w, \bar{w})\, u, \bar{z} - \bar{w})$$

which is equal to zero by the constraint of (18).

To prove the second part, let z^0 be an optimal point of (1). Then by Theorem 3 of [1] there exists a $u^0 \in S^*$ such that the constraint of (18) is satisfied by $(z^0, \bar{z}^0)$ and u^0. Also by Theorem 3 of [1], $\operatorname{Re}(u^0, g(z^0, \bar{z}^0)) = 0$. Therefore, (11) holds with equality and hence $(z^0, \bar{z}^0)$ and u^0 are an optimal solution of (18).

A converse dual theorem generalizing the result of Hanson [5], to complex space is now established. For this purpose, associate with an analytic function $\alpha : C^m \to C^n$ a second analytic function $\bar{\alpha} : C^m \to C^n$ defined by $\bar{\alpha}(z) = \overline{\alpha(\bar{z})}$. Alternatively, the components of $\bar{\alpha}(z)$ are represented by the power series obtained from the power series for the components of $\alpha(z)$ by replacing the coefficients and the point about which the function is being expanded by their complex conjugates. Thus if $\alpha(z)$ is analytic in a neighborhood of z^0, $\bar{\alpha}(z)$ will be analytic in a neighborhood of $\bar{z}^0$.

THEOREM 4. *Let $(z^0, \bar{z}^0, u^0)$ be an optimal solution of* (18), *where* $\operatorname{Re} f(z, \bar{z})$ *and $g(z, \bar{z})$ satisfy the hypothesis of Theorem* 1. *Assume that there exists a function $\alpha : C^m \to C^n$ analytic in a neighborhood of u^0, such that $\alpha(u^0) = z^0$ and such that in a neighborhood of u^0, $(\alpha(u), \bar{\alpha}(\bar{u}), u)$ satisfies*

$$\nabla_z f(z, \bar{z}) + \overline{\nabla_{\bar{z}} f(z, \bar{z})} = [D_z{}^T g(z, \bar{z})]\, \bar{u} + [D_{\bar{z}}{}^H g(z, \bar{z})]\, u. \tag{21}$$

Then $(z^0, \bar{z}^0)$ is an optimal solution of (1).

Proof. Since $(\alpha(u^0), \bar{\alpha}(\bar{u}^0), u^0) = (z^0, \bar{z}^0, u^0)$ is an optimal solution of (18) and $(\alpha(u), \bar{\alpha}(u), u)$ satisfies (21), $(\alpha(u^0), \bar{\alpha}(\bar{u}^0), u^0)$ is a local maximum of the problem

$$\begin{aligned} &\text{maximize} && \operatorname{Re}[f(\alpha(u), \bar{\alpha}(\bar{u})) - (u, g(\alpha(u), \bar{\alpha}(\bar{u})))] \\ &\text{subject to} && u \in S^* \end{aligned} \tag{22}$$

The necessary conditions of Theorem 3 of [1] must therefore be satisfied at $(\alpha u^0), \bar{\alpha}(\bar{u}^0), u^0)$, i.e., there exists a $w \in S^{**} = S$ such that

$$\begin{aligned} &- D_u{}^T\alpha(u^0)\, \nabla_z f(z^0, \bar{z}^0) - \overline{D_u{}^T\bar{\alpha}(\bar{u}^0)\, \nabla_{\bar{z}} f(z^0, \bar{z}^0)} + \overline{g(z^0, \bar{z}_0)} \\ &+ D_u{}^T\alpha(u^0)\, D_z{}^T g(z^0, \bar{z}^0)\, \bar{u}^0 + D_u{}^H\bar{\alpha}(\bar{u}^0)\, D_{\bar{z}}{}^H g(z^0, \bar{z}^0)\, u^0 = \bar{w} \end{aligned} \tag{23}$$

and

$$\operatorname{Re}(w, u^0) = 0. \tag{24}$$

Noting that $D_u\alpha(u^0) = \overline{D_u\bar{\alpha}(\bar{u}^0)}$ and using (21), we find that (23) and (24) become

$$g(z^0, \bar{z}^0) = w \in S, \tag{25}$$

and

$$\operatorname{Re}(u^0, g(z^0, \bar{z}^0)) = 0 \tag{26}$$

Equation (25) implies that $(z^0, \bar{z}^0)$ is feasible for (1) and Theorem 1 implies that $(z^0, \bar{z}^0)$ is optimal for (1).

7. Examples

7.1. *Duality in Complex Linear Programming*

Let $A \in C^{m \times n}$, $b \in C^m$, $c \in C^n$ and let S be a polyhedral cone in C^n. Consider the complex linear programming problem

$$\begin{aligned} &\text{minimize} && \operatorname{Re} b^H y \\ &\text{subject to} && A^H y - c \in S^*. \end{aligned} \tag{27}$$

Since the objective function and the constraints are linear, they are both convex and concave. Thus the dual according to Section 6 is

$$\begin{aligned} &\text{maximize} && \operatorname{Re} b^H y - \operatorname{Re}[u^H(A^H y - c)] \\ &\text{subject to} && b = Au, \quad u \in S^*. \end{aligned} \tag{28}$$

Using the constraint to simplify the objective function, (28) becomes

$$\begin{aligned} &\text{maximize} && \operatorname{Re} u^H c \\ &\text{subject to} && Au = b, \quad u \in S^{**} = S. \end{aligned} \tag{29}$$

The dual problems (27) and (28) were first presented in [4].

7.2. *Duality in Complex Quadratic Programming*

Let $B \in C^{n \times n}$ be a positive definite Hermitian matrix. Let $A \in C^{m \times n}$, $b \in C^m$, $c \in C^n$ and let $S \subset C^n$ and $T \subset C^m$ be polyhedral cones. Consider the problem

$$\begin{aligned} &\text{minimize} && \operatorname{Re}(\tfrac{1}{2} x^H B x + c^H x) \\ &\text{subject to} && Ax - b \in T, \quad x \in S. \end{aligned} \tag{30}$$

Problem (30) may be rewritten in the form of (1) as

$$\begin{aligned} &\text{minimize} && \operatorname{Re}[\tfrac{1}{2} x^H B x + c^H x] \\ &\text{subject to} && \begin{pmatrix} A \\ I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \in T + S. \end{aligned} \tag{31}$$

The constraint of (31) is linear and hence concave with respect to S. Convexity of $\frac{1}{2} x^H B x$ with respect to R_+ is, from (5), equivalent to the nonnegativity of

$$\tfrac{1}{2} x_1{}^H B x_1 - \tfrac{1}{2} x_2{}^H B x_2 - \operatorname{Re}[(x_1 - x_2)^H B x_2] \tag{32}$$

which follows from Lemma 1 of [7]. Thus the objective function is convex with respect to R_+, and from Section 6, it follows that the dual of (31) is

$$\begin{aligned} &\text{maximize} && \operatorname{Re}\left[\frac{1}{2} y^H B y - \begin{pmatrix} Ax - b \\ x \end{pmatrix}^H \begin{pmatrix} u \\ v \end{pmatrix}\right] \\ &\text{subject to} && By + c = A^H u + v \\ & && \begin{pmatrix} u \\ v \end{pmatrix} \in (T \times S)^* = T^* \times S^*. \end{aligned} \tag{33}$$

Using the constraint to simplify the objective function and rewriting the constraints in an equivalent form (33) becomes

$$\begin{aligned} &\text{maximize} && \operatorname{Re}(-\tfrac{1}{2} y^H B y + b^H u) \\ &\text{subject to} && c + By - A^H u \in S^* \\ & && u \in T^*. \end{aligned} \tag{34}$$

The dual problems (31) and (34) were obtained in [2] by using Dorn's technique of linearizing the objective function.

7.3. *Problem of the form* (1) *with* $m = n = 1$.

Consider the following:

$$\begin{aligned} &\text{minimize} && \operatorname{Re} i(z - \bar{z}) \\ &\text{subject to} && 2z - z\bar{z} \in S = \left\{ w : 0 \leqslant \arg w \leqslant \frac{\Pi}{4} \right\}. \end{aligned} \tag{35}$$

The objective function is linear and hence convex and the constraint function is concave with respect to S. The sufficient conditions (7) and (8) become

$$2i = (2 - \bar{z})\,\bar{u} - \bar{z}u \tag{36}$$

and

$$u(2\bar{z} - z\bar{z}) + \bar{u}(2z - z\bar{z}) = 0 \tag{37}$$

where

$$u \in S^* = \left\{w : -\frac{\Pi}{4} \leqslant \arg w \leqslant \frac{\Pi}{2}\right\}$$

which are equivalent to

$$2\bar{u} - 2i = \bar{z}(\bar{u} + u) \tag{38}$$

and

$$\bar{z}u + z\bar{u} = \frac{z\bar{z}}{2}(u + \bar{u}). \tag{39}$$

A solution to (38) and (39) may be obtained with $z = 0$, but this implies, by (38), that $u = -i$ which is not an element of S^*. Therefore $z \neq 0$. Substituting the expression for $(u + \bar{u})$ from (38) into (39) yields

$$u = \frac{-z}{\bar{z}}\,i. \tag{40}$$

With (40), Eq. (38) becomes

$$(z - \bar{z})(z + \bar{z} - 2) = 0. \tag{41}$$

Therefore either z is real or $\operatorname{Re} z = 1$ and $\operatorname{Im} z > 0$. Assume z is real and $z = x$; then (39) implies

$$x(2 - x)(u + \bar{u}) = 0. \tag{42}$$

It has already been noted that x cannot be zero. If $x = 2$, (38) implies $u = -i \notin S^*$. Also, if $u + \bar{u} = 0$, (38) implies $u = -i$. Therefore at a point satisfying (36) and (37), $\operatorname{Re} z = 1$ and $\operatorname{Im} z > 0$.

The dual cone of S is

$$S^* = \left\{w : -\frac{\Pi}{4} \leqslant \arg w \leqslant \frac{\Pi}{2}\right\}.$$

Thus in order that $\operatorname{Re}(u, g(z, \bar{z}))$ be equal to zero, i.e., the vectors u and $g(z, \bar{z})$ be perpendicular, u must lie on the half line $\{w : \arg w = -\Pi/4)$ and $g(z, \bar{z})$ must lie on the half line $\{w : \arg w = \Pi/4\}$. From (40) it follows that $|u| = 1$. Therefore,

$$u = \frac{\sqrt{2}}{2}(1 - i) \tag{43}$$

and

$$z = 1 + i(\sqrt{2} - 1) \tag{44}$$

satisfy the sufficient conditions for an optimal point of (35). Geometrically, (35) is equivalent to maximizing the imaginary part of z with z constrained to the feasible region shown in the figure.

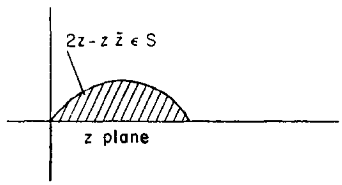

FIGURE 1

From Section 6 the dual of (35) is, after simplifying the objective function with the constraint,

$$\begin{aligned} &\text{maximize} && -2\operatorname{Re}\bar{z}u \\ &\text{subject to} && 2i = 2\bar{u} - \bar{z}(u + \bar{u}) \\ &&& u \in S^* \end{aligned} \tag{45}$$

The dual feasible solution given by (43) and (44) result in a value of the dual objective function of $2 - 2\sqrt{2}$ which equals the optimal value of the primal objective function. Therefore, as required by Theorem 3,

$$z = 1 + i(\sqrt{2} - 1), \qquad u = \frac{\sqrt{2}}{2}(1 - i)$$

is an optimal solution of (45).

REFERENCES

1. R. A. ABRAMS AND A. BEN-ISRAEL, Nonlinear programming in complex space: Necessary conditions, *SIAM J. Control*, Vol. 9, No. 4, Nov. 1971.
2. R. A. ABRAMS AND A. BEN-ISRAEL, A duality theorem for complex quadratic programming, *J. Optimization Theory Appl.* **4** (1969), 244–252.
3. R. A. ABRAMS AND A. BEN-ISRAEL, "Complex Mathematical Programming," in "Developments in Operations Research" edited by B. Aui-Itzhak, Gorden and Breach, New York, 1971 (pp. 3–28).
   Proceedings of the Third Annual Israel Conference on Operations Research, Tel-Aviv, July 1969, to appear.
4. A. BEN-ISRAEL, Linear equations and inequalities on finite dimensional, real or complex, vector spaces: A unified theory, *J. Math. Anal. Appl.* **27** (1969), 367–389.

5. M. A. Hanson, A duality theorem in nonlinear programming with nonlinear constraints, *Austral. J. Statist.* **3** (1961), 64–71.
6. M. A. Hanson and B. Mond, Duality for nonlinear programming in complex space, *J. Math. Anal. Appl.* **28** (1969), 52–58.
7. M. A. Hanson and B. Mond, Quadratic programming in complex space, *J. Math. Anal. Appl.* **20** (1967).
8. H. W. Kuhn and A. W. Tucker, Nonlinear programming, *in* "Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability," (J. Neyman, Ed.), pp. 481–493, University of California Press, Berkeley, Calif., 1951.
9. N. Levinson, Linear programming in complex space, *J. Math. Anal. Appl.* **14** (1966), 44–62.
10. O. L. Mangasarian, "Nonlinear Programming," McGraw-Hill, New York, 1969.
11. P. Wolfe, A duality theorem for nonlinear programming, *Quart. Appl. Math.* **19** (1961), 239–244.
12. H. Weyl, The elementary theory of convex polyhedra, *in* "Contributions to the Theory of Games," (H. W. Kuhn and A. W. Tucker, Eds.), Vol. I, pp. 3–18, Annals of Math. Studies No. 24, Princeton University Press, Princeton, N. J., 1950.