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Self-Injective Rings

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Throughout this paper we suppose that all rings and modules are unitary. By J we denote always the Jacobson radical of a ring S. The left (right, resp.) annihilator of X is denoted by $l(X)$ [$r(X)$, resp.].

A module M over a ring S is called completely faithful if there is a direct sum of isomorphic copies of M which has the module S as a homomorphic image. Thus, a module $_SM$ is completely faithful if and only if there are $w_i \in \mathrm{Hom}_S(M, S)$, $i = 1,..., n$, such that $\sum_{i=1}^n \mathrm{Im}(w_i) = S$. Such a module is called also a generator of the category of all left S-modules, and is important especially in connection with the Morita theorems. (See [*4*; Section 3].)

We call a ring a left *PF*-ring if every faithful left module over the ring is completely faithful. Note that every completely faithful module is faithful as is readily seen.

Azumaya has proved that every left *PF*-ring is left self-injective, and also that every *QF*-ring is left *PF*.

In the present paper we shall show the following two theorems:

THEOREM. *Let S be a left self-injective ring. Then, $l(A) = 0$ for an ideal A of S implies $A = S$ if and only if* (i) *S/J is a direct sum of a finite number of left self-injective simple rings, and* (ii) *$r(l(J)) = J$.*

THEOREM. *A ring S is left PF if and only if* (i) *S is left self-injective,* (ii) *S/J is Artinean, and* (iii) *every nonzero left ideal of S contains a minimal left ideal.*

1. We denote the canonical epimorphism $S \to S/J$ for a ring S by j.

Let S be a left self-injective ring. Then it is well known that S satisfies the following

CONDITION 1.1. For any left ideal A of S there is an idempotent e of S such that Se is an essential extension of A.

In connection with this we have the following

LEMMA 1.2. *Let S be a left self-injective ring, and A an ideal. If Se, $e = e^2$, is essential over A as a left S-module, then $j(e)$ is central in $j(S)$.*

Proof. Suppose that Se and $S(1 - e)$ contain mutually isomorphic left ideals B and C, respectively. If $B \neq 0$, $A \cap B \neq 0$ since Se is essential over A, and hence $A \cap C \neq 0$ because the isomorphism $B \to C$ is given by the right multiplication of an element of S. This implies that $Se \cap S(1 - e) \neq 0$, a contradiction. Thus $B = C = 0$. It follows then by the first part of the proof of [5; Lemma 5.5], that $j(e)j(S)j(1 - e) = 0$. Since $j(S)$ is semiprime, this means that $j(e)$ is central, as desired.

The following is a characterization of the radical J of a left self-injective ring S.

THEOREM 1.3. *Let S be a left self-injective ring, and A a left (or right) ideal. Then $A \subset J$ if and only if A does not contain any nonzero idempotent.*

Proof. $j(S)$ is regular in the sense that, for any element p, there exists q with $pqp = p$ by [5; Lemma 4.1]. Let $A \not\subset J$. Then $j(A)$ contains a nonzero idempotent $j(x)$. By [5; Corollary 3.2] there is an idempotent e of S such that $j(e) = j(x)$. $1 - e + x$ is then inversible, and so $Sex = Se(1 - e + x)$ is isomorphic to Se. Now S satisfies the following condition.

CONDITION 1.4. Any left ideal of S isomorphic to a left ideal generated by an idempotent is generated by an idempotent.

Thus, in our case, Sex is generated by $f = f^2$. Since $j(x) \neq 0$, $e \neq 0$ and $f \neq 0$. If A is a left ideal, we may suppose that $x \in A$, and hence we have $A \ni f$. In case A is a right ideal, let $f = yx$. Then A contains a nonzero idempotent $xyxy$, completing the proof.

The following is an immediate consequence of the above theorem.

COROLLARY 1.5. *Let S be a left self-injective ring, and A a left ideal. Then A is essential* (in ${}_SS$) *if and only if $r(A) \subset J$.*

Proof. A is not essential if and only if A is contained in Se, $e = e^2 \neq 1$, since A satisfies Condition 1.1. And it is the case if and only if $r(A)$ contains a nonzero idempotent, that is, $r(A) \not\subset J$.

Let S be left self-injective. Then $l(J)$ is the core of the left S-module S in the sense of Lesieur and Croisot [3]. Thus we have

COROLLARY 1.6. *Let S be a left self-injective ring. Then the core of the left S-module S is essential in S if and only if $r(l(J)) = J$.*

2. Let S be a semiprime ring, and A an ideal. Then for any left ideal B $A \cap B = 0$ if and only if $BA = 0$. Therefore an ideal A is essential in S as a left ideal if and only if $l(A) = 0$.

LEMMA 2.1. *Let S be a left self-injective ring, and A an ideal. Suppose that $r(l(J)) = J$. Then $l(A) = 0$ implies that $l(j(A)) = 0$ in $j(S)$.*

Proof. By [5; Theorem 4.8], $j(S)$ is also left self-injective. Hence there exists an idempotent $j(e)$ of $j(S)$ such that $j(S)j(e)$ is an essential extension of $j(A)$. By Lemma 1.2, $j(e)$ is central. By [5; Corollary 3.2] we can suppose that e is an idempotent of S. $A \subset Se + J = eS + J$, and so

$$l(A) \supset S(1 - e) \cap l(J),$$

hence $S(1 - e) \cap l(J) = 0$. By Corollary 1.6, the (left) core $l(J)$ is essential in S as a left ideal. Thus, $S(1 - e) = 0$, and $e = 1$. This implies that $j(A)$ is an essential left ideal of $j(S)$. Since evidently $j(S)$ is semiprime, we have then that $l(j(A)) = 0$, as desired.

Let S be a left self injective regular ring, and suppose that the sum of principal left ideals Sx_i is direct. Then there is a system (e_i) of orthogonal idempotents such that $Sx_i = Se_i$ for every i. (See [6; Theorem 2.2]). We shall use this fact to show the following.

LEMMA 2.2. *Let S be a left self-injective ring, and A an ideal containing J. Then $l(j(A)) = 0$ in $j(S)$ means that $l(A) = 0$ in S.*

Proof. By assumption, $j(A)$ is essential left ideal of $j(S)$. Let $B = \sum^{\oplus} j(S)j(x_i)$ be a maximal left ideal which is a direct sum of principal left ideals contained in $j(A)$. Then $j(A)$ is essential over B, and hence B is an essential left ideal of $j(S)$. There is a system $(j(e_i))$ of orthogonal idempotents of $j(S)$ with $j(S)j(x_i) = j(S)j(e_i)$ for every i. By [5; Theorem 4.9], we may suppose, with no loss in generality, that (e_i) is a system of orthogonal idempotents of S, and that S is essential over $\sum Se_i$. Since $A \supset J$, A contains every e_i. If $l(A) \neq 0$, $l(A) \cap \sum Se_i \neq 0$. However $l(A) \cap \sum Se_i = 0$ since $l(A)\, e_i = 0$ for every i. This contradiction shows that $l(A) = 0$, completing the proof.

THEOREM 2.3. *Let S be a left self-injective ring. Then the following two conditions are equivalent:*

(i) $l(A) = 0$ *for an ideal A implies that $A = S$.*

(ii) *$j(S)$ is a direct sum of a finite number of left self-injective simple rings, and $r(l(J)) = J$.*

Proof. (1) Suppose (i) and that $J = 0$. Then it is readily seen that S is completely reducible as an (S, S)-bimodule, and hence it is a direct sum of a finite number of simple rings. Each of the simple rings is obviously left self-injective.

(2) Suppose Condition (i). If $l(j(A)) = 0$ for an ideal A containing J, then $l(A) = 0$, by Lemma 2.2. By assumption, $A = S$, and so $j(A) = j(S)$. Thus, by part (1) of this proof, $j(S)$ is a direct sum of a finite number of left self-injective simple rings. Let e be an idempotent of S such that Se is essential over $l(J)$. Then $j(e)$ is central by Lemma 1.2, and $J + eS$ is an ideal. $l(J + eS) = l(J) \cap S(1 - e) = 0$. Hence $J + eS = S$, and $eS = S$, so $e = 1$. This means that $l(J)$ is an essential left ideal, and therefore $r(l(J)) = J$ by Corollary 1.6.

(3) Suppose Condition (ii), and let A be an ideal with $l(A) = 0$. By Lemma 2.1, $l(j(A)) = 0$. This implies that $j(A)$ is essential in $j(S)$. Thus by assumption, $j(A) = j(S)$, and $S = A + J$, therefore $S = A$, completing the proof.

COROLLARY 2.4. *Let S be a left self-injective ring satisfying the conditions in Theorem 2.3. Then $j(S)$ is Artinean if and only if every nonzero left ideal contains a uniform left ideal, that is, a nonzero left ideal which is essential over every nonzero left ideal contained in it.*

Proof. If $j(S)$ is Artinean, for any nonzero left ideal A of S let e be an idempotent such that Se is essential over A. $j(S)j(e)$ contains a primitive idempotent $j(f)$. By [5; Lemma 3.1], we may suppose that f is an idempotent of S with $fe = f$. Then $Sf \cap A$ is nonzero, and is uniform, as is readily seen.

Conversely, let us suppose that any nonzero left ideal contains a uniform left ideal. Let $j(g)$ be the unit element of any one of the simple rings whose direct sum is $j(S)$. g may be supposed to be an idempotent of S. Sg contains a uniform left ideal B, by assumption. Then there is an idempotent h such that Sh is essential over B, and that $hg = h$. Then $j(h)j(g) = j(h)$ and $j(h)$ is a primitive idempotent of $j(S)$. Since $j(S)$ is regular by [5; Lemma 4.1], $j(S)j(h)$ is a minimal left ideal of $j(S)$. Thus, $j(S)j(g)$ is a simple ring with a minimal left ideal, and so it is a simple Artinean ring. Therefore $j(S)$ is Artinean, as desired.

2.5. Note that, in the above corollary, if the left S-module S is finite-dimensional, that is, if S does not contain any direct sum of an infinite number of nonzero left ideals, then every nonzero left ideal contains a uniform left ideal. (See [3; Propriété 1.4]).

2.6. In Corollary 2.4 if $j(S)$ is Artinean, $l(J)$ is the sum of all minimal annihilator left ideals. Any nonzero annihilator left ideal contains a minimal annihilator left ideal.

In fact, for any idempotent e of S such that $j(e)$ is primitive, $Se \cap l(J)$ is a minimal annihilator left ideal. And any minimal annihilator left ideal has this form since any maximal right ideal is $J + (1 - e)S$, e being an idempotent such that $j(e)$ is primitive.

3. For any modules A and B denote by $\mathrm{Im}(A, B)$ the sum of $\mathrm{Im}(v)$ for all $v \in \mathrm{Hom}(A, B)$. Then a module M is completely faithful if and only if $\mathrm{Im}(M, S) = S$, S being the coefficient ring. In other words, in case S satisfies the condition that $r(C) = 0$ for an ideal C implies that $S = C$, then a left S-module M is completely faithful if and only if the dual module is faithful.

The following is known:

3.1. Every completely faithful left S-module M is faithful.

This is evident from the inclusion $l(M) \subset l(\mathrm{Im}(M, S))$ which holds for any left S-module M.

PROPOSITION 3.2. *Let S be a left self-injective ring, and suppose that every nonzero left ideal contains a minimal left ideal. Denote the left socle of S by P. Then $P \cap l(M) = P \cap l(\mathrm{Im}(M, S))$ for any left S-module M.*

Proof. $l(M) \subset l(\mathrm{Im}(M, S))$, and so it is enough to see that

$$P \cap l(M) \supset P \cap l(\mathrm{Im}(M, S)).$$

Since S is left self-injective, $\mathrm{Im}(M, S) = \sum_{x \in M} \mathrm{Im}(Sx, S)$. Now

$$\mathrm{Im}(Sx, S) = \mathrm{Im}(S/l(x), S) = Sr(l(x))$$

for every $x \in M$, where r denotes the right annihilator in S. Hence $\mathrm{Im}(Sx, S) \supset r(l(x))$, and so $l(\mathrm{Im}(M, S)) \subset \cap\, l(r(l(x)))$. By the left self-injectivity of S, $l(r(l(x)))$ is essential over $l(x)$ for every $x \in M$. Hence $P \cap l(r(l(x))) = P \cap l(x)$. Taking the intersection for all $x \in M$ we have $P \cap l(\mathrm{Im}(M, S)) \subset P \cap l(M)$, as desired.

The following is due to Azumaya.

3.3. If there is a completely faithful injective left S-module, then S is left self-injective.

In fact, if M is a completely faithful left S-module, there is an epimorphism from M^n to S where M^n is the direct sum of a certain finite number of isomorphic copies of M. Since the left S-module S is projective, S is a direct summand of M^n. If M is injective, so is M^n, and hence S is left self-injective, as desired.

Recall the definition of left PF-rings. A ring S is left PF if and only if every faithful left S-module is completely faithful. Azumaya has proved that every QF-ring is (left) PF. This result is contained in the following.

THEOREM 3.4. *A ring S is left PF if and only if it is left self-injective, $j(S)$ is Artinean, and every nonzero left ideal contains a minimal left ideal.*

Proof (Sufficiency). The left socle of S is $r(J)$, and is an essential left ideal by assumption. Hence by Corollary 1.5, $r(r(J)) \subset J$. Since $r(J)$ is a completely reducible left ideal, it is the smallest among all essential left ideals. Thus, by [5; Lemma 4.1], $r(J)\,x = 0$ for any $x \in J$. Thus, $r(J) \subset l(J)$. $J \subset r(l(J)) \subset r(r(J)) \subset J$, and $J = r(l(J))$. It follows from Lemma 2.3 that only ideal A with $l(A) = 0$ is S. Let M be a faithful left S-module. Then $l(M) = 0$, and hence $l(\mathrm{Im}(M, S)) = 0$ by Proposition 3.2. Since $\mathrm{Im}(M, S)$ is an ideal of S, it follows that $\mathrm{Im}(M, S) = S$; that is, M is completely faithful.

(Necessity). The left S-module S has an injective extension K, which is completely faithful by assumption. Hence by Statement 3.3, S is left self-injective. Let A be an ideal with $l(A) = 0$. Then A is a faithful left S-module, and hence it is completely faithful. By the left self-injectivity of S, this means that $A = S$. Thus, in view of Corollary 2.4, it is enough to show that every left ideal $\neq 0$ contains a minimal left ideal. Let P be the left socle of S. Then there is an idempotent e such that Se is essential over the left ideal P. $j(e)$ is central, by Lemma 1.2. Let M be the direct sum of Se and $S/(A_i + Se)$ for all left ideals A_i essential in $S(1-e)$. Suppose first that M is faithful. Then it is completely faithful. Hence $S = \mathrm{Im}(M, S) = \mathrm{Im}(Se, S) + \sum_i \mathrm{Im}(S/(A_i + Se), S) = SeS + Sr(A_i + Se)$. Since $A_i + Se$ is an essential left ideal, $r(A_i + Se) \subset J$ for every i. Thus, $j(S) = j(SeS) = j(Se)$ since $j(e)$ is central. This shows that $j(e) = j(1)$, and $e = 1$. Therefore P is an essential left ideal, and hence every nonzero left ideal contains a minimal left ideal. Next suppose that M is not faithful. Then $0 \neq l(M) \subset S(1-e) \cap (\cap (A_i + Se))$. Since $S(1-e) \supset A_i$, $S(1-e) \cap (A_i + Se) = A_i$, and so $\cap A_i \neq 0$. $\cap A_i$ is completely reducible by the next lemma, and is contained in the left socle P. Since $\cap A_i \subset S(1-e)$, this is a contradiction, completing the proof.

LEMMA 3.5. *Let M be a module, and suppose that the intersection D of all essential submodules is nonzero. Then D is completely reducible.*

Proof. Let N be a submodule of D, and let H be a maximal submodule of M which is disjoint to N. Then $N + H$ is essential, and contains D. Thus, $D = N \oplus (D \cap H)$. This shows that every submodule of D is a direct summand of D, as desired.

Let S be a left PF-ring. Counting the number of nonisomorphic simple left S-modules and that of nonisomorphic minimal left ideals, it is readily seen that for any simple left S-module there is a minimal left ideal which is isomorphic to the module. Thus, the proof of [2; Theorem 2] shows the following.

3.6. Let S be a left PF-ring, and A a left ideal. Then there is no left ideal B such that $A \subset B \subset l(r(A))$, and such that B/A is simple.

As an immediate consequence of this every maximal left ideal of a left

PF-ring is an annihilator left ideal. Thus, by [*3*; Propriété 3.2] every element of $l(J)\,e$ generates a minimal left ideal of the left *PF*-ring S if e is a primitive idempotent. Thus, $l(J)\,e \subset r(J)$, and so $l(J) \subset r(J)$. In the proof of Theorem 3.4 we have just seen that $r(J) \subset l(J)$. Therefore:

PROPOSITION 3.7. *The left socle of a left PF-ring coincides with the right socle.*

In [*2*; Theorem 1], Ikeda and Nakayama showed the following.

3.8. Let S be a left self-injective ring, and A a finitely generated right ideal. Then $r(l(A)) = A$.

This has several consequences:

PROPOSITION 3.9. *Let S be a left PF-ring, and P the socle.*

(i) *P is an essential right ideal, that is, every nonzero right ideal contains a minimal right ideal.*

(ii) *Every minimal right ideal is an annihilator right ideal, and has the form eP, e being a primitive idempotent. Conversely, eP is a minimal right ideal for any primitive idempotent e.*

[Note that the right–left symmetry of the above (i) and (ii) also holds for a left *PF*-ring S, as is easily seen from Theorem 3.4.]

Proof. Let A be a minimal right ideal. Then by Statement 3.8, $A = r(l(A))$. Since every maximal left ideal of S is an annihilator left ideal, $l(A)$ must be a maximal left ideal: $l(A) = J \oplus S(1 - e)$ for some primitive idempotent e. Thus, $A = r(l(A)) = r(J) \cap eS = eP$.

Conversely, let e be a primitive idempotent. Then $eP = r(J + S(1 - e))$, where $J + S(1 - e)$ is a maximal left ideal. Since $eP(\subset P)$ contains a minimal right ideal which is also an annihilator right ideal as we have just seen, eP itself is a minimal right ideal.

Let B be a nonzero right ideal. Then it contains a finitely generated right ideal $B' \neq 0$, and $l(B')$ is contained in a maximal left ideal C. $0 \neq r(C) \subset r(l(B')) = B'$, and $0 \neq r(C) \subset B' \cap P \subset B \cap P$. Thus B contains a minimal right ideal.

PROPOSITION 3.10. *Let S be a left PF-ring, and P the socle. Then $l(P) = r(P) = J$.*

Proof. By Theorem 2.3, $r(l(J)) = J$; that is, $r(P) = J$. If $l(r(J)) \neq J$, then, by Theorem 1.3, $l(r(J))$ contains a nonzero idempotent e. Thus, $er(J) = 0$, and so $eS \cap r(J) = 0$, contradicting (i) of Proposition 3.9. Hence $l(r(J)) = J$; that is, $l(P) = J$, as desired.

PROPOSITION 3.11. *Let S be a left PF-ring, and e an idempotent. If a right ideal A is isomorphic to eS, then A is also generated by an idempotent.*

Proof. Let $v : eS \to A$ be the given isomorphism, and set $v(e) = x$. If $x \in J$, then, by Proposition 3.10, $Se = l(r(e)) = l(r(x)) \subset l(r(J)) = J$. Hence $e = 0$, and $A = 0S$. Suppose that e is the sum of k orthogonal primitive idempotents.

Induction for k. Let $x \notin J$. Then, by Theorem 1.3, A contains a nonzero idempotent f. Set $B = A \cap (1 - f)\, S$. Then $A = B \oplus fS$, and

$$eS = v^{-1}(B) \oplus v^{-1}(fS).$$

$v^{-1}(B)$ is generated by an idempotent h. Thus,

$$A = v(hS) \oplus fS.$$

Set $C = A \cap (1 - f)\, S$. $A = C \oplus fS$. Since $C \approx v(hS) \approx hS$, C is generated by an idempotent d. Therefore $A = bS$ for $b = f + d - df$, as desired.

PROPOSITION 3.12. *Let S be a left PF-ring, and A a right ideal such that $r(l(A))$ is essential over A. Then there exists an idempotent e such that eS is essential over A. In particular, if A is a finitely generated right ideal, A has an essential extension generated by an idempotent.*

Proof. Suppose that $j(S)$ is the sum of n minimal left ideals, and that $P \cap A$ is the sum of k minimal right ideals, where P denotes the socle of S. There is an idempotent e such that $l(A) + J = S(1 - e) + J$. Then $P \cap A = P \cap r(l(A)) = r(J + l(A)) = P \cap eS$. Thus, by Proposition 3.9, e is the sum of k orthogonal primitive idempotents. There is an element x of $l(A)$ with $j(x) = j(1 - e)$. Then $x + e$ has the inverse, and

$$S(1 - e) \approx S(1 - e)\, x.$$

Hence $S(1 - e)\, x$ is generated by an idempotent f by Condition 1.4. f is the sum of $n - k$ orthogonal primitive idempotents, and we have $fA = 0$. $A \subset (1 - f)\, S$. Since $1 - f$ is the sum of k orthogonal primitive idempotents, $P \cap (1 - f)\, S$ is the sum of k minimal right ideals, by Proposition 3.9. This shows that $(1 - f)\, S$ is essential over A, as desired.

By the proof of [5; Theorem 1.2], the following is a consequence of Propositions 3.11 and 3.12.

PROPOSITION 3.13. *Let S be a left PF-ring.*

(i) *Let A be a right ideal such that $r(l(A))$ is essential over A, and e an idempotent such that $A \subset eS$. Then there is an idempotent f such that $ef = f$, and that A is essential in fS.*

(ii) *Let $gS \cap hS = 0$ for idempotents g and h, $gS + hS$ is generated by an idempotent.*

4. Azumaya [*1*] called an (S, T)-bimodule ${}_SM_T$ quasi-Frobenius if (i) $r(A)$ is 0 or simple for any maximal left ideal A of S, and (ii) $l(B)$ is 0 or simple for any maximal right ideal B of T. Note that any left *PF*-ring S is a quasi-Frobenius module over (S, S). This follows directly from what we have seen.

Whether or not a left *PF*-ring is always right *PF* is an open problem.

Professor Azumaya kindly informed me that he independently obtained a theorem essentially the same as Theorem 3.4. His proof will appear in Nagoya J. Math.

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