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Vorticity-Velocity Formulation for the Stationary Navier-Stokes Equations: The Three-Dimensional Case

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Abstract—In this article, we propose a mixed method for the vorticity-velocity formulation of the stationary Stokes and Navier-Stokes equations in space dimension three, the unknowns being the vorticity and the velocity of the fluid.

Keywords—Incompressible fluid, Navier-Stokes equations, Vorticity-Velocity formulation, Mixed formulation.

1. INTRODUCTION

There are various formulations for the three-dimensional Navier-Stokes equations, each of them having both advantages and disadvantages.

In the last decade, the velocity-vorticity formulation of the Navier-Stokes equations appeared to an increasing number of people as an attractive alternative to the usual primitive variables formulation. In contrast with the stream function-vorticity formulation that cannot easily be extended to three-dimensional flows, the vorticity-velocity formulation is valid for both two and three-dimensional flows.

In [1], the author gives a closed system of Navier-Stokes equations in vorticity-velocity variables which is equivalent to the usual primitive variables formulation.

In that formulation, the vorticity ω is supplemented with the following two boundary conditions:

$$\operatorname{div} \omega = \nabla.\omega = 0 \quad \text{on } \partial\Omega, \qquad n.\omega = n.\nabla_S \times b \quad \text{on } \partial\Omega,$$

where b is the velocity boundary condition and the relation $n.\omega = n.\nabla_S \times b$ on $\partial\Omega$ is the component of the equation $\omega = \nabla \times u$ normal to the boundary.

The author in [1] notes that these conditions cannot be apparently imposed simultaneously in the framework of a variational formulation, hence, a disadvantage of the vorticity-velocity formulation.

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In this article, we propose a mixed method for the vorticity-velocity formulation of the stationary Navier-Stokes equations in the three-dimensional case in which these boundaries conditions are indeed simultaneously satisfied. We show the existence and uniqueness of solution of the variational problem, and prove for the steady-state case, the equivalence of this formulation with the Navier-Stokes equations (1).

2. VARIATIONAL FORMULATION

Let Ω be a bounded, simply-connected domain in $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let us consider the stationary incompressible Navier-Stokes equations in primitive variables:

$$\begin{aligned} -\nu\Delta u + (u.\nabla)\,u + \text{grad}\,p &= \overline{f} && \text{in } \Omega,\\ \text{div}\,u &= 0 && \text{in } \Omega,\\ u &= b && \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The vorticity-velocity formulation can be used to rewrite the system (1). The vorticity ω is defined by $\omega = \nabla \times u = \text{curl}\ u$. The equation of motion then reads:

$$\begin{aligned} -\nu\Delta\omega + \nabla \times (\omega \times u) = f &= \text{curl}\,\overline{f} && \text{in } \Omega,\\ -\Delta u &= \nabla \times \omega && \text{in } \Omega,\\ u &= b && \text{on } \partial\Omega,\\ \text{div}\,u = \nabla.u &= 0 && \text{on } \partial\Omega,\\ \text{div}\,\omega = \nabla.\omega &= 0 && \text{on } \partial\Omega,\\ n.\omega &= n.\nabla_S \times b && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

assuming the following conditions on the data: $\int_{\partial\Omega} b.n\,ds = 0$.

We consider the following spaces.

Hereafter, F denoting any vector space, we denote by $\mathbb{F}$ the space $F \times F \times F$. Let $\Psi = \{\phi \in \mathbb{H}^1(\Omega),\ \nabla \times \phi \in \mathbb{H}^1_0(\Omega), \text{div}\,\phi = 0,\ \phi \times n = 0 \text{ on } \partial\Omega, \int_{\partial\Omega} \phi.n\,ds = 0\}$. The definition of the space Ψ is motivated by the properties of the curl operator given in [4, Appendix I], and [2, Theorem 3.6]. $X_1 = \{\theta \in \mathbb{L}^2(\Omega), \exists\,\psi \in \Psi, -\Delta\psi = \theta\}$, $X = \mathbb{H}^1_0(\Omega) \times X_1$, $M = \mathbb{H}^1_0(\Omega)$. We suppose that $\overline{f} \in \mathbb{H}^{-1}(\Omega)$. Since X_1 is a closed subspace of $\mathbb{L}^2(\Omega)$, X is a closed subspace of $\mathbb{H}^1_0(\Omega) \times \mathbb{L}^2(\Omega)$.

We consider the following bilinear forms $a(.,.)$, and $b(.,.)$ defined respectively on $X \times X$ and $X \times M$ by: for $\overline{u} = (u,\omega)$, $\overline{v} = (v,\theta) \in X$, $\lambda \in M$,

$$a(\overline{u},\overline{v}) = \nu(\omega,\theta), \qquad b(\overline{u},\lambda) = (-\Delta u - \nabla \times \omega, \lambda),$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and $(.,.)$ is the scalar product in $\mathbb{L}^2(\Omega)$.

For the nonlinearity, we consider the trilinear form $j : X \times X \times X \longrightarrow \mathbb{R}$ defined by: for $\overline{u} = (u,\phi)$, $\overline{v} = (v,\theta)$, $\overline{w} = (\omega,\tau) \in X$,

$$j(\overline{u},\overline{v},\overline{w}) = \int_\Omega (\phi \times v).\omega\,\mathrm{d}x.$$

The trilinear form j is well defined since $\phi \in L^2(\Omega)$, $v \in \mathbb{H}^1_0(\Omega) \subset \mathbb{L}^6(\Omega)$, $\omega \in \mathbb{H}^1_0(\Omega) \subset \mathbb{L}^3(\Omega)$. Moreover, we have $j(\overline{u},\overline{v},\overline{v}) = 0\ \forall\,\overline{u},\overline{v} \in X$. We consider also the trilinear form a_1 defined on $X \times X \times X$ by:

$$a_1\left(\overline{u},\overline{v},\overline{w}\right) = j\left(\overline{u},\overline{v},\overline{w}\right) + a\left(\overline{v},\overline{w}\right), \qquad \forall\,\overline{u},\overline{v},\overline{w} \in X.$$

Finally, let

$$\begin{aligned} V = V(0) &= \{\overline{u} = (u,\omega) \in X,\ b(\overline{u},\mu) = 0, \forall\,\mu \in M = \mathbb{H}^1_0(\Omega)\}\\ &= \{\overline{u} = (u,\omega) \in X,\ -\Delta u = \nabla \times \omega\}. \end{aligned}$$

PROPOSITION 1. *For all* $\overline{u} = (u, \omega) \in V \cap (\mathbb{H}^2(\Omega))^2$, *we have*

$$\begin{aligned} \text{div } u = \nabla.u &= 0 && \text{on } \partial\Omega, \\ \text{div } \omega = \nabla.\omega &= 0 && \text{on } \partial\Omega, \\ n.\omega &= n.\nabla_S \times \gamma_0(u) && \text{on } \partial\Omega, \end{aligned}$$

where $\gamma_0(u)$ *is the trace operator defined by* $\gamma_0(u) = u|_{\partial\Gamma}$.

We consider the following problem, called problem (Q_1):

$$\begin{aligned} &\text{To find } (\overline{u}, \lambda) \in X \times M \text{ such that,} \\ &a_1(\overline{u}, \overline{u}, \overline{v}) + b(\overline{v}, \lambda) = \langle g, \overline{v} \rangle \qquad \forall \overline{v} \in X, \\ &b(\overline{u}, \mu) = 0 \qquad \forall \mu \in M, \end{aligned} \tag{Q_1}$$

where $g = (\overline{f}, 0) = ((\overline{f}_1, \overline{f}_2, \overline{f}_3), 0) \in X'$. To (Q_1), we associate the problem:

$$\begin{aligned} &\text{To find } \overline{u} \in V \text{ such that,} \\ &a_1(\overline{u}, \overline{u}, \overline{v}) = \langle g, \overline{v} \rangle \qquad \forall \overline{v} \in V. \end{aligned} \tag{P_1}$$

THEOREM 2. *Problem* (P_1) *has at least one solution* $\overline{u}$. *Moreover, there exists* $\nu_0 = \nu_0(\Omega, g)$ *such that for* $\nu > \nu_0$, *problem* (P_1) *has a unique solution* $\overline{u} = (u, \omega)$, *which is exactly the solution of* (2).

THEOREM 3. *For any solution* $\overline{u}$ *of* (P_1), *there exists* $\lambda \in M$ *such that* $(\overline{u}, \lambda)$ *is solution of* (Q_1). *Moreover, we have the relation* $(0, 0, \nu\,\omega) = (0, 0, \nabla \times \lambda)$ *in* X'.

The details of the proof will appear in [6]. The following section is devoted to some indications on the proof.

3. SKETCH OF THE PROOFS

The proofs of the previous results follow from the framework given in [2] provided we check that all hypotheses hold. The following properties are easy to check:

p_1: The bilinear form $a(.,.)$ is V-elliptic.

p_2: The bilinear form b satisfies the inf-sup condition; that is

$$\exists\, \beta > 0, \qquad \text{such that } \forall\, \mu \in M, \qquad \sup_{v \in X - \{0\}} \frac{b(v, \mu)}{\|v\|_X} \geq \beta \|\mu\|_M.$$

p_3: $\exists\, \alpha > 0$ such that $a_1(v, v, v) \geq \alpha \|v\|_X^2$ $\forall\, v \in V$.

p_4: The space V is separable.

p_5: $\forall\, v \in V$, the mapping $u \longmapsto a_1(u, u, v)$ is sequentially weakly continuous on V which means:

$$u_n \rightharpoonup u \text{ in } V, \text{ implies } \lim_{n \to \infty} a_1(u_n, u_n, v) = a_1(u, u, v) \qquad \forall\, v \in V.$$

p_6: The bilinear form $a_1(\omega, ., .)$ is uniformly *V-elliptic* with respect to ω; i.e., $\exists \alpha' > 0$ such that

$$a_1(\omega, v, v) \geq \alpha' \|v\|_X^2 \qquad \forall\, v, \omega \in V.$$

p_7: The mapping $\omega \longmapsto \pi A_1(\omega)$ is locally Lipschitz-continuous in V, which means that there exists a continuous and monotocally increasing function $L : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that:

$$\begin{gathered} \forall\, \mu > 0, \qquad |a_1(\omega_1, u, v) - a_1(\omega_2, u, v)| \leq L(\mu) \times \|u\|_X \times \|v\|_X \times \|\omega_1 - \omega_2\|_X, \\ \forall\, u, v \in V, \quad \forall\, \omega_1, \omega_2 \in \mathcal{S}(\mu), \quad \text{where } \mathcal{S}(\mu) = \{\omega \in V \text{ such that } \|\omega\|_X \leq \mu\}. \end{gathered}$$

Theorem 2 follows from properties $p_1, p_2, p_3, p_4, p_5, p_6, p_7$, and the general framework given in [2].

The first part of Theorem 3 follows from the property p_2, and the second from the variational problem (Q_1), by taking the particular case of $\overline{v} = (0, \theta) \in X$.

We give hereafter the interpretation of (P_1) in the homogeneous case. Let us suppose $u = 0$ on $\partial\Omega$. Let (u, ω) be the solution of (2). Let us prove that $\overline{u} = (u, \omega)$ is a solution of (P_1). We have $\omega = \nabla \times u$, and div $u = 0$, since (2) is equivalent to the primitive variables formulation (1). Moreover, $\exists\, \psi \in \Psi$ such that $u = \nabla \times \psi = \text{curl}\, \psi$, div $\psi = 0$. Since $\omega = \nabla \times u$, we have $-\Delta\, \psi = \omega$ and $-\Delta u = \nabla \times \omega$; which means $\overline{u} = (u, \omega) \in V$. Since (1) and (2) are equivalent, there exists p such that

$$-\nu\Delta u + (u.\bigtriangledown)u + \text{grad}\; p = \overline{f}.$$

The nonlinear term is expressed in the form

$$(u.\bigtriangledown)u = (\nabla \times u) \times u + \text{grad}\left(\frac{1}{2}u^2\right).$$

Then u satisfies:

$$\nu\nabla \times \nabla \times u + (\nabla \times u) \times u + \text{grad}\left(\frac{1}{2}u^2\right) = \overline{f}.$$

Multiplying this equality by $\nabla \times \varphi$ for $\varphi \in \Psi$, we obtain

$$\nu(\nabla \times \nabla \times \nabla \times \psi, \nabla \times \varphi) + ((\nabla \times u) \times u, \nabla \times \varphi) = \left(\overline{f}, \nabla \times \varphi\right),$$

which gives

$$\nu(\omega, \theta) + (\omega \times u, \nabla \times \varphi) = \left(\overline{f}, \nabla \times \varphi\right),$$

which means that

$$a\left(\overline{u}, \overline{v}\right) + j\left(\overline{u}, \overline{u}, \overline{v}\right) = \left(g, \overline{v}\right) \quad \forall \overline{v} \in V,$$

since for $\overline{v} = (v, \theta) \in V$. Let now $\varphi \in \Psi$ such that $-\Delta\varphi = \theta$; then we have

$$-\nu(\omega, \Delta\varphi) = \nu(\omega, \theta) = a\left(\overline{u}, \overline{v}\right), \qquad \langle \overline{f}, \nabla \times \varphi\rangle = \langle g, \overline{v}\rangle,$$

$$\langle(\omega \times u), \nabla \times \varphi\rangle = \langle(\omega \times u), v\rangle = j\left(\overline{u}, \overline{u}, \overline{v}\right),$$

which proves that $\overline{u}$ is a solution of (P_1). We conclude that if $\nu > \nu_0$, (P_1) is equivalent to (2). The nonhomogeneous case is quite similar; for more details, the reader is referred to [5] in which we also consider the three-dimensional nonstationary Stokes equations.

We have similar results in the two-dimensional case. In this case, the space Ψ is replaced by:

$$\Psi_1 = \left\{\psi \in L^2(\Omega),\ \nabla \times \psi \in H_0^1(\Omega) \times H_0^1(\Omega),\ \psi = 0 \text{ on } \partial\Omega\right\}.$$

The reader is referred to [6] for a variational formulation for the full two-dimensional Navier-Stokes equations in the vorticity-velocity variables.

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