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On new inequalities of Simpson's type for s-convex functions

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ABSTRACT

In this paper, we establish some new inequalities of Simpson's type based on s-convexity. Some applications to special means of real numbers are also given.

1. Introduction

The following inequality, named Simpson's inequality, is one of the best known results in the literature.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 .$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [1–3].

In [2], Dragomir et al. proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 2. *Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $f' \in L[a, b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{3}\left\|f'\right\|_1 \tag{1.1}$$

where $\left\|f'\right\|_1 = \int_a^b \left|f'(x)\right| \mathrm{d}x$.

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by $\frac{5}{36}L(b-a)$.

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Also, the following inequality was obtained in [2].

Theorem 3. *Suppose $f : [a, b] \to \mathbb{R}$ is an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p \tag{1.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [4], Sarikaya et al. obtained inequalities for differentiable convex mappings which are connected with Simpson's inequality, and they used the following lemma to prove this.

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$=\frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]\mathrm{d}t. \tag{1.3}$$

The main inequality in [4], pointed out, is as follows.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right\}, \tag{1.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [5], some inequalities of the Hermite–Hadamard type for differentiable convex mappings were presented as follows.

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|$ is convex on $[a, b]$, then the following equality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)}{4}\left[\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]. \tag{1.5}$$

Definition 1 (*[6]*). Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex (in the second sense), or f belongs to the class K_s^2, if

$$f(\alpha x+(1-\alpha)y) \le \alpha^s f(x)+(1-\alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

An s-convex function was introduced in Breckner's paper [6] and a number of properties and connections with s-convexity in the first sense are discussed in paper [7]. Of course, convexity means just s-convexity when $s = 1$.

In [8], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 6 (*[8]*). *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{s+1}. \tag{1.6}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.6).

For recent results, refinement, generalizations and new Hermite–Hadamard type inequalities see [8–10].

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are s-convex functions.

2. Main results

The following theorems give a new result of Simpson's inequality for s-convex functions.

Theorem 7. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq (b-a)\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.1}$$

Proof. From Lemma 1 and since $|f'|$ is s-convex on $[a,b]$, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leq \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]\mathrm{d}t \\
&\leq \frac{b-a}{2}\int_0^1\left(\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|+\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|+\left(\frac{1+t}{2}\right)^s\left|f'(a)\right|+\left(\frac{1-t}{2}\right)^s\left|f'(b)\right|\right]\right)\mathrm{d}t \\
&= \frac{b-a}{2^{s+1}}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[(1+t)^s+(1-t)^s\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\mathrm{d}t.
\end{aligned} \tag{2.2}$$

It is easy to observe that

$$\begin{aligned}
&\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[(1+t)^s+(1-t)^s\right] \\
&= \int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)\left[(1+t)^s+(1-t)^s\right]\mathrm{d}t+\int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)\left[(1+t)^s+(1-t)^s\right]\mathrm{d}t \\
&= J_1+J_2.
\end{aligned} \tag{2.3}$$

By simple computation,

$$\begin{aligned}
J_1 &= \int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)\left[(1+t)^s+(1-t)^s\right]\mathrm{d}t \\
&= \int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)(1+t)^s\,\mathrm{d}t+\int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)(1-t)^s\,\mathrm{d}t \\
&= \left(\frac{5\,(1+t)^{s+1}}{6(s+1)}-\frac{(1+t)^{s+2}}{2(s+2)}\right)\Bigg|_0^{\frac{2}{3}}+\left(\frac{(1-t)^{s+1}}{6(s+1)}-\frac{(1-t)^{s+2}}{2(s+2)}\right)\Bigg|_0^{\frac{2}{3}} \\
&= \frac{5^{s+2}-2\times 3^{s+2}+1}{2\times 3^{s+2}(s+1)(s+2)},
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
J_2 &= \int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)\left[(1+t)^s+(1-t)^s\right]\mathrm{d}t \\
&= \int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)(1+t)^s\,\mathrm{d}t+\int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)(1-t)^s\,\mathrm{d}t \\
&= \left(\frac{(1+t)^{s+2}}{2(s+2)}-\frac{5\,(1+t)^{s+1}}{6(s+1)}\right)\Bigg|_{\frac{2}{3}}^1+\left(-\frac{(1-t)^{s+1}}{6(s+1)}+\frac{(1-t)^{s+2}}{2(s+2)}\right)\Bigg|_{\frac{2}{3}}^1 \\
&= \frac{(s-4)6^{s+1}+5^{s+2}+1}{2\times 3^{s+2}(s+1)(s+2)}.
\end{aligned} \tag{2.5}$$

Using (2.4) and (2.5) in (2.3) and the above observations in (2.2), we get (2.1) which completes the proof. □

Corollary 1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{5(b-a)}{72}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].$$

Corollary 2. *In Corollary 1, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \le \frac{5(b-a)}{72}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{2.6}$$

Corollary 1 was proved by Sarikaya et al. in [4]. Hence, our results in Theorem 7 are generalizations of the corresponding results of Sarikaya et al. in [4].

Remark 1. We note that the obtained midpoint inequality (2.6) is better than the inequality (1.5).

Remark 2. We note that the obtained midpoint inequality (2.1) is the same midpoint in Theorem 5 [1].

In the following theorem, we shall propose a new upper bound for the right-hand side of Simpson's inequality for s-convex mapping, which is better than the inequality obtained in Theorem 6 [1].

Theorem 8. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds,*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\quad\le \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q + |f'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{2.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and by Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\quad\le \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2} - \frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right| + \left|\frac{1}{3} - \frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\right]\mathrm{d}t \\ &\quad\le \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2} - \frac{1}{3}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\ &\quad\quad\left. + \left(\int_0^1\left|\frac{1}{3} - \frac{t}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|f'|^q$ is s-convex on $[a, b]$, by using in (1.6), we get

$$\int_0^1\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^q\mathrm{d}t \le \frac{|f'(b)|^q + |f'(\frac{a+b}{2})|^q}{s+1},$$

and

$$\int_0^1\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q\mathrm{d}t \le \frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{s+1}.$$

Hence

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\quad\le \frac{b-a}{2}\left(\int_0^1\left|\frac{t}{2} - \frac{1}{3}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q + |f'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

By simple computation,

$$\int_0^1 \left|\frac{t}{2}-\frac{1}{3}\right|^p = \int_0^{\frac{2}{3}} \left(\frac{1}{3}-\frac{t}{2}\right)^p \mathrm{d}t + \int_{\frac{2}{3}}^1 \left(\frac{t}{2}-\frac{1}{3}\right)^p \mathrm{d}t = \frac{2(1+2^{p+1})}{6^{p+1}(p+1)}.$$

Thus, we get (2.7) which completes the proof. □

Corollary 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4. *In Corollary 3, if $f'(\frac{a+b}{2})=0$, then we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}\left(\frac{\left|f'(b)\right|+\left|f'(a)\right|}{2}\right).$$

Remark 3. We note that the obtained midpoint inequality (2.8) is better than the inequality (1.5).

Corollary 5. *In Corollary 4, if $f(a)=f(\frac{a+b}{2})=f(b)$ and $p=q=2$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{6\sqrt{2}}\left(\frac{\left|f'(b)\right|+\left|f'(a)\right|}{2}\right). \tag{2.8}$$

Corollary 6. *In Theorem 8, if $f(a)=f(\frac{a+b}{2})=f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

We shall give another version of Simpson's type inequality for s-convex functions as follows.

Theorem 9. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{(2^{s+1}-1)\left|f'(b)\right|^q+\left|f'(a)\right|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right\}, \tag{2.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and by Hölder's integral inequality, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]\mathrm{d}t$$

$$\leq \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $|f'|^q$ is s-convex on $[a,b]$, we know that for $t\in[0,1]$ and $s\in(0,1]$

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q\leq\left(\frac{1+t}{2}\right)^s|f'(b)|^q+\left(\frac{1-t}{2}\right)^s|f'(a)|^q.$$

Hence

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1+t}{2}\right)^s|f'(b)|^q+\left(\frac{1-t}{2}\right)^s|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1+t}{2}\right)^s|f'(a)|^q+\left(\frac{1-t}{2}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$=\frac{b-a}{2}\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{(2^{s+1}-1)|f'(b)|^q+|f'(a)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right\},$$

where we have used the facts that

$$\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^p=\int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{t}{2}\right)^p dt+\int_{\frac{2}{3}}^1\left(\frac{t}{2}-\frac{1}{3}\right)^p dt=\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}.$$

This completes the proof. □

Remark 4. If we take $s=1$ in (2.9), then we obtain (1.4).

Theorem 10. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q+\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q+\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q\right)^{\frac{1}{q}}\right\}. \tag{2.10}$$

Proof. From Lemma 1 and by the power mean inequality, we get

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{2}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|+\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt$$
$$\leq\frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\Bigg\}.$$

Since $\left|f'\right|^q$ is s-convex on $[a,b]$, we know that for $t\in[0,1]$ and $s\in(0,1]$

$$\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q\le\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|^q+\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|^q.$$

Hence

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{2}\left\{\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s\left|f'(b)\right|^q+\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|\left[\left(\frac{1+t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{1-t}{2}\right)^s\left|f'(b)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right\}$$
$$=\frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2\times5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s+1}(s+1)(s+2)}\left|f'(b)\right|^q+\frac{(2s+1)3^{s+1}+2}{3\times6^{s+1}(s+1)(s+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2s+1)3^{s+1}+2}{3\times6^{s+1}(s+1)(s+2)}\left|f'(b)\right|^q+\frac{2\times5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s+1}(s+1)(s+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Remark 5. In Theorem 10, we take $q=1$; then Theorem 10 reduces to Theorem 7.

Corollary 7. *In Theorem 10, if $f(a)=f(\frac{a+b}{2})=f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{b-a}{72}(5)^{1-\frac{1}{q}}\left\{\left(\frac{2\times5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s-1}(s+1)(s+2)}\left|f'(b)\right|^q+\frac{(2s+1)3^{s+1}+2}{3\times6^{s-1}(s+1)(s+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2s+1)3^{s+1}+2}{3\times6^{s-1}(s+1)(s+2)}\left|f'(b)\right|^q+\frac{2\times5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s-1}(s+1)(s+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right\}.$$

3. Applications to special means

In [1], the following result is given.

Let $g:I\to I_1\subseteq[0,\infty)$ be a non-negative convex function on I. Then $g^s(x)$ is s-convex on I, $0<s<1$.

For arbitrary positive real numbers a,b $(a\ne b)$, we shall consider the following special means.

(1) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\ge0.$$

(2) The logarithmic mean:

$$L=L(a,b):=\begin{cases}a & \text{if } a=b\\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\ne b\end{cases},\quad a,b>0.$$

(3) The p-logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\ne b\\ a & \text{if } a=b\end{cases},\quad p\in\mathbb{R}\setminus\{-1,0\};\ a,b>0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality

$$L \leq A.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

(1) Let $f : [a, b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^s$, $s \in (0, 1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = L_s^s(a, b),$$
$$\frac{f(a)+f(b)}{2} = A(a^s, b^s),$$
$$f\left(\frac{a+b}{2}\right) = A^s(a, b).$$

(a) From Theorem 7, we obtain

$$\left|\frac{1}{3}A\left(a^s, b^s\right) + \frac{2}{3}A^s(a, b) - L_s^s(a, b)\right| \leq 2s(b-a)\,\frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}A(a^{s-1}, b^{s-1}).$$

For instance, if $s = 1$ then we get

$$|A(a, b) - L(a, b)| \leq \frac{5}{36}(b-a).$$

(b) From Theorem 8, we have

$$\left|\frac{1}{3}A\left(a^s, b^s\right) + \frac{2}{3}A^s(a, b) - L_s^s(a, b)\right| \leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}}}$$
$$\times\left\{\left(b^{q(s-1)} + [A(a, b)]^{q(s-1)}\right)^{\frac{1}{q}} + \left(a^{q(s-1)} + [A(a, b)]^{q(s-1)}\right)^{\frac{1}{q}}\right\},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$|A(a, b) - L(a, b)| \leq \frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}, \quad p > 1.$$

(c) From Theorem 9, we get

$$\left|\frac{1}{3}A\left(a^s, b^s\right) + \frac{2}{3}A^s(a, b) - L_s^s(a, b)\right| \leq \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\frac{s}{2^{s/q}(s+1)^{\frac{1}{q}}}$$
$$\times\left\{\left(\left(2^{s+1}-1\right)b^{q(s-1)} + a^{q(s-1)}\right)^{\frac{1}{q}} + \left(\left(2^{s+1}-1\right)a^{q(s-1)} + b^{q(s-1)}\right)^{\frac{1}{q}}\right\},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$|A(a, b) - L(a, b)| \leq \frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}, \quad p > 1.$$

(2) Let $f : [a, b] \subseteq [0, \infty) \to \mathbb{R}$, $(0 < a < b)$, $f(x) = \frac{1}{x^s} \in K_s^2$, $s \in (0, 1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = L_{-s}^{-s}(a, b),$$
$$\frac{f(a)+f(b)}{2} = A(a^{-s}, b^{-s}),$$
$$f\left(\frac{a+b}{2}\right) = A^{-s}(a, b).$$

(a) From Theorem 7, we obtain

$$\left|\frac{1}{3}A\left(a^{-s}, b^{-s}\right) + \frac{2}{3}A^{-s}(a, b) - L_{-s}^{-s}(a, b)\right| \leq 2s(b-a)\,\frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}A(a^{-s-1}, b^{-s-1}).$$

For instance, if $s = 1$ then we get

$$\left|\frac{1}{3}A\left(a^{-1}, b^{-1}\right) + \frac{2}{3}A^{-1}(a, b) - L_{-1}^{-1}(a, b)\right| \le \frac{5}{36}(b-a)A(a^{-2}, b^{-2}).$$

(b) From Theorem 8, we have

$$\left|\frac{1}{3}A\left(a^{-s}, b^{-s}\right) + \frac{2}{3}A^{-s}(a, b) - L_{-s}^{-s}(a, b)\right|$$
$$\le \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \frac{s}{(s+1)^{\frac{1}{q}}}\left\{\left(b^{-q(s+1)} + [A(a, b)]^{-q(s+1)}\right)^{\frac{1}{q}} + \left(a^{-q(s+1)} + [A(a, b)]^{-q(s+1)}\right)^{\frac{1}{q}}\right\},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$\left|\frac{1}{3}A\left(a^{-1}, b^{-1}\right) + \frac{2}{3}A^{-1}(a, b) - L_{-1}^{-1}(a, b)\right|$$
$$\le \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \frac{1}{2^{\frac{1}{q}}}\left\{\left(b^{-2q} + [A(a, b)]^{-2q}\right)^{\frac{1}{q}} + \left(a^{-2q} + [A(a, b)]^{-2q}\right)^{\frac{1}{q}}\right\},$$

where $p > 1$.

(c) From Theorem 9, we get

$$\left|\frac{1}{3}A\left(a^{-s}, b^{-s}\right) + \frac{2}{3}A^{-s}(a, b) - L_{-s}^{-s}(a, b)\right|$$
$$\le \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \frac{s}{2^{\frac{s-1}{q}}(s+1)^{\frac{1}{q}}}$$
$$\times\left\{\left[A\left((2^{s+1}-1)b^{-q(s+1)}, a^{-q(s+1)}\right)\right]^{\frac{1}{q}} + \left[A\left((2^{s+1}-1)a^{-q(s+1)}, b^{-q(s+1)}\right)\right]^{\frac{1}{q}}\right\},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$\left|\frac{1}{3}A\left(a^{-1}, b^{-1}\right) + \frac{2}{3}A^{-1}(a, b) - L_{-1}^{-1}(a, b)\right|$$
$$\le \frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3b^{-2q}+a^{-2q}}{4}\right)^{\frac{1}{q}} + \left(\frac{3a^{-2q}+b^{-2q}}{4}\right)^{\frac{1}{q}}\right\},$$

where $p > 1$.

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