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Indagationes Mathematicae 23 (2012) 256–268
indagationes mathematicae
www.elsevier.com/locate/indag

Polynomial maps with invertible sums of Jacobian matrices and directional derivatives☆

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Received 17 June 2011; received in revised form 18 October 2011; accepted 30 November 2011

Communicated by J. Top

Abstract

Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map with $\deg F = d \geq 2$. We prove that F is invertible if $m = n$ and $\sum_{i=1}^{d-1} (\mathcal{J}F)|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$, which is trivially the case for invertible quadratic maps.

More generally, we prove that for affine lines $L = \{\beta + \mu\gamma \mid \mu \in \mathbb{C}\} \subseteq \mathbb{C}^n$ ($\gamma \neq 0$), $F|_L$ is linearly rectifiable, if and only if $\sum_{i=1}^{d-1} (\mathcal{J}F)|_{\alpha_i} \cdot \gamma \neq 0$ for all $\alpha_i \in L$. This appears to be the case for all affine lines L when F is injective and $d \leq 3$.

We also prove that if $m = n$ and $\sum_{i=1}^{n} (\mathcal{J}F)|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$, then F is a composition of an invertible linear map and an invertible polynomial map $X + H$ with linear part X, such that the subspace generated by $\{(\mathcal{J}H)|_\alpha \mid \alpha \in \mathbb{C}^n\}$ consists of nilpotent matrices.

Keywords: Jacobian matrix; Jacobian Conjecture; Polynomial embedding; Linearly rectifiable

1. Introduction

Denote by $\mathcal{J}F$ the Jacobian matrix of a polynomial map $F: \mathbb{C}^n \to \mathbb{C}^n$. The Jacobian Conjecture states that F is invertible if $\mathcal{J}F$ is invertible, or equivalently if $(\mathcal{J}F)|_\alpha$ is invertible

☆ Supported by NSF of China (No. 11071097, No. 11026039) and "211 Project" and "985 Project" of Jilin University.
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0019-3577/$ - see front matter
doi:10.1016/j.indag.2011.11.007

for all $\alpha \in \mathbb{C}^n$. The conjecture has been reduced to polynomial maps of the form $F = X + H$, where H is homogeneous (of degree 3) and $\mathcal{J}H$ is nilpotent, by Bass et al. [1], and independently by Yagzhev in [13]. Subsequent reductions are to the case where for the polynomial map $F = X + H$ above, each component of H is a cube of a linear form, by Drużkowski in [7], and to the case where $\mathcal{J}H$ is symmetric, by de Bondt and van den Essen in [2], but these reductions cannot be applied simultaneously; see also [3]. More details about the Jacobian Conjecture can be found in [8,4].

Invertibility of a polynomial map F has been examined by several authors under certain conditions on the evaluated Jacobian matrices $(\mathcal{J}F)|_\alpha, \alpha \in \mathbb{C}^n$. With an extra assumption that $F - X$ is cubic homogeneous, Yagzhev proved in [13] that if $(\mathcal{J}F)|_{\alpha_1} + (\mathcal{J}F)|_{\alpha_2}$ is invertible for all $\alpha_1, \alpha_2 \in \mathbb{C}^n$, then the polynomial map F is invertible. The Jacobian matrix $\mathcal{J}H$ of a polynomial map H is called *strongly nilpotent* if $(\mathcal{J}H)|_{\alpha_1} \cdot (\mathcal{J}H)|_{\alpha_2} \cdot \cdots \cdot (\mathcal{J}H)|_{\alpha_n} = 0$ for all $\alpha_i \in \mathbb{C}^n$. Van den Essen and Hubbers proved in [9] that $\mathcal{J}H$ is strongly nilpotent if and only if there exists $T \in GL_n(\mathbb{C})$ such that $T^{-1}\mathcal{J}(H)T$ is strictly upper triangular, if and only if the polynomial map $F = X + H$ is linearly triangularizable (so F is invertible). This result was generalized by Yu in [14], where he additionally observed that $\mathcal{J}H$ is already strongly nilpotent if $(\mathcal{J}H)|_{\alpha_1} \cdot (\mathcal{J}H)|_{\alpha_2} \cdot \cdots \cdot (\mathcal{J}H)|_{\alpha_m} = 0$ for some $m \in \mathbb{N}$.

In [11] Sun introduced the notion of additive-nilpotency to extend that of strong nilpotency. The Jacobian matrix $\mathcal{J}H$ of a polynomial map H is *additive-nilpotent*, if $\sum_{i=1}^m (\mathcal{J}H)|_{\alpha_i}$ is nilpotent for each positive integer m and all $\alpha_i \in \mathbb{C}^n$. By expanding $(\sum_{i=1}^m (\mathcal{J}H)|_{\alpha_i})^n$, one can see that strong nilpotency implies additive-nilpotency. Sun proved that a polynomial map $F = X + H$ is invertible if the Jacobian matrix $\mathcal{J}H$ is additive-nilpotent, which generalizes results in [9,12–14]. In case $F = X + H$ with JH additive-nilpotent, we have that $\sum_{i=1}^m (\mathcal{J}F)|_{\alpha_i} = mI_n + \sum_{i=1}^m (\mathcal{J}H)|_{\alpha_i}$ is invertible for all positive integers m and all $\alpha_i \in \mathbb{C}^n$. Therefore, instead of looking at a polynomial map $F = X + H$ such that $\mathcal{J}H$ is nilpotent, we look at a polynomial map F in general, and assume that $\det \sum_{i=1}^{d-1} (\mathcal{J}F)|_{\alpha_i} \neq 0$ for all $\alpha_i \in \mathbb{C}^n$, where $d = \deg F$.

More generally, we consider a polynomial map $F: \mathbb{C}^n \to \mathbb{C}^m$ and assume that $\sum_{i=1}^{d-1} (\mathcal{J}F)|_{\alpha_i} \cdot \gamma \neq 0$ only holds for $\alpha_i \in \mathbb{C}^n$ which are contained in a certain line, where $\gamma \neq 0$ is the direction of that line, in order to prove that F is injective on that line. In the particular case that $m = n$ and the assumption holds for all lines in $\mathbb{C}^n$, F is injective and hence invertible. This generalizes results of Wang in [12], Yagzhev in [13], van den Essen and Hubbers in [9] and Sun in [11].

Observe that if $F = X + H$ is a polynomial map such that $\mathcal{J}H$ is additive-nilpotent, then $\sum_{i=1}^m (\mathcal{J}\tilde{F})|_{\alpha_i}$ is invertible for all $m \in \mathbb{N}$ and all $\alpha_i \in \mathbb{C}^n$, where $\tilde{F} = L_1 \circ F \circ L_2$ is a composition of F and invertible linear maps L_1 and L_2. Conversely, it is interesting to describe the polynomial maps such that sums of the evaluated Jacobian matrices are invertible. We prove the invertibility of a polynomial map $F: \mathbb{C}^n \to \mathbb{C}^n$ such that $\sum_{i=1}^n (\mathcal{J}F)|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$, and characterize such a polynomial map as a composition of an invertible linear map and an invertible polynomial map $X + H$ such that $\mathcal{J}H$ is additive-nilpotent.

2. Additive properties of the derivative on lines

Let $F|_G$ denote substituting X by G in F.

Lemma 2.1. *Assume $\lambda_1, \lambda_2, \ldots, \lambda_{d-1} \in \mathbb{C}$ such that $\sum_{i \in I} \lambda_i \neq 0$ for all nonempty $I \subseteq \{1, 2, \ldots, d-1\}$, and $P \in \mathbb{C}[[T]]$ with constant term $\lambda_1 + \lambda_2 + \cdots + \lambda_{d-1}$. Then there are*

$r_1, r_2, \ldots, r_{d-1} \in \mathbb{C}$ such that

$$P - \sum_{i=1}^{d-1} \lambda_i \exp(r_i T)$$

is divisible by T^d, where $\exp(T) = \sum_{j=0}^{\infty} \frac{1}{j!} T^j$.

Proof. Write

$$P = \sum_{j=0}^{\infty} \frac{p_j}{j!} T^j.$$

Then we must find a solution $(Y_1, Y_2, \ldots, Y_{d-1}) = (r_1, r_2, \ldots, r_{d-1}) \in \mathbb{C}^{d-1}$ of

$$\sum_{i=1}^{d-1} \lambda_i Y_i^j = p_j \quad (j = 0, 1, \ldots, d-1). \tag{2.1}$$

The equation for $j = 0$ is fulfilled by assumption, and finding a solution of (2.1) is the same as finding a solution $(Y_1, Y_2, \ldots, Y_d) = (r_1, r_2, \ldots, r_d)$ of

$$\sum_{i=1}^{d-1} \lambda_i Y_i^j = p_j Y_d^j \quad (j = 1, \ldots, d-1) \tag{2.2}$$

for which $r_d = 1$. Since $(Y_1, Y_2, \ldots, Y_d) = 0$ is a solution of (2.2), it follows from Krull's Height Theorem that the dimension of the set of solutions $(r_1, r_2, \ldots, r_d) \in \mathbb{C}^d$ of (2.2) is at least one. Hence there exists a nonzero solution $(r_1, r_2, \ldots, r_d) \in \mathbb{C}^d$ of (2.2).

If $r_d \neq 0$, then $r_d^{-1}(r_1, r_2, \ldots, r_d)$ is a solution of (2.2) as well, because the equations of (2.2) are homogeneous. Hence $r_d^{-1}(r_1, r_2, \ldots, r_{d-1})$ is a solution of (2.1) in that case. So assume that $r_d = 0$. Then $\sum_{i=1}^{d-1} \lambda_i r_i^j = 0$ for all j. Take $e \leq d-1$ and nonzero $s_1 < s_2 < \cdots < s_e$ such that $\{0, r_1, r_2, \ldots, r_{d-1}\} = \{0, s_1, s_2, \ldots, s_e\}$. Then $e \geq 1$ because $(r_1, r_2, \ldots, r_d) \neq 0$, and

$$0 = \sum_{i=1}^{d-1} \lambda_i r_i^j = \sum_{k=1}^{e} s_k^j \sum_{r_i = s_k} \lambda_i$$

for all j such that $1 \leq j \leq e$. This means that the vector v defined by $v_k := \sum_{r_i = s_k} \lambda_i$ for all k satisfies $Mv = 0$, where M is the Vandermonde matrix with entries $M_{jk} = s_k^j$. Since v_k is nonzero for all k by assumption, this contradicts $\det M \neq 0$. $\square$

Let $f \in \mathbb{C}[X] = \mathbb{C}[X_1, X_2, \ldots, X_n]$ be a polynomial of degree d and $\beta, \gamma \in \mathbb{C}^n$. Set $g(T) := f(\beta + T\gamma)$ and $D := \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial X_i}$. Notice that $T \mapsto D$ induces an isomorphism of $\mathbb{C}[T]$ and $\mathbb{C}[D]$. By the chain rule,

$$\begin{aligned} \frac{d^i}{dT^i}\big(f(\beta + T\gamma)\big) &= \frac{d^{i-1}}{dT^{i-1}}\big((\mathcal{J} f)|_{\beta+T\gamma} \cdot \gamma\big) \\ &= \frac{d^{i-1}}{dT^{i-1}}\big((Df)(\beta + T\gamma)\big) = (D^i f)(\beta + T\gamma) \end{aligned}$$

follows for all $i \in \mathbb{N}$ by induction on i. Using the Taylor series at 0 of g, we see that for all $c \in \mathbb{C}$,

$$
\begin{aligned}
f(\beta + c\gamma) = g(c) &= \sum_{i=0}^{\infty} \frac{(c-0)^i}{i!} \left(\frac{d^i}{dT^i} g(T) \right) \bigg|_{T=0} \\
&= \sum_{i=0}^{\infty} \frac{c^i}{i!} \left(\frac{d^i}{dT^i} f(\beta + T\gamma) \right) \bigg|_{T=0} \\
&= \sum_{i=0}^{\infty} \frac{c^i}{i!} \big((D^i f)(\beta + T\gamma) \big) \Big|_{T=0} \\
&= \big((\exp cD) f \big) \big|_{\beta + T\gamma} \Big|_{T=0} = \big((\exp cD) f \big) \big|_{\beta}. \qquad (2.3)
\end{aligned}
$$

Proposition 2.2. *Let $F: \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map of degree d and $\lambda_i \in \mathbb{C}$ for all i, such that $\sum_{i \in I} \lambda_i \neq 0$ for all nonempty $I \subseteq \{1, 2, \ldots, d-1\}$. Assume $\beta, \gamma \in \mathbb{C}^n$ such that $\gamma \neq 0$. If every sum of $d-1$ directional derivatives of $F|_{\beta + \mathbb{C}\gamma}$ along γ is nonzero ($\lambda_i = 1$ for all i below), or more generally,*

$$
\sum_{i=1}^{d-1} \lambda_i \cdot (\mathcal{J}F)|_{\alpha_i} \cdot \gamma \neq 0
$$

for all $\alpha_i \in \{\beta + \mu\gamma \mid \mu \in \mathbb{C}\}$, then $F(\beta) \neq F(\beta + \gamma)$.

Proof. Set $D := \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial X_i}$ and $P(T) := \big(\sum_{i=1}^{d-1} \lambda_i\big) T^{-1}(\exp(T) - 1)$. By (2.3),

$$
\begin{aligned}
\left(\sum_{i=1}^{d-1} \lambda_i \right) \cdot \big(F_j(\beta + \gamma) - F_j(\beta) \big) &= \left(\sum_{i=1}^{d-1} \lambda_i \right) \cdot \Big(\big(\exp(D) - 1 \big) F_j \Big) \Big|_{\beta} \\
&= \big(D P(D) F_j \big) \big|_{\beta} = \big(P(D)(DF_j) \big) \big|_{\beta}
\end{aligned}
$$

for all j. Choose r_i as in Lemma 2.1 for all i. From the definition of D and (2.3) with $c = r_i$ and $f = DF_j$,

$$
\begin{aligned}
\sum_{i=1}^{d-1} \lambda_i \cdot (\mathcal{J}F_j)|_{\beta + r_i\gamma} \cdot \gamma &= \sum_{i=1}^{d-1} \lambda_i (DF_j)(\beta + r_i\gamma) \\
&= \left(\sum_{i=1}^{d-1} \lambda_i \exp(r_i D)(DF_j) \right) \Bigg|_{\beta}
\end{aligned}
$$

follows for all j. Since $P(T) - \sum_{i=1}^{d-1} \lambda_i \exp(r_i T)$ is divisible by T^d and DF_j has degree at most $d-1$, we have

$$
P(D)(DF_j) = \sum_{i=1}^{d-1} \lambda_i \exp(r_i D)(DF_j)
$$

for all j. By substituting $X = \beta$ on both sides, we obtain

$$
\left(\sum_{i=1}^{d-1} \lambda_i \right) \cdot \big(F_j(\beta + \gamma) - F_j(\beta) \big) = \sum_{i=1}^{d-1} \lambda_i (\mathcal{J}F_j)|_{\beta + r_i\gamma} \cdot \gamma
$$

for all j, which gives the desired result. □

Corollary 2.3. *Let $F:\mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map of degree d and $\lambda_i \in \mathbb{C}$ for all i, such that $\sum_{i\in I}\lambda_i \neq 0$ for all nonempty $I \subseteq \{1,2,\ldots,d-1\}$. If $\operatorname{rk}(\sum_{i=1}^{d-1}\lambda_i(\mathcal{J}F)|_{\alpha_i}) = n$ for all $\alpha_i \in \mathbb{C}^n$, then F is injective.*

If additionally $n = m$, then F is an invertible polynomial map.

Proof. Assume $F(\beta) = F(\beta+\gamma)$ for some $\beta,\gamma \in \mathbb{C}^n$. By Proposition 2.2, there are $\alpha_i \in \mathbb{C}^n$ such that

$$\sum_{i=1}^{d-1}\lambda_i \cdot (\mathcal{J}F)|_{\alpha_i} \cdot \gamma = 0$$

and in particular $\operatorname{rk}\left(\sum_{i=1}^{d-1}\lambda_i \cdot (\mathcal{J}F)\Big|_{\alpha_i}\right) \neq n$.

If $n = m$, then a special case of the Cynk–Rusek Theorem in [6] (see also [13, Lemma 3] and [5]) tells us that F is an invertible polynomial map in case it is injective, which is the case here. □

Remark 2.4. When $d = 2$ or $d = 3$, Corollary 2.3 gives a result of Wang [12, Theorem 1.2.2] and one of Yagzhev [13, Theorem 1(ii)], respectively. Corollary 2.3 also generalizes [11, Theorem 2.2.1, Corollary 2.2.2].

Remark 2.5. Now you might think that for Proposition 2.2, the condition that there are $d-1$ collinear α_i's with the additive property therein is weaker than a similar property for $s\alpha_i$'s, where $s \in \mathbb{N}$ is arbitrary. This is however not the case.

Theorem 2.6. *Let $F:\mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map of degree $\leq d$ and $\beta,\gamma \in \mathbb{C}^n$. Then the following statements are equivalent.*

(1) *There exists $\lambda_1,\lambda_2,\ldots,\lambda_{d-1} \in \mathbb{C}$ satisfying $\sum_{i\in I}\lambda_i \neq 0$ for all nonempty $I \subseteq \{1,2,\ldots,d-1\}$, such that*

$$\sum_{i=1}^{d-1}\lambda_i \cdot (\mathcal{J}F)|_{\alpha_i} \cdot \gamma \neq 0$$

for all $\alpha_i \in \{\beta+\mu\gamma \mid \mu \in \mathbb{C}\}$.

(2) *$F|_{\beta+\mathbb{C}\gamma}$ is linearly rectifiable (in particular injective), that is, there exists a vector $v \in \mathbb{C}^m$ such that*

$$\sum_{j=1}^{m} v_j \cdot \frac{d}{dT}\big(F_j(\beta+T\gamma)\big) = 1. \tag{2.4}$$

(3) *For all $s \in \mathbb{N}$,*

$$\sum_{i=1}^{s}\lambda_i \cdot (\mathcal{J}F)|_{\alpha_i} \cdot \gamma \neq 0$$

for all $\lambda_i \in \mathbb{C}$ such that $\lambda_1+\lambda_2+\cdots+\lambda_s \neq 0$, and all $\alpha_i \in \{\beta+\mu\gamma \mid \mu \in \mathbb{C}\}$.

Proof. Since (3) ⇒ (1) is trivial, only two implications remain.

(2) ⇒ (3). Assume that (2) is satisfied. Take $s \in \mathbb{N}$, $\lambda_1,\lambda_2,\ldots,\lambda_s \in \mathbb{C}$ such that $\lambda_1+\lambda_2+\cdots+\lambda_s \neq 0$, and $\alpha_i \in \{\beta+\mu\gamma \mid \mu \in \mathbb{C}\}$. Each α_i is of the form $\alpha_i = \beta + r_i\gamma$ for

some $r_i \in \mathbb{C}$. By the chain rule,

$$\begin{aligned} v^t \cdot \left(\sum_{i=1}^{s} \lambda_i \cdot (\mathcal{J}F)|_{\alpha_i} \cdot \gamma\right) &= \sum_{i=1}^{s} \lambda_i \cdot \left(\sum_{j=1}^{m} v_j \cdot (\mathcal{J}F_j)|_{\beta+r_i\gamma} \cdot \gamma\right) \\ &= \sum_{i=1}^{s} \lambda_i \cdot \left(\sum_{j=1}^{m} v_j \frac{d}{dT}\big(F_j(\beta+T\gamma)\big)\right)\Big|_{T=r_i} \\ &= \sum_{i=1}^{s} \lambda_i \cdot 1|_{T=r_i} = \sum_{i=1}^{s} \lambda_i \neq 0, \end{aligned}$$

which gives (3).

(1) $\Rightarrow$ (2). Assume that (2) does not hold. We will derive a contradiction by showing that (1) does not hold either.

Since $\deg_T \frac{d}{dT} F_j(\beta + T\gamma) \le d-1$ for all j, the $\mathbb{C}$-space U that is generated by

$$\frac{d}{dT}F_1(\beta+T\gamma), \frac{d}{dT}F_2(\beta+T\gamma), \ldots, \frac{d}{dT}F_m(\beta+T\gamma)$$

has dimension $s \le d-1$, for $1 \notin U$. Take a basis of U of monic $u_1, u_2, \ldots, u_s \in \mathbb{C}[T]$ such that $0 < \deg u_1 < \deg u_2 < \cdots < \deg u_s < d$. Write u_{ji} for the coefficient of T^i of u_j.

Next, define p_i for $i = 0, 1, \ldots, d-1$ as follows.

$$p_i := \begin{cases} -\sum_{k=0}^{i-1} p_k u_{jk} & \text{if } u_j \text{ has degree } i, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_{d-1} & \text{if no } u_j \text{ has degree } i. \end{cases}$$

Set $P := \sum_{k=1}^{d-1} \frac{p_k}{k!} T^k$ and choose r_i as in Lemma 2.1 for all i. Looking at the term expansion of u_j, we see that

$$P\Big(\frac{d}{dT}\Big)u_j = \sum_{k=0}^{\infty} \frac{p_k}{k!} \cdot \sum_{l=0}^{\infty} \frac{(k+l)!}{l!} u_{jk} T^l,$$

whence for $i = \deg u_j$,

$$\left(P\Big(\frac{d}{dT}\Big)u_j\right)\Big|_{T=0} = \sum_{k=0}^{\infty} p_k u_{jk} = p_i + \sum_{k=0}^{i-1} p_k u_{jk} = 0$$

and similarly for each i

$$\left(\exp\Big(r_i \frac{d}{dT}\Big)u_j\right)\Big|_{T=0} = \sum_{k=0}^{\infty} r_i^k u_{jk} = u_j(r_i) = u_j\big|_{T=r_i}$$

follow for all j.

By Lemma 2.1, $P - \sum_{i=1}^{d-1} \lambda_i \exp(r_i T)$ is divisible by T^d. Since $\deg u_j < d$ for all j,

$$0 = \left(P\Big(\frac{d}{dT}\Big)u_j\right)\Big|_{T=0} = \sum_{i=1}^{d-1} \lambda_i \cdot \left(\exp\Big(r_i \frac{d}{dT}\Big)u_j\right)\Big|_{T=0} = \sum_{i=1}^{d-1} \lambda_i u_j\big|_{T=r_i}.$$

Since $\frac{d}{dT}F_j(\beta + T\gamma)$ is a $\mathbb{C}$-linear combination of $u_1, u_2, \ldots, u_s$ for all j, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{d-1} \lambda_i \cdot \Big(\mathcal{J}_T\big(F(\beta + T\gamma)\big)\Big)\Big|_{T=r_i} \\ &= \sum_{i=1}^{d-1} \lambda_i \cdot \big((\mathcal{J}F)|_{\beta+T\gamma} \cdot \gamma\big)\big|_{T=r_i} \\ &= \sum_{i=1}^{d-1} \lambda_i \cdot \big(\mathcal{J}F\big)\big|_{\beta+r_i\gamma} \cdot \gamma, \end{aligned}$$

which is a contradiction. □

Remark 2.7. The definition of rectifiable in [8, Definition 5.3.3], which is there for the formulation of the Abhyankar–Moh–Suzuki theorem [8, Theorem 5.3.5], is about the existence of an invertible polynomial map G (called F^{-1} in [8, Definition 5.3.3]) such that $G(\phi_1(T), \phi_2(T), \ldots, \phi_m(T)) = (T, 0, \ldots, 0)$. The definition of linearly rectifiable is more specific in the sense that $\deg G_1 = 1$ is required.

In (2.4), we have $\phi(T) = F(\beta + T\gamma)$ and $G_1 = \sum_{j=1}^{m} v_j(Y_j - F_j(\beta))$, and one can show that G can be extended to an automorphism (both in (2.4) and any other situation where G_1 is known), because for all $i \geq 2$ we can choose $G_i = Y_j - \phi_j(G_1)$ for some $j \in \{i-1, i\}$.

Remark 2.8. For the map $F = (X_1 + (X_2 + X_1^2)^2, X_2 + X_1^2)$, only images of lines parallel to the X_2-axis are linearly rectifiable. But all images of lines are linearly rectifiable when $F = (X_1 + (X_2 + X_1^2)^2 - (X_3 + X_1^2)^2, X_2 + X_1^2, X_3 + X_1^2)$ or any other invertible cubic map over $\mathbb{C}$. This follows from the proposition below.

Proposition 2.9. *Let $F: \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map of degree ≤ 3, and $\beta, \gamma \in \mathbb{C}^n$ such that $\gamma \neq 0$. If $F|_{\beta+\mathbb{C}\gamma}$ is injective and $(\mathcal{J}F)|_\alpha \cdot \gamma \neq 0$ for all $\alpha \in \{\beta + \mu\gamma \mid \mu \in \mathbb{C}\}$, then $F|_{\beta+\mathbb{C}\gamma}$ is linearly rectifiable, that is, there exists a $v \in \mathbb{C}^m$ such that (2.4) holds.*

Proof. Assume $F|_{\beta+\mathbb{C}\gamma}$ is not linearly rectifiable. Then there exist monic $u_1, u_2 \in \mathbb{C}[T]$ such that $\deg u_i = i$ and for all j, $\frac{d}{dT}F_j(\beta + T\gamma)$ is linearly dependent over $\mathbb{C}$ of u_1 and u_2. If the constant term u_{10} of u_1 is nonzero, then u_{10} will become zero after replacing β by $\beta - u_{10}\gamma$ and adapting u_1 and u_2 accordingly. So assume $u_{10} = 0$ and let u_{20} be the constant term of u_2. By taking the integral of u_1 and u_2 from $T = -\sqrt{-3u_{20}}$ to $T = +\sqrt{-3u_{20}}$, we see that $F(\beta - \sqrt{-3u_{20}}\gamma) = F(\beta + \sqrt{-3u_{20}}\gamma)$, thus either $F|_{\beta+\mathbb{C}\gamma}$ is not injective or $u_{20} = 0$. If $u_{20} = 0$, then $(\mathcal{J}F)|_{X=\beta} \cdot \gamma = 0$ because both u_1 and u_2 are divisible by T. This completes the proof of Proposition 2.9. □

Corollary 2.10. *Assume $F: \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map of degree ≤ 3 which satisfies the Keller condition $\det \mathcal{J}F \in \mathbb{C}^*$. Then F is invertible, if and only if $F|_L$ is linearly rectifiable for every affine line $L \subseteq \mathbb{C}^n$, if and only if $\big((\mathcal{J}F)|_\alpha + (\mathcal{J}F)|_\beta\big)(\alpha - \beta) \neq 0$ for all $\alpha, \beta \in \mathbb{C}^n$ with $\alpha \neq \beta$.*

Proof. By Proposition 2.9, F is invertible, if and only if $F|_L$ is linearly rectifiable for every affine line $L \subseteq \mathbb{C}^n$. By Theorem 2.6, the latter is equivalent to $\big((\mathcal{J}F)|_\alpha + (\mathcal{J}F)|_\beta\big)(\alpha - \beta) \neq 0$ for all $\alpha, \beta \in \mathbb{C}^n$ with $\alpha \neq \beta$, as desired. □

Remark 2.11. Notice that in the proof of Lemma 2.1, we solve $d-1$ equations in $d-1$ variables to obtain $r_1, r_2, \ldots, r_{d-1}$. In case $\lambda_1 = \lambda_2 = \cdots = \lambda_{d-1}$, it suffices to solve only one equation in only one variable to obtain $r_1, r_2, \ldots, r_{d-1}$.

Lemma 2.1′. *Let $P \in \mathbb{C}[[T]]$ with constant term $d-1$. Then there are $r_1, r_2, \ldots, r_{d-1} \in \mathbb{C}$, which are roots of a polynomial whose coefficients are polynomials in those of P, such that*

$$P - \sum_{i=1}^{d-1} \exp(r_i T)$$

is divisible by T^d, where $\exp(T) = \sum_{j=0}^{\infty} \frac{1}{j!} T^j$.

Proof. Write

$$P = \sum_{j=0}^{\infty} \frac{p_j}{j!} T^j.$$

Then we must find a solution $(Y_1, Y_2, \ldots, Y_{d-1}) = (r_1, r_2, \ldots, r_{d-1})$ of

$$\sum_{i=1}^{d-1} Y_i^j = p_j \quad (j = 0, 1, \ldots, d-1).$$

By Newton's identities for symmetric polynomials, there exists a polynomial $f \in \mathbb{C}[T][X_1, X_2, \ldots, X_{d-1}]$ which is injective as a function of $\mathbb{C}^{d-1}$ to $\mathbb{C}[T]$, such that

$$f\left(\sum_{i=1}^{d-1} X_i, \sum_{i=1}^{d-1} X_i^2, \ldots, \sum_{i=1}^{d-1} X_i^{d-1}\right) = \prod_{i=1}^{d-1} (T + X_i).$$

Notice that $g := f(p_1, \ldots, p_{d-1})$ is a monic polynomial of degree $d-1$ in T. Hence we can decompose g as

$$g = \prod_{i=1}^{d-1} (T + r_i) = f\left(\sum_{i=1}^{d-1} r_i, \sum_{i=1}^{d-1} r_i^2, \ldots, \sum_{i=1}^{d-1} r_i^{d-1}\right),$$

and the injectivity of f gives the desired result. □

3. Additive properties of the Jacobian determinant

Proposition 3.1. *Let $F: \mathbb{C}^n \to \mathbb{C}^n$ be a quadratic polynomial map such that* $\det \mathcal{J}F \in \mathbb{C}$. *Then for all $s \in \mathbb{N}$,*

$$\det\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \det\left(\sum_{i=1}^{s} b_i \cdot \mathcal{J}F\right) = \left(\sum_{i=1}^{s} b_i\right)^n \cdot \det \mathcal{J}F$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and all $b_1, b_2, \ldots, b_s \in \mathbb{C}$.

Proof. Since the entries of $\mathcal{J}F$ are affinely linear, we have

$$\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i} = \sigma \cdot \left(\mathcal{J}F\right)\Big|_{\sigma^{-1} \sum_{i=1}^{s} b_i \alpha_i}$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and all $b_1, b_2, \ldots, b_s \in \mathbb{C}$, in case $\sigma := \sum_{i=1}^{s} b_i \neq 0$. Taking determinants on both sides, it follows from $\det \mathcal{J}F \in \mathbb{C}$ that

$$\det\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \det(\sigma \cdot \mathcal{J}F) = \sigma^n \cdot \det \mathcal{J}F$$

when $\sigma \neq 0$, and by continuity also in case $\sigma = 0$, as desired. □

Lemma 3.2. *Assume $f \in \mathbb{C}[X]$ has degree $\leq d$. If f vanishes on the set $S := \{a \in \mathbb{N}^n \mid a_1 + a_2 + \cdots + a_n \leq d\}$, then $f = 0$.*

Proof. Notice that we can write f in the form

$$f(X_1, \ldots, X_{n-1}, X_n) = f(X_1, \ldots, X_{n-1}, 0) + X_n \cdot g(X_1, \ldots, X_{n-1}, X_n - 1)$$

where $g \in \mathbb{C}[X]$. By induction on n, $f(X_1, \ldots, X_{n-1}, 0) = 0$. Furthermore, if $a \in S$ and $a_n \geq 1$, then

$$g(a_1, a_2, \ldots, a_{n-1}, a_n - 1) = \frac{f(a) - f(a_1, a_2, \ldots, a_{n-1}, 0)}{a_n} = 0.$$

Thus by induction on d, $g = 0$. Hence $f = 0$ as well. □

Corollary 3.3. *Let $f \in \mathbb{C}[X]$ be a polynomial of degree $\leq d$. If $f(a) = 0$ for all $a \in \mathbb{N}^n$ such that $\sum_{i=1}^{n} a_i = d$, then $\sum_{i=1}^{n} x_i - d \mid f$. If additionally f is homogeneous, then $f = 0$.*

Proof. If we substitute $X_n = d - \sum_{i=1}^{n-1} X_i$ in f, then we get a polynomial of degree $\leq d$ which is zero on account of Lemma 3.2. Hence $X_n = d - \sum_{i=1}^{n-1} X_i$ is a zero of $f \in \mathbb{C}(X_1, X_2, \ldots, X_{n-1})[X_n]$ and f is divisible over $\mathbb{C}(X_1, X_2, \ldots, X_{n-1})$ by $\sum_{i=1}^{n} X_i - d$. By Gauss' Lemma, f is divisible over $\mathbb{C}[X]$ by $\sum_{i=1}^{n} X_i - d$, which is only homogeneous if $d = 0$. Hence $f = 0$ when f is homogeneous. □

Lemma 3.4. *Let $F: \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map and $P: \mathrm{Mat}_{m,n}(\mathbb{C}) \to \mathbb{C}$ be a polynomial of degree $\leq d$ in the entries of its input matrix. Fix $\mu \in \mathbb{C}$ and assume that*

$$P\left(\sum_{i=1}^{d} (\mathcal{J}F)|_{\alpha_i}\right) = \mu$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{C}^n$. Then for all $s \in \mathbb{N}$

$$P\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \mu = P(d\mathcal{J}F)$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and all $b_1, b_2, \ldots, b_s \in \mathbb{C}$ such that $\sum_{i=1}^{s} b_i = d$.

If additionally P is homogeneous, then

$$P\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \left(\frac{1}{d}\sum_{i=1}^{s} b_i\right)^{\deg P} \mu = \left(\sum_{i=1}^{s} b_i\right)^{\deg P} P(\mathcal{J}F)$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and all $b_1, b_2, \ldots, b_s \in \mathbb{C}$.

Proof. Since $P\big(\sum_{i=1}^{d}(\mathcal{J}F)|_{\alpha_i}\big) = \mu$ is constant,

$$\mu = P\left(\sum_{i=1}^{d}(\mathcal{J}F)|_{\alpha_i}\right) = P(d\mathcal{J}F)$$

for all $\alpha_i \in \mathbb{C}^n$. Take $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and let

$$f(Y_1, Y_2, \ldots, Y_s) := P\left(\sum_{i=1}^{s} Y_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) - \mu.$$

Then $\deg f(Y_1, Y_2, \ldots, Y_s) \le d$, and for all $b \in \mathbb{N}^s$ such that $\sum_{i=1}^{s} b_i = d$, we have

$$\begin{aligned} f(b) &= P\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) - \mu \\ &= P\left(\sum_{i=1}^{s}\sum_{j=1}^{b_i}(\mathcal{J}F)|_{\alpha_i}\right) - \mu = 0. \end{aligned}$$

By Corollary 3.3, $\sum_{i=1}^{s} Y_i - d \;\Big|\; f(Y_1, Y_2, \ldots, Y_s) - \mu$, whence

$$0 \;\left|\; P\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) - \mu\right.$$

for all $b \in \mathbb{C}^s$ such that $\sum_{i=1}^{s} b_i = d$. This gives the first assertion of Lemma 3.4.

Assume P is homogeneous. Then

$$\begin{aligned} g(Y_1, Y_2, \ldots, Y_s) &:= P\left(\sum_{i=1}^{s} Y_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) - \left(\frac{1}{d}\sum_{i=1}^{s} Y_i\right)^{\deg P} \mu \\ &= P\left(\sum_{i=1}^{s} Y_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) - \left(\sum_{i=1}^{s} Y_i\right)^{\deg P} P(\mathcal{J}F) \end{aligned}$$

is homogeneous as well. Since g vanishes on b for all $b \in \mathbb{N}^s$ such that $\sum_{i=1}^{s} b_i = d$, we obtain from Corollary 3.3 that $g = 0$, which completes the proof of Lemma 3.4. □

Theorem 3.5. *Let $m \ge n$ and $F: \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that for a fixed $\mu \in \mathbb{C}$, we have*

$$\det\left(\sum_{i=1}^{m}(\mathcal{J}F)|_{\alpha_i}\right) = \mu$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C}^n$. Then $\mu = \det(m\mathcal{J}F) = m^n \det(\mathcal{J}F)$ and for all $s \in \mathbb{N}$

$$\det\left(\sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \left(\frac{1}{m}\sum_{i=1}^{s} b_i\right)^n \mu = \left(\sum_{i=1}^{s} b_i\right)^n \det(\mathcal{J}F)$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{C}^n$ and all $b_1, b_2, \ldots, b_s \in \mathbb{C}$. Furthermore, F is an invertible polynomial map in case $\det \mathcal{J}F \ne 0$.

Proof. To obtain the first assertion, take $P = \det$, $d = m$ and $m = n$ in Lemma 3.4. By taking $s = \deg F - 1$ and $b_i = 1$ for all i in this assertion, it follows from Corollary 2.3 that F is an invertible polynomial map in case $\det \mathcal{J}F \ne 0$. □

Theorem 3.6. *Assume $H:\mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map and define*

$$M(\alpha_1, \alpha_2, \ldots, \alpha_s) := (\mathcal{J}H)|_{\alpha_1} + (\mathcal{J}H)|_{\alpha_2} + \cdots + (\mathcal{J}H)|_{\alpha_s}.$$

If for some $m \geq d$, the sum of the principal minors of size d of $M(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is zero for all $\alpha_i \in \mathbb{C}^n$, then for all $s \in \mathbb{N}$, the sum of the principal minors of size d of

$$b_1 \cdot (\mathcal{J}H)|_{\alpha_1} + b_2 \cdot (\mathcal{J}H)|_{\alpha_2} + \cdots + b_s \cdot (\mathcal{J}H)|_{\alpha_s} \tag{3.1}$$

is zero as well, for all $b_i \in \mathbb{C}$ and all $\alpha_i \in \mathbb{C}^n$. If for some $m \geq d$, the trace of $M(\alpha_1, \alpha_2, \ldots, \alpha_m)^d$ is zero for all $\alpha_i \in \mathbb{C}^n$, then for all $s \in \mathbb{N}$, the trace of the d-th power of (3.1) is zero as well, for all $b_i \in \mathbb{C}$ and all $\alpha_i \in \mathbb{C}^n$.

Proof. Take for P the sum of the principal minors of size m or the trace of the m-th power, respectively. By Lemma 3.4, $P((3.1))$ is divisible by $\mu := P(m\mathcal{J}H) = P(M(\alpha_1, \alpha_2, \ldots, \alpha_m)) = 0$. □

Remark 3.7. Let $F = X + H$ such that the Jacobian matrix $\mathcal{J}H$ is additive-nilpotent. Then for all $m \in \mathbb{N}$, $\sum_{i=1}^{m} (\mathcal{J}F)|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$. We shall show below that the converse holds when H does not have linear terms. But the converse is not true in general. For example, let $F(X) = X + H$, where $H = (-X_1 + X_2, X_1 - X_2 + X_2^2)$. Then

$$\mathcal{J}H = \begin{pmatrix} -1 & 1 \\ 1 & 2X_2 - 1 \end{pmatrix} \quad \text{and} \quad \mathcal{J}F = \begin{pmatrix} 0 & 1 \\ 1 & 2X_2 \end{pmatrix}$$

such that $\mathcal{J}H$ is not even nilpotent and $\sum_{i=1}^{2} (\mathcal{J}F)|_{\alpha_i}$ is invertible for all $\alpha_1, \alpha_2 \in \mathbb{C}^2$.

Proposition 3.8. *Assume $F:\mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map of the form $F = L + H$, such that L is invertible and $\deg L = 1$. Then for all $s \in \mathbb{N}$, all $b_i \in \mathbb{C}$, and all $\alpha_i \in \mathbb{C}^n$, the following statements are equivalent to each other.*

(1) *For all $\mu \in \mathbb{C}$, we have*

$$\det\left(\mu \cdot \mathcal{J}L + \sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \left(\mu + \sum_{i=1}^{s} b_i\right)^n \cdot \det(\mathcal{J}L).$$

(2) $\sum_{i=1}^{s} b_i \cdot \left(\mathcal{J}(L^{-1} \circ H)\right)\big|_{\alpha_i}$ *is nilpotent.*

Proof. Assume (1). Since the equality of (1) holds for all $\mu \in \mathbb{C}$, we obtain

$$\det\left(T \cdot \mathcal{J}L + \sum_{i=1}^{s} b_i \cdot (\mathcal{J}F)|_{\alpha_i}\right) = \left(T + \sum_{i=1}^{s} b_i\right)^n \cdot \det(\mathcal{J}L),$$

which is equivalent to

$$\det\left(\left(T - \sum_{i=1}^{s} b_i\right) \cdot \mathcal{J}L + \sum_{i=1}^{s} b_i \cdot (\mathcal{J}L + \mathcal{J}H)|_{\alpha_i}\right) = T^n \cdot \det(\mathcal{J}L)$$

and

$$\det\left(T \cdot \mathcal{J}L + \sum_{i=1}^{s} b_i \cdot (\mathcal{J}H)|_{\alpha_i}\right) = T^n \cdot \det(\mathcal{J}L).$$

By dividing both sides by $\det(\mathcal{J}L)$, we obtain

$$\det\left(T+\sum_{i=1}^{s} b_i\cdot(\mathcal{J}L)^{-1}\cdot(\mathcal{J}H)|_{\alpha_i}\right)=T^n,$$

which implies (2). The converse is similar. □

Let $F = X + H$ such that $\mathcal{J}H$ is additive-nilpotent. Then $\sum_{i=1}^{m}(\mathcal{J}\tilde{F})|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$ and all positive integers m, where $\tilde{F} = L_1 \circ F \circ L_2$ for invertible linear maps L_1 and L_2. We next prove that the converse holds.

Theorem 3.9. *For a polynomial map $F: \mathbb{C}^n \to \mathbb{C}^n$ the following statements are equivalent.*

(1) $\sum_{i=1}^{n}(\mathcal{J}F)|_{\alpha_i}$ *is invertible for all* $\alpha_i \in \mathbb{C}^n$*;*
(2) $F = L \circ (X + H)$*, where H has no linear terms, the linear part L of F is invertible and $\mathcal{J}H$ is additive-nilpotent;*
(3) $F = (X + H) \circ L$*, where H has no linear terms, the linear part L of F is invertible and $\mathcal{J}H$ is additive-nilpotent;*
(4) $F = L_1 \circ (X + H) \circ L_2$*, where L_1 and L_2 are invertible maps of degree one and $\mathcal{J}H$ is additive-nilpotent.*

Proof. Since (3) ⇒ (4) is trivial, the following three implications remain to be proved.

(4) ⇒ (1). Since $\mathcal{J}H$ is additive-nilpotent, (1) holds with $X + H$ instead of F. Since (1) is not affected by compositions with translations and invertible linear maps, and F can be obtained from $X + H$ in that manner, (1) follows.

(1) ⇒ (2). By the fundamental theorem of algebra, the determinant of $\sum_{i=1}^{n}(\mathcal{J}F)|_{\alpha_i}$ is a nonzero constant which does not depend on $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let L be the linear part of F. By Theorem 3.5, we obtain that $\det \mathcal{J}F = \det(\mathcal{J}F)|_0 = \det \mathcal{J}L$ and that (1) of Proposition 3.8 holds for all $s \in \mathbb{N}$, all $b_i \in \mathbb{C}$, and all $\alpha_i \in \mathbb{C}^n$. Hence the Jacobian matrix of $H := L^{-1}\circ(F-L)$ is additive-nilpotent on account of Proposition 3.8, which gives the desired result.

(2) ⇒ (3). This follows from the fact that $F = L \circ (X + H) = \big(X + (L \circ H \circ L^{-1})\big) \circ L$ and the Jacobian matrix of $L \circ H \circ L^{-1}$ is also additive-nilpotent. □

Remark 3.10. A polynomial map $F = (F_1, \ldots, F_n)$ is called *triangular* if its Jacobian matrix is triangular, that is, either above or below the main diagonal, all entries of $\mathcal{J}F$ are zero. The Jacobian matrix $\mathcal{J}F$ of a triangular invertible polynomial map F can only have nonzero constants on the main diagonal, and thus for all invertible linear maps L_1 and L_2, $\sum_{i=1}^{n}\big(\mathcal{J}(L_1\circ F\circ L_2)\big)\big|_{\alpha_i}$ is invertible for all $\alpha_i \in \mathbb{C}^n$. However, a polynomial map satisfying the conditions of Theorem 3.9 is not necessarily a composition of a triangular map and two linear maps. Indeed, in [10], it was shown that in dimension 5 and up, Keller maps $X + H$ with H quadratic homogeneous do not necessarily have the property that $\mathcal{J}H$ is strongly nilpotent. But on account of Proposition 3.1, such maps satisfy property (1) of Theorem 3.9.

In [4], all those maps such that $\mathcal{J}H$ is not strongly nilpotent are determined in dimension 5. H is either of the form

$$H = L^{-1}\circ\left(\begin{pmatrix}0\\ \lambda X_1^2\\ X_2X_4\\ X_1X_3 - X_2X_5\\ X_1X_4\end{pmatrix}+\begin{pmatrix}0\\ 0\\ p(X_1,X_2)\\ q(X_1,X_2)\\ r(X_1,X_2)\end{pmatrix}\right)\circ L,$$

where $\lambda \in \{0, 1\}$, L is linear and $p, q, r \in \mathbb{C}[X_1, X_2]$, or of the form

$$H = L^{-1} \circ \left(\begin{pmatrix} 0 \\ X_1X_3 \\ X_2^2 - X_1X_4 \\ 2X_2X_3 - X_1X_5 \\ X_3^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 X_1^2 \\ \lambda_3 X_1^2 \\ \lambda_4 X_1^2 \\ \lambda_5 X_1^2 \end{pmatrix} \right) \circ L,$$

where L is linear and $\lambda_i \in \mathbb{C}$. One can show that in both cases, the columns of $\mathcal{J}(L \circ H \circ L^{-1})$ are linearly independent over $\mathbb{C}$, something that cannot be counteracted with compositions with invertible linear maps. Hence the columns of $\mathcal{J}(L_1 \circ H \circ L_2)$ are linearly independent over $\mathbb{C}$ for all invertible maps L_i. $\mathcal{J}(L_1 \circ H \circ L_2)$ is exactly the linear part of $\mathcal{J}(L_1 \circ F \circ L_2)$, thus $\mathcal{J}(L_1 \circ F \circ L_2)$ can only be triangular if its main diagonal is not constant on one of its ends. This is however not possible since $L_1 \circ F \circ L_2$ is invertible.

Acknowledgments

The authors would like to thank the referees for their careful reading and constructive comments.

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