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Well-posedness of the fifth order Kadomtsev–Petviashvili I equation in anisotropic Sobolev spaces with nonnegative indices ☆

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Abstract

In this paper we establish the local and global well-posedness of the real valued fifth order Kadomtsev–Petviashvili I equation in the anisotropic Sobolev spaces with nonnegative indices. In particular, our local well-posedness improves Saut–Tzvetkov's one and our global well-posedness gives an affirmative answer to Saut–Tzvetkov's L^2-data conjecture.

Résumé

Dans cet article, on montre que l'équation de Kadomtsev–Petviashvili I d'ordre 5 à valeurs réelles est localement et globalement bien posée dans les espaces de Sobolev anisotropes d'indices positifs. En particulier, notre resultat local améliore celui de Saut–Tzvetkov tandis que notre résultat global répond par l'affirmative à la conjecture de Saut–Tzvetkov sur les données L^2.

Keywords: Fifth KP-I equation; Anisotropic Sobolev space; Bourgain space; Dyadic decomposed Strichartz estimate

1. Introduction

In their J. Math. Pures Appl. (2000) paper on the initial value problem (IVP) of the real valued fifth order Kadomtsev–Petviashvili I (KP-I) equation (for $(\alpha, t, x, y) \in \mathbb{R}^4$):

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(0, x, y) = \phi(x, y), \end{cases} \tag{1}$$

J.C. Saut and N. Tzvetkov obtained the following result (cf. [17, Theorems 1 and 2]):

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Saut–Tzvetkov's Theorem. (i) *The IVP* (1) *is locally well-posed for initial data ϕ satisfying*:

$$\|\phi\|_{L^2(\mathbb{R}^2)} + \big\||-i\partial_x|^s\phi\big\|_{L^2(\mathbb{R}^2)} + \big\||-i\partial_y|^k\phi\big\|_{L^2(\mathbb{R}^2)} < \infty \quad \text{with } s-1,\ k \geqslant 0; \qquad \frac{\hat{\phi}(\xi,\eta)}{|\xi|} \in \mathcal{S}'(\mathbb{R}^2). \tag{2}$$

(ii) *The IVP* (1) *is globaly well-posed for initial data ϕ satisfying*:

$$\|\phi\|_{L^2(\mathbb{R}^2)} < \infty; \qquad \frac{1}{2}\int_{\mathbb{R}^2} \big|\partial_x^2\phi\big|^2 + \frac{\alpha}{2}\int_{\mathbb{R}^2} |\partial_x\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \big|\partial_x^{-1}\partial_y\phi\big|^2 - \frac{1}{6}\int_{\mathbb{R}^2} \phi^3 < \infty. \tag{3}$$

Here and henceforth, $|-i\partial_x|^s$ and $|-i\partial_y|^s$ are defined via the Fourier transform:

$$\widehat{|-i\partial_x|^s\phi}(\xi,\eta) = |\xi|^s\hat{\phi}(\xi,\eta) \quad \text{and} \quad \widehat{|-i\partial_y|^s\phi}(\xi,\eta) = |\eta|^s\hat{\phi}(\xi,\eta).$$

Since they simultaneously found in [17, Theorem 3] that the condition,

$$\|\phi\|_{L^2(\mathbb{R}^2)} < \infty; \qquad |\xi|^{-1}\hat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbb{R}^2), \tag{4}$$

ensures the global well-posedness for the real valued fifth order Kadomtsev–Petviashvili II (KP-II) equation (for $(\alpha,t,x,y) \in \mathbb{R}^4$):

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(0,x,y) = \phi(x,y), \end{cases} \tag{5}$$

they made immediately a conjecture in [17, Remarks, p. 310] which is now reformulated in the following form:

Saut–Tzvetkov's L^2-data Conjecture. *The IVP* (1) *is globally well-posed for initial data ϕ satisfying* (4).

In the above and below, as "local well-posedness" we refer to finding a Banach space $(X, \|\cdot\|_X)$ – when the initial data $\phi \in X$ there exists a time T depending on $\|\phi\|_X$ such that (1) has a unique solution u in $C([-T,T];X) \cap Y$ (where Y is one of the Bourgain spaces defined in Section 2) and u depends continuously on ϕ (in some reasonable topology). If this existing time T can be extended to the positive infinity, then "local well-posedness" is said to be "global well-posedness". Of course, the choice of a Banach space relies upon the boundedness of the fundamental solution to the corresponding homogenous equation or the conservation law for equation itself.

In our current paper, we settle this conjecture through improving the above-cited Saut–Tzvetkov's theorem. More precisely, we have the following:

Theorem 1.1. *The IVP* (1) *is not only locally but also globally well-posed for initial data ϕ satisfying*:

$$\phi \in H^{s_1,s_2}(\mathbb{R}^2) \quad \text{with } s_1, s_2 \geqslant 0; \qquad |\xi|^{-1}\hat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbb{R}^2). \tag{6}$$

Here and henceafter, the symbol,

$$H^{s_1,s_2}(\mathbb{R}^2) = \big\{ f \in \mathcal{S}'(\mathbb{R}^2):\ \|f\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \big\|(1+|\xi|^2)^{\frac{s_1}{2}}(1+|\eta|^2)^{\frac{s_2}{2}}\hat{f}(\xi,\eta)\big\|_{L^2(\mathbb{R}^2)} < \infty \big\},$$

stands for the anisotropic Sobolev space with nonnegative indices $s_1, s_2 \in [0,\infty)$. Obviously, if $s_1 = s_2 = 0$ then $H^{s_1,s_2}(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ and hence (6) goes back to (4) which may be regarded as the appropriate constraint on the initial data ϕ deriving the global well-posedness of the IVP for the fifth order KP-I equation. And yet the understanding of Theorem 1.1 is not deep enough without making three more observations below:

• **Observation 1.** The classification of the fifth order KP equations is determined by the dispersive function:

$$\omega(\xi,\mu) = \pm\xi^5 - \alpha\xi^3 + \frac{\mu^2}{\xi}, \tag{7}$$

where the signs $\pm$ in (7) produce the fifth order KP-I and KP-II equations, respectively. The forthcoming estimates play an important role in the analysis of the fifth order KP equations—for the fifth order KP-I equation, we have:

$$|\xi|^2 > |\alpha| \quad \Rightarrow \quad \big|\nabla\omega(\xi,\mu)\big| = \left|\left(5\xi^4 + 3\alpha\xi^2 - \frac{\mu^2}{\xi^2}, 2\frac{\mu}{\xi}\right)\right| \gtrsim |\xi|^2; \tag{8}$$

and for the fifth order KP-II equation, we have:

$$|\xi|^2 > |\alpha| \quad \Rightarrow \quad \left|\nabla\omega(\xi,\mu)\right| = \left|\left(5\xi^4 + 3\alpha\xi^2 + \frac{\mu^2}{\xi^2}, 2\frac{\mu}{\xi}\right)\right| \gtrsim |\xi|^4. \tag{9}$$

By (9), we can get more smooth effect estimates than by (8). These imply that we can get a well-posedness (in other words, a lower regularity) for the fifth order KP-II equation better than that for the fifth order KP-I equation. Another crucial concept is the resonance function:

$$\begin{aligned} R(\xi_1,\xi_2,\mu_1,\mu_2) &= \omega(\xi_1+\xi_2,\mu_1+\mu_2) - \omega(\xi_1,\mu_1) - \omega(\xi_2,\mu_2) \\ &= \frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left((\xi_1+\xi_2)^2\left[5\left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right) - 3\alpha\right] \mp \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}\right)^2\right). \end{aligned} \tag{10}$$

Evidently, the fifth order KP-II equation (corresponding to "+" in (10)) always enjoys:

$$\left|R(\xi_1,\xi_2,\mu_1,\mu_2)\right| \gtrsim \left(\max\left\{|\xi_1|, |\xi_2|, |\xi_1+\xi_2|\right\}\right)^4 \min\left\{|\xi_1|, |\xi_2|, |\xi_1+\xi_2|\right\}. \tag{11}$$

Nevertheless, this last inequality (11) is no longer true for the fifth order KP-I equation.

In the foregoing and following the notation $A \lesssim B$ (i.e., $B \gtrsim A$) means: there exists a constant $C > 0$ independent of A and B such that $A \leqslant CB$. In addition, if there exist two positive constants c and C such that $10^{-3} < c < C < 10^3$ and $cA \leqslant B \leqslant CB$ then the notation $A \sim B$ will be used.

• **Observation 2.** Perhaps it worths pointing out that the well-posedness of the fifth order KP-II equation is relatively easier to establish but also its result is much better than that of the fifth order KP-I equation. Although the study of the well-posedness for the fifth order KP-II equation (without the third order partial derivative term) usually focuses on the critical cases (which means $s_1 + 2s_2 = -2$ by a scaling argument), in [16] Saut and Tzvetkov only obtained the local well-posedness for the fifth order KP-II equation in the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{4}$, $s_2 \geqslant 0$ with a modification of the low frequency, and furthermore in [17] they removed this modification and obtained the global well-posedness in $L^2(\mathbb{R}^2)$. On the other hand, in [9] Isaza, López and Mejía established the local well-posedness for $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{5}{4}, s_2 \geqslant 0$ and the global well-posedness for $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{4}{7}, s_2 \geqslant 0$. More recently, Hadac [4] also gained the same local well-posedness in a broader context. Meanwhile in the fifth order KP-I equation case, the attention is mainly paid on those spaces possessing conservation law such as $L^2(\mathbb{R}^2)$ and the energy space:

$$E^1\left(\mathbb{R}^2\right) = \left\{f \in L^2\left(\mathbb{R}^2\right):\ \left\|\left(1+|\xi|^2+|\xi|^{-1}|\mu|\right)\hat{f}(\xi,\mu)\right\|_{L^2(\mathbb{R}^2)} < \infty\right\}.$$

To obtain the local well-posedness of KP-I in $E^1(\mathbb{R}^2)$, in [17], besides the above-mentioned results Saut and Tzvetkov also got the local well-posedness in $\tilde{H}^{s,k}(\mathbb{R}^2)$ with $s-1, k \geqslant 0$;

$$\tilde{H}^{s,k}\left(\mathbb{R}^2\right) = \left\{f \in L^2\left(\mathbb{R}^2\right):\ \left\|\left(1+|\xi|^s+|\xi|^{-1}|\eta|^k\right)\hat{f}(\xi,\eta)\right\|_{L^2(\mathbb{R}^2)} < \infty\right\}.$$

For the energy case $\tilde{H}^{2,1}(\mathbb{R}^2) = E^1(\mathbb{R}^2)$, they obtained the global well-posedness of (1). In [6], Ionescu and Kenig got the global well-posedness for the fifth order periodic KP-I equation (without the third order dispersive term) in the standard energy space $E^1(\mathbb{R}^2)$. Recently, in [3] Chen, Li and Miao obtained the local well-posedness in:

$$E^s\left(\mathbb{R}^2\right) = \left\{f \in L^2\left(\mathbb{R}^2\right):\ \left\|\left(1+|\xi|^2+|\xi|^{-1}|\mu|\right)^s\hat{f}(\xi,\mu)\right\|_{L^2(\mathbb{R}^2)} < \infty\right\}, \quad 0 < s \leqslant 1.$$

• **Observation 3.** The well-posedness for the IVP of the third order KP equations in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u \mp \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(0,x,y) = \phi(x,y), \end{cases} \tag{12}$$

in which the sign $\mp$ give the third order KP-I and KP-II equations respectively, is an important background material of the investigation of the well-posedness for the fifth order KP equations. Molinet, Saut and Tzvetkov showed in [14,15] that, for the third order KP-I equation one cannot obtain the local well-posedness in any type of nonisotropic L^2-based Sobolev space or in the energy space using Picard's iteration—see also [13]; while Iório and Nunes [8] applied a compactness method to deduce the local well-posedness for the third KP-I equation with data being in the normal Sobolev space $H^s(\mathbb{R}^2), s > 2$ and obeying a "zero-mass" condition. On the other hand, the global well-posedness for the third order KP-I equation was discussed by using the classical energy method in [11] where Kenig established the global well-posedness in:

$$\left\{f \in L^2(\mathbb{R}^2):\ \|f\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-1}\partial_y f\right\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^2 f\right\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-2}\partial_y^2 f\right\|_{L^2(\mathbb{R}^2)} < \infty\right\}.$$

As far as we know, the best existing result on the third order KP-I equation is due to Ionescu, Kenig and Tataru [7] which gives the global well-posedness for the third order KP-I equation in the energy space:

$$\left\{f \in L^2(\mathbb{R}^2):\ \|f\|_{L^2\mathbb{R}^2} + \left\|\partial_x^{-1}\partial_y f\right\|_{L^2(\mathbb{R}^2)} + \|\partial_x f\|_{L^2(\mathbb{R}^2)} < \infty\right\}.$$

Relatively speaking, the results on the third order KP-II equation are nearly perfect. In [2], Bourgain proved the global well-posedness of the third order KP-II equation in $L^2(\mathbb{R}^2)$—the assertion was then extended by Takaoka and Tzvetkov [19] and Isaza and Mejía [10] from $L^2(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{3}$, $s_2 \geqslant 0$. In [18], Takaoka obtained the local well-posedness for the third order KP-II equation in $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{2}$, $s_2 = 0$ under an additional low frequency condition $|-i\partial_x|^{-\frac{1}{2}+\varepsilon}\phi \in L^2(\mathbb{R}^2)$ which was removed successfully in Hadac's recent paper [4]. These results are very close to the critical index $s_1 + 2s_2 = -\frac{1}{2}$ (which follows from the scaling argument)—see also Hadac, Herr and Koch [5] for a small data global well-posedness and scattering result in the homogeneous Sobolev space $\dot{H}^{-1/2,0}(\mathbb{R}^2)$ as well as for arbitrarily large initial data local well-posedness in both $\dot{H}^{-1/2,0}(\mathbb{R}^2)$ and $H^{-1/2,0}(\mathbb{R}^2)$.

The rest of this paper is devoted to an argument for Theorem 1.1. In Section 2 we collect some useful and basically known linear estimates for the fifth order KP-I equation. In Section 3 we present the necessary and crucial bilinear estimates in order to set up the local (and hence global) well-posedness—this part is partially motivated by [17] though—the main difference between their treatment and ours is how to dispose the "high–high interaction"—their method exhausts no geometric structure of the resonant set of the fifth order KP-I equation while ours does fairly enough. In Section 4 we complete the argument through applying the facts verified in Sections 2 and 3 and Picard's iteration principle to the integral equation corresponding to (1).

2. Linear estimates

We begin with the IVP of linear fifth order KP-I equation:

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u = 0, \\ u(0,x,y) = \phi(x,y). \end{cases} \tag{13}$$

By the Fourier transform $\hat{(\cdot)}$, the solution of (13) can be defined as

$$u(t)(x,y) = \big(S(t)\phi\big)(x,y) = \int_{\mathbb{R}^2} e^{i(x\xi + y\mu + t\omega(\xi,\mu))}\hat{\phi}(\xi,\mu)\,d\xi\,d\mu.$$

By Duhamel's formula, (1) can be reduced to the integral representation below:

$$u(t) = S(t)\phi - \frac{1}{2}\int_0^t S(t-t')\partial_x\big(u^2(t')\big)\,dt'. \tag{14}$$

So, in order to get the local well-posedness we will apply a Picard fixed point argument in a suitable function space to the following integral equation:

$$u(t) = \psi(t)S(t)\phi - \frac{\psi_T(t)}{2}\int_0^t S(t-t')\partial_x\big(u^2(t')\big)\,dt', \tag{15}$$

where t belongs to $\mathbb{R}$, ψ is a time cut-off function satisfying:

$$\psi \in C_0^\infty(\mathbb{R}); \qquad \operatorname{supp}\psi \subset [-2,2]; \qquad \psi = 1 \quad \text{on } [-1,1],$$

and $\psi_T(\cdot)$ represents $\psi(\cdot/T)$ for a given time $T \in (0,1)$. Consequently, we need to define an appropriate Bourgain type space, which is associated with the fifth order KP-I equation. To this end, for $s_1, s_2 \geqslant 0$ and $b \in \mathbb{R}$ the notation $X_b^{s_1,s_2}$ is used as the Bourgain space with norm:

$$\|u\|_{X_b^{s_1,s_2}} = \left\| \langle \tau - \omega(\xi,\mu) \rangle^b \langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \hat{u}(\tau,\xi,\mu) \right\|_{L^2(\mathbb{R}^3)},$$

where $\langle\cdot\rangle$ stands for $(1+|\cdot|^2)^{1/2} \sim 1+|\cdot|$. Furthermore, for an interval $I \subset \mathbb{R}$ the localized Bourgain space $X_b^{s_1,s_2}(I)$ can be defined via requiring:

$$\|u\|_{X_b^{s_1,s_2}(I)} = \inf_{w \in X_b^{s_1,s_2}} \left\{ \|w\|_{X_b^{s_1,s_2}} \colon w(t) = u(t) \text{ on interval } I \right\}.$$

The following two results are known.

Proposition 2.1. *(See [17].) If*

$$T \in (0,\infty); \quad s_1, s_2 \geqslant 0; \quad -\frac{1}{2} < b' \leqslant 0 \leqslant b \leqslant b'+1,$$

then

$$\left\| \psi S(t)\phi \right\|_{X_b^{s_1,s_2}} \lesssim \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}; \tag{16}$$

$$\left\| \psi(t/T) \int_0^t S(t-t')h(t')\,dt' \right\|_{X_b^{s_1,s_2}} \lesssim T^{1-b+b'} \|h\|_{X_{b'}^{s_1,s_2}}, \tag{17}$$

for any $\|h\|_{X_{b'}^{s_1,s_2}} < \infty$.

Proposition 2.2. *(See [1].) If* $r \in [2,\infty)$, *then there exists a constant* $c > 0$ *independent of* $T \in (0,1)$ *such that*

$$\left\| |-i\partial_x|^{\frac{1}{2}-\frac{1}{r}} \big(S(t)\phi\big)(x,y) \right\|_{L_T^{\frac{2r}{r-2}} L^r(\mathbb{R}^2)} \leqslant c\|\phi\|_{L^2(\mathbb{R}^2)}, \tag{18}$$

where

$$\|f\|_{L_T^{\frac{2r}{r-2}} L^r(\mathbb{R}^2)} = \left(\int_{-T}^{T} \left(\int_{\mathbb{R}^2} \left| f(x,y,t) \right|^r dx\,dy \right)^{\frac{2}{r-2}} dt \right)^{\frac{r-2}{2r}}.$$

To reach our bilinear inequalities in Section 3, we will use $(\cdot)^\vee$ for the inverse Fourier transform, and take the dyadic decomposed Strichartz estimates below into account.

Proposition 2.3. *Let* η *be a bump function with compact support in* $[-2,2] \subset \mathbb{R}$ *and* $\eta = 1$ *on* $(-1,1) \subset \mathbb{R}$. *For each integer* $j \geqslant 1$ *set* $\eta_j(x) = \eta(2^{-j}x) - \eta(2^{1-j}x)$, $\eta_0(x) = \eta(x)$, $\eta_j(\xi,\mu,\tau) = \eta_j(\tau - \omega(\xi,\mu))$, *and* $f_j(\xi,\mu,\tau) = (\eta_j(\xi,\mu,\tau)|\hat{f}|(\xi,\mu,\tau))^\vee$ *for any given* $f \in L^2(\mathbb{R}^3)$. *Then for given* $r \in [2,\infty)$ *and any* $T \in (0,1)$ *we have*:

$$\left\| |-i\partial_x|^{\frac{1}{2}-\frac{1}{r}} f_j \right\|_{L_T^{\frac{2r}{r-2}} L^r(\mathbb{R}^2)} \lesssim 2^{\frac{j}{2}} \|f_j\|_{L^2(\mathbb{R}^3)}; \tag{19}$$

in particular,

$$\left\| |-i\partial_x|^{\frac{1}{4}} f_j \right\|_{L_T^4 L^4(\mathbb{R}^2)} \lesssim 2^{\frac{j}{2}} \|f_j\|_{L^2(\mathbb{R}^3)}. \tag{20}$$

Proof. Note first that

$$f_j(x,y,t) = \int_{\mathbb{R}^3} e^{i(x\xi+y\mu+t\tau)} |\hat{f}| \eta_j(\xi,\mu,\tau)\,d\xi\,d\mu\,d\tau.$$

So, changing variables and using $\widehat{f_\lambda}(\xi,\mu) = |\hat{f}|(\xi,\mu,\lambda+\omega)$ we can write:

$$\begin{aligned} f_j(x,y,t) &= \int_{\mathbb{R}^3} e^{i(x\xi+y\mu+t(\lambda+\omega))}|\hat{f}|(\xi,\mu,\lambda+\omega)\eta_j(\lambda)\,d\xi\,d\mu\,d\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda}\eta_j(\lambda)\biggl[\int_{\mathbb{R}^2} e^{i(x\xi+y\mu+t\omega)}|\hat{f}|(\xi,\mu,\lambda+\omega)\,d\xi\,d\mu\biggr]d\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda}\eta_j(\lambda)S(t)f_\lambda(x,y)\,d\lambda. \end{aligned}$$

Now the estimate (19) follows from Minkowski's inequality, the Strichartz estimate (18) and the Cauchy–Schwarz inequality. □

The following well-known elementary inequalities are also useful – see for example [17, Proposition 2.2].

Proposition 2.4. *Let $\gamma > 1$. Then*

$$\int_{\mathbb{R}} \frac{dt}{\langle t\rangle^\gamma \langle t-a\rangle^\gamma} \lesssim \langle a\rangle^{-\gamma}, \tag{21}$$

and

$$\int_{\mathbb{R}} \frac{dt}{\langle t\rangle^\gamma |t-a|^{\frac{1}{2}}} \lesssim \langle a\rangle^{-\frac{1}{2}}, \tag{22}$$

hold for any $a \in \mathbb{R}$.

3. Bilinear estimates

Although there were many works on the so-called bilinear estimates, we have found that the Kenig–Ponce–Vega's bilinear estimation approach introduced in [12] is quite suitable for our purpose. With the convention: when $a \in \mathbb{R}$ the number $a\pm$ equals $a \pm \epsilon$ for arbitrarily small number $\epsilon > 0$, we can state our bilinear estimate as follows:

Theorem 3.1. *If $s_1, s_2 \geqslant 0$ and functions u, v have compact time support on $[-T,T]$ with $0 < T < 1$, then*

$$\bigl\|\partial_x(uv)\bigr\|_{X^{s_1,s_2}_{-\frac{1}{2}+}} \lesssim \|u\|_{X^{s_1,s_2}_{\frac{1}{2}+}}\|v\|_{X^{s_1,s_2}_{\frac{1}{2}+}}. \tag{23}$$

Proof. In what follows, we derive (23) using the duality; that is, we are required to dominate the integral,

$$\int_{A^*} \frac{|\xi|\langle\xi\rangle^{s_1}\langle\mu\rangle^{s_2}}{\langle\tau-\omega(\xi,\mu)\rangle^{\frac{1}{2}-}} g(\xi,\mu,\tau)|\hat{u}|(\xi_1,\mu_1,\tau_1)|\hat{v}|(\xi_2,\mu_2,\tau_2)\,d\xi_1\,d\mu_1\,d\tau_1\,d\xi_2\,d\mu_2\,d\tau_2, \tag{24}$$

where $g \geqslant 0$, $\|g\|_{L^2(\mathbb{R}^2)} \leqslant 1$ and

$$A^* = \bigl\{(\xi_1,\mu_1,\tau_1,\xi_2,\mu_2,\tau_2) \in \mathbb{R}^6\colon\ \xi_1+\xi_2=\xi,\ \mu_1+\mu_2=\mu,\ \tau_1+\tau_2=\tau\bigr\}.$$

Let

$$\sigma = \tau-\omega(\xi,\mu); \qquad \sigma_1 = \tau_1-\omega(\xi_1,\mu_1); \qquad \sigma_2 = \tau_2-\sigma(\xi_2,\mu_2).$$

Define two functions below:

$$f_1(\xi_1,\mu_1,\tau_1) = \langle\xi_1\rangle^{s_1}\langle\mu_1\rangle^{s_2}\langle\sigma_1\rangle^{\frac{1}{2}+}\bigl|\hat{u}(\xi_1,\mu_1,\tau_1)\bigr|,$$

and

$$f_2(\xi_2,\mu_2,\tau_2) = \langle\xi_2\rangle^{s_1}\langle\mu_2\rangle^{s_2}\langle\sigma_2\rangle^{\frac{1}{2}+}\bigl|\hat{v}(\xi_2,\mu_2,\tau_2)\bigr|.$$

Then we need to bound the integral,

$$\int_{A^*} K(\xi_1,\mu_1,\tau_1,\xi_2,\mu_2,\tau_2)g(\xi,\mu,\tau)f_1(\xi_1,\mu_1,\tau_1)f_2(\xi_2,\mu_2,\tau_2)\,d\xi_1\,d\mu_1\,d\tau_1\,d\xi_2\,d\mu_2\,d\tau_2, \tag{25}$$

from above by using a constant multiple of $\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}$. Here

$$K(\xi_1,\mu_1,\tau_1,\xi_2,\mu_2,\tau_2)=\left(\frac{|\xi_1+\xi_2|}{\langle\sigma\rangle^{\frac12-}\langle\sigma_1\rangle^{\frac12+}\langle\sigma_2\rangle^{\frac12+}}\right)\left(\frac{\langle\xi_1+\xi_2\rangle^{s_1}}{\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_1}}\right)\left(\frac{\langle\mu_1+\mu_2\rangle^{s_2}}{\langle\mu_1\rangle^{s_2}\langle\mu_2\rangle^{s_2}}\right).$$

It is clear that for $s_1,s_2\geqslant 0$ we always have:

$$K(\xi_1,\mu_1,\tau_1,\xi_2,\mu_2,\tau_2)\lesssim\frac{|\xi_1+\xi_2|}{\langle\sigma\rangle^{\frac12-}\langle\sigma_1\rangle^{\frac12+}\langle\sigma_2\rangle^{\frac12+}}.$$

Keeping a further assumption $|\xi_1|\geqslant|\xi_2|$ (which follows from symmetry) in mind, we are about to fully control the integral in (25) through handling two situations:

• **Situation 1—Low frequency** $|\xi_1+\xi_2|\lesssim\max\{10,|\alpha|\}$.

∘ *High + High → Low* $|\xi_1|,|\xi_2|\gtrsim\max\{10,|\alpha|\}$. We first deduce a dyadic decomposition. Employing η_j in Proposition 2.3, we have $\sum_{j\geqslant 0}\eta_j=1$, and consequently (25) can be bounded from above by a constant multiple of

$$\sum_{j\geqslant 0}2^{-j(\frac12-)}\int_{A^*}\eta_j(\sigma)g(\xi,\mu,\tau)\left(\frac{f_1(\xi_1,\mu_1,\tau_1)}{\langle\sigma_1\rangle^{\frac12+}}\right)\left(\frac{f_2(\xi_2,\mu_2,\tau_2)}{\langle\sigma_2\rangle^{\frac12+}}\right)d\xi_1\,d\mu_1\,d\tau_1\,d\xi_2\,d\mu_2\,d\tau_2. \tag{26}$$

We may assume that for each natural number j,

$$G_j(x,y,t)=\mathcal{F}^{-1}\big(\eta_j(\sigma)g(\xi,\mu,\tau)\big)(x,y,t),$$

has support compact in the interval $[-T,T]$ whenever it acts as a time-dependent function, where $\mathcal{F}^{-1}$ also denotes the inverse Fourier transform. In fact, if we consider the following functions generated by $\mathcal{F}^{-1}$:

$$F_l(x,y,t)=\mathcal{F}^{-1}\left(\frac{f_l(\xi_l,\mu_l,\tau_k)}{\langle\sigma_l\rangle^{\frac12+}}\right)(x,y,t)\quad\text{for } l=1,2,$$

then the integral in (26) can be written as an L^2 inner product $\langle G_j,F_1F_2\rangle$. Since u and v (acting as time-dependent functions) have compact support in $[-T,T]$, so does F_1F_2. As a result, the inner product $\langle G_j,F_1F_2\rangle$ can be restricted on the interval $[-T,T]$, namely, we may assume that G_j has the same compact support (with respect to time) as F_1F_2's. Now, an application of (20) yields that the sum in (26) is bounded by a constant multiple of

$$\begin{aligned}
&\sum_{j\geqslant 0}2^{-j(\frac12-)}\langle G_j,F_1F_2\rangle\\
&\quad\lesssim\sum_{j,j_1,j_2\geqslant 0}\big(2^{-j(\frac12-)}2^{-j_1(\frac12+)}2^{-j_2(\frac12+)}\\
&\qquad\times\big\||-i\partial_x|^{\frac14}\big(\eta_{j_1}(\sigma_1)f_1\big)^\vee\big\|_{L^4_TL^4(\mathbb{R}^2)}\big\||-i\partial_x|^{\frac14}\big(\eta_{j_2}(\sigma_2)f_2\big)^\vee\big\|_{L^4_TL^4(\mathbb{R}^2)}\big\|\eta_j(\sigma)g\big\|_{L^2(\mathbb{R}^3)}\big)\\
&\quad\lesssim\sum_{j,j_1,j_2\geqslant 0}\big(2^{-j(\frac12-)}2^{-j_1[(\frac12+)-\frac12]}2^{-j_2[(\frac12+)-\frac12]}\big\|\eta_{j_1}(\sigma_1)f_1\big\|_{L^2(\mathbb{R}^3)}\big\|\eta_{j_2}(\sigma_2)f_2\big\|_{L^2(\mathbb{R}^3)}\big\|\eta_j(\sigma)g\big\|_{L^2(\mathbb{R}^3)}\big)\\
&\quad\lesssim\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

∘ *Low + Low → Low* $|\xi_1|,|\xi_2|\lesssim\max\{15,|\alpha|\}$. Via changing variables and using the Cauchy–Schwarz inequality we can bound (25) with

$$\int K_{ll}\left(\int\big|f_1(\xi_1,\mu_1,\tau_1)f_2(\xi-\xi_1,\mu-\mu_1,\tau-\tau_1)\big|^2\,d\tau_1\,d\xi_1\,d\mu_1\right)^{\frac12}g(\xi,\mu,\tau)\,d\xi\,d\mu\,d\tau,$$

where

$$K_{ll}=\frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{d\tau_1\,d\xi_1\,d\mu_1}{\langle\tau_1-\omega(\xi_1,\mu_1)\rangle^{1+}\langle\tau-\tau_1-\omega(\xi-\xi_1,\mu-\mu_1)\rangle^{1+}}\right)^{\frac{1}{2}}.$$

We need only to control K_{ll} using a constant independent of ξ,μ,τ. By (21) we have:

$$K_{ll}\lesssim\frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{d\xi_1\,d\mu_1}{\langle\tau-\omega(\xi,\mu)-\omega(\xi-\xi_1,\mu-\mu_1)\rangle^{1+}}\right)^{\frac{1}{2}}.$$

An elementary computation with the change of variables,

$$\nu=\tau-\omega(\xi,\mu)-\omega(\xi-\xi_1,\mu-\mu_1),$$

shows

$$\left|\frac{d\nu}{d\mu_1}\right|\gtrsim|\xi|^{\frac{1}{2}}\left|\sigma+\xi\xi_1(\xi-\xi_1)\left(5\xi^2-5\xi\xi_1+5\xi_1^2-3\alpha\right)-\nu\right|^{\frac{1}{2}},$$

and consequently

$$K_{ll}\lesssim\frac{|\xi|^{\frac{3}{4}}}{\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{d\xi_1\,d\nu}{\langle\nu\rangle^{1+}|\sigma+\xi\xi_1(\xi-\xi_1)(5\xi^2-5\xi\xi_1+5\xi_1^2-3\alpha)-\nu|^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

By (22) we further get:

$$K_{ll}\lesssim\left(\int\limits_{|\xi_1|\lesssim\max\{15,|\alpha|\}}\frac{d\xi_1}{\langle\sigma+\xi\xi_1(\xi-\xi_1)(5\xi^2-5\xi\xi_1+5\xi_1^2-3\alpha)\rangle^{\frac{1}{2}}}\right)^{\frac{1}{2}}\lesssim 1.$$

• **Situation 2—High frequency** $|\xi_1+\xi_2|\gtrsim\max\{10,|\alpha|\}$.

◦ *High + Low → High* $|\xi_2|\lesssim\max\{10,|\alpha|\}\lesssim|\xi|\sim|\xi_1|$. As above, we apply the Cauchy–Schwarz inequality to bound the integral in (25) from above with a constant multiple of

$$\int K_{hl}\left(\int\left|f_1(\xi_1,\mu_1,\tau_1)f_2(\xi-\xi_1,\mu-\mu_1,\tau-\tau_1)\right|^2d\tau_1\,d\xi_1\,d\mu_1\right)^{\frac{1}{2}}g(\xi,\mu,\tau)\,d\xi\,d\mu\,d\tau,$$

where

$$K_{hl}=\frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{d\tau_1\,d\xi_1\,d\mu_1}{\langle\tau_1-\omega(\xi_1,\mu_1)\rangle^{1+}\langle\tau-\tau_1-\omega(\xi-\xi_1,\mu-\mu_1)\rangle^{1+}}\right)^{\frac{1}{2}},$$

but also we have the following estimate:

$$K_{hl}\lesssim\frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{d\xi_1\,d\mu_1}{\langle\tau-\omega(\xi,\mu)-\omega(\xi-\xi_1,\mu-\mu_1)\rangle^{1+}}\right)^{\frac{1}{2}}.$$

Under the change of variables,

$$\kappa=\xi\xi_1(\xi-\xi_1)\left(5\xi^2-5\xi\xi_1+5\xi_1^2-3\alpha\right);\qquad\nu=\tau-\omega(\xi,\mu)-\omega(\xi-\xi_1,\mu-\mu_1),$$

the Jacobian determinant J enjoys

$$J\lesssim\frac{|\kappa|^{\frac{1}{2}}}{|\xi|^{\frac{7}{2}}|\sigma+\kappa-\nu|^{\frac{1}{2}}(|\xi|^5-2|\kappa|)^{\frac{1}{2}}}.$$

As a by-product of the last inequality and (22), we obtain:

$$K_{hl}\lesssim\frac{1}{|\xi|^{\frac{3}{4}}\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{|\kappa|^{\frac{1}{2}}\,d\kappa\,d\nu}{|\sigma+\kappa-\nu|^{\frac{1}{2}}(|\xi|^5-2|\kappa|)^{\frac{1}{2}}\langle\nu\rangle^{1+}}\right)^{\frac{1}{2}}\lesssim\frac{1}{|\xi|^{\frac{3}{4}}\langle\sigma\rangle^{\frac{1}{2}-}}\left(\int\frac{|\kappa|^{\frac{1}{2}}d\kappa}{\langle\sigma+\kappa\rangle^{\frac{1}{2}}(|\xi|^5-2|\kappa|)^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

Since $|\xi-\xi_1|\lesssim\max\{10,|\alpha|\}$, we have $|\kappa|\lesssim|\xi|^4$, whence getting

$$K_{hl}\lesssim\frac{1}{|\xi|^2\langle\sigma\rangle^{\frac{1}{2}-}}\biggl(\int\limits_{|\kappa|\lesssim|\xi|^4}\frac{d\kappa}{\langle\sigma+\kappa\rangle^{\frac{1}{2}}}\biggr)^{\frac{1}{2}}\lesssim 1.$$

∘ *High + High → High* $|\xi_1|,|\xi_2|\gtrsim\max\{10,|\alpha|\}$. Since $|\xi_1|\geqslant|\xi_2|$, we have $|\xi_1|\gtrsim|\xi_1+\xi_2|$. Under this circumstance, we will deal with two cases in the sequel.

⋄ *Case* (i) $\max\{|\sigma|,|\sigma_2|\}\gtrsim|\xi_1|^2$. Decomposing the integral according to $|\xi_1|\sim 2^m$ where $m=1,2,\ldots,$ we can run the dyadic decomposition:

$$|\sigma|\sim 2^j,\qquad|\sigma_1|\sim 2^{j_1},\qquad|\sigma_2|\sim 2^{j_2}\quad\text{for } j,j_1,j_2=0,1,2,\ldots.$$

If $|\sigma|\geqslant|\sigma_2|\geqslant|\xi_1|^2$, then an application of (20) yields that the integral in (25) is bounded from above by a constant multiple of,

$$\begin{aligned}
&\sum_{m\geqslant1}\sum_{j\geqslant2m}\sum_{j_1,j_2\geqslant0}\bigl(2^{\frac{3m}{4}}2^{-j(\frac12-)}2^{-j_1(\frac12+)}2^{-j_2(\frac12+)}\bigl\|\eta_j(\sigma)g\bigr\|_{L^2(\mathbb{R}^3)}\\
&\qquad\times\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_m(\xi_1)\eta_{j_1}(\sigma_1)f_1\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_{j_2}(\sigma_2)f_2\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigr)\\
&\quad\lesssim\sum_{m\geqslant1}\sum_{j\geqslant2m}\sum_{j_1,j_2\geqslant0}\bigl(2^{-j(\frac12-)}2^{\frac{3m}{4}}2^{-j_1[(\frac12+)-\frac12]}2^{-j_2[(\frac12+)-\frac12]}\bigl\|\eta_{j_1}(\sigma_1)f_1\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_{j_2}(\sigma_2)f_2\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_j(\sigma)g\bigr\|_{L^2(\mathbb{R}^3)}\bigr)\\
&\quad\lesssim\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

If $|\sigma_2|\geqslant|\sigma|\geqslant|\xi_1|^2$, then a further use of (20) derives that the integral in (25) is bounded from above by a constant multiple of,

$$\begin{aligned}
&\sum_{m\geqslant1}\sum_{j_2\geqslant2m,j\geqslant0}\sum_{j_1\geqslant0}\bigl(2^{\frac{m}{2}}2^{-j(\frac12-)}2^{-j_1(\frac12+)}2^{-j(\frac12+)}\bigl\|\eta_{j_2}(\sigma_2)f_2\bigr\|_{L^2(\mathbb{R}^3)}\\
&\qquad\times\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_m(\xi_1)\eta_{j_1}(\sigma_1)f_1\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_j(\sigma)g\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigr)\\
&\quad\lesssim\sum_{m\geqslant1}\sum_{j_2\geqslant2m}\sum_{j_1,j_2\geqslant0}\bigl(2^{-j_2(\frac14+)}2^{\frac{m}{2}}2^{-j_1[(\frac12+)-\frac12]}2^{-j[(\frac34-)-\frac12]}\bigl\|\eta_{j_1}(\sigma_1)f_1\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_{j_2}(\sigma_2)f_2\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_j(\sigma)g\bigr\|_{L^2(\mathbb{R}^2)}\bigr)\\
&\quad\lesssim\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

⋄ *Case* (ii) $\max\{|\sigma|,|\sigma_2|\}\lesssim|\xi_1|^2$. In this case, we need to consider the size of the resonance function even more carefully. This consideration will be done via splitting the estimate into two pieces according to the size of resonance function.

▹ *Subcase* (i) $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\gtrsim|\xi_1|^4$. This means that the resonant interaction does not happen and consequently $|\sigma_1|\gtrsim|\xi_1|^4$. The dyadic decomposition and (20) are applied to deduce that the integral in (25) is bounded from above by a constant multiple of,

$$\begin{aligned}
&\sum_{m\geqslant1}\sum_{j_1\geqslant4m}\sum_{2m\geqslant j,j_2\geqslant0}\bigl(2^{\frac34m}2^{-j(\frac12-)}2^{-j_1(\frac12+)}2^{-j_2(\frac12+)}\bigl\|\eta_m(\xi_1)\eta_{j_1}(\sigma_1)f_1\bigr\|_{L^2(\mathbb{R}^3)}\\
&\qquad\times\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_{j_2}(\sigma_2)f_2\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigl\||-i\partial_x|^{\frac14}\bigl(\eta_j(\sigma)g\bigr)^\vee\bigr\|_{L^4_TL^4(\mathbb{R}^2)}\bigr)\\
&\quad\lesssim\sum_{m\geqslant1}\sum_{j_1\geqslant4m}\sum_{2m\geqslant j,j_2\geqslant0}\bigl(2^{\frac{3m}{4}}2^{-j_1(\frac14+)}2^{-j[(\frac34-)-\frac12]}2^{-j_2[(\frac12+)-\frac12]}\\
&\qquad\times\bigl\|\eta_{j_1}(\sigma_1)f_1\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_{j_2}(\sigma_2)f_2\bigr\|_{L^2(\mathbb{R}^3)}\bigl\|\eta_j(\sigma)g\bigr\|_{L^2(\mathbb{R}^3)}\bigr)\\
&\quad\lesssim\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

▹ *Subcase* (ii) $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \lesssim |\xi_1|^4$. This means that the resonant interaction does happen. By the definition of the resonant function we have:

$$\left|\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right|^2 > 2^{-1}|\xi_1+\xi_2|^2\left|5\left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)-3\alpha\right|.$$

Let

$$\theta_1=\tau_1-\omega(\xi_1,\mu_1); \qquad \theta_2=\tau_2-\omega(\xi_2,\mu_2),$$

and A_{j,j_1,j_2} be the image of set of all points $(\xi_1,\xi_2,\mu_1,\mu_2,\tau_1,\tau_2)\in A^*$ that satisfy the following three conditions:

$$\min\{|\xi|,|\xi_1|,|\xi_2|\} \gtrsim \max\{10,|\alpha|\};$$
$$|\sigma|\sim 2^j, \qquad |\sigma_1|\sim 2^{j_1}, \qquad |\sigma_2|\sim 2^{j_2};$$
$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \lesssim |\xi_1|^4,$$

under the transformation $(\xi_1,\mu_1,\tau_1,\xi_2,\mu_2,\tau_2)\mapsto(\xi_1,\mu_1,\theta_1,\xi_2,\mu_2,\theta_2)$. If in addition,

$$f_{j_1}=\eta_{j_1}(\sigma_1)f_1(\xi_1,\mu_1,\tau_1); \qquad f_{j_2}=\eta_{j_2}(\sigma_2)f_2(\xi_2,\mu_2,\tau_2),$$

then the integral in (25) is controlled from above by a constant multiple of,

$$\sum_{j>0}\sum_{j_1,j_2\geqslant 0}\Bigg(2^{-j(\frac12-)}2^{-j_1(\frac12+)}2^{-j_2(\frac12+)}\int\limits_{A_{j,j_1,j_2}}\Big[|\xi|g\big(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2)\big)$$
$$\times\eta_j\big(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\big)$$
$$\times f_{j_1}\big(\xi_1,\mu_1,\theta_1+\omega(\xi_1,\mu_1)\big)f_{j_2}\big(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\mu_2)\big)\Big]\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\,d\theta_1\,d\theta_2\Bigg). \tag{27}$$

To get the desired estimate, we are led to dominate the following sum for each fixed natural number j:

$$\sum_{j_1,j_2\geqslant 0}\Bigg(2^{-j_1(\frac12+)}2^{-j_2(\frac12+)}\int\limits_{A_{j,j_1,j_2}}\Big[|\xi|g\big(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2)\big)$$
$$\times\eta_j\big(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\big)$$
$$\times f_{j_1}\big(\xi_1,\mu_1,\theta_1+\omega(\xi_1,\mu_1)\big)f_{j_2}\big(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\mu_2)\big)\Big]\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\,d\theta_1\,d\theta_2\Bigg). \tag{28}$$

This will be accomplished via considering two more settings.

⋆ *Subsubcase* (i)

$$\left|5\left(\xi_1^4-\xi_2^4\right)-3\alpha\left(\xi_1^2-\xi_2^2\right)-\left[\left(\frac{\mu_1}{\xi_1}\right)^2-\left(\frac{\mu_2}{\xi_2}\right)^2\right]\right|>2^j.$$

Under this circumstance, we change the variables

$$\begin{cases} u=\xi_1+\xi_2,\\ v=\mu_1+\mu_2,\\ w=\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2),\\ \mu_2=\mu_2, \end{cases} \tag{29}$$

and then obtain its Jacobian determinant

$$J_\mu=\begin{vmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 5\xi_1^4-3\alpha\xi_1^2-\frac{\mu_1^2}{\xi_1^2} & 5\xi_2^4-3\alpha\xi_2^2-\frac{\mu_2^2}{\xi_2^2} & 2\frac{\mu_1}{\xi_1} & 2\frac{\mu_2}{\xi_2}\\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$=5\left(\xi_1^4-\xi_2^4\right)-3\alpha\left(\xi_1^2-\xi_2^2\right)-\left[\left(\frac{\mu_1}{\xi_1}\right)^2-\left(\frac{\mu_2}{\xi_2}\right)^2\right]. \tag{30}$$

Suppose now $A^{(1)}_{j,j_1,j_2}$ is the image of the subset of all points $(\xi_1,\mu_1,\theta_1,\xi_2,\mu_2,\theta_2)\in A_{j,j_1,j_2}$ obeying the just-assumed subsubcase (i) condition under the transformation (29). Then it is not hard to deduce that $|J_\mu|\gtrsim 2^j$ and so that the sum in (28) is

$$\lesssim \sum_{j_1,j_2\geqslant 0} 2^{-j_1(\frac{1}{2}+)}2^{-j_2(\frac{1}{2}+)} \int\limits_{A^{(1)}_{j,j_1,j_2}} \frac{|u|g(u,v,w)}{|J_\mu|} H(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\,d\theta_1\,d\theta_2, \tag{31}$$

where $H(u,v,w,\mu_2,\theta_1,\theta_2)$ is just $\eta_j f_{j_1} f_{j_2}$ with respect to the transformation (29). For the fixed variables: $\theta_1,\theta_2,\xi_1,\xi_2,\mu_1$, we calculate the set length, denoted by Δ_{μ_2}, where the free variable μ_2 can range. More precisely, if

$$f(\mu)=\theta_1+\theta_2-\frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left((\xi_1+\xi_2)^2\left[5\left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)-3\alpha\right]-\left(\frac{\mu_1}{\xi_1}-\frac{\mu}{\xi_2}\right)^2\right),$$

then $|f'(\mu_2)|>|\xi_1|^2\gtrsim|u|^2$, and hence $\Delta_{\mu_2}\lesssim 2^j|u|^{-2}$ follows from:

$$\begin{aligned}
&\left|\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\right| \\
&\quad=\left|\theta_1+\theta_2-\frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left((\xi_1+\xi_2)^2\left[5\left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)-3\alpha\right]-\left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right)^2\right)\right|\sim 2^j.
\end{aligned}$$

By the Cauchy–Schwarz inequality and the inverse change of variables we have:

$$\begin{aligned}
&\int\limits_{A^{(1)}_{j,j_1,j_2}} |u|g(u,v,w)|J_\mu|^{-1}H(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\,d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{\frac{j}{2}}\int |u|g(u,v,w)\left(\int |J_\mu|^{-2}H^2(u,v,w,\mu_2,\theta_1,\theta_2)\,d\mu_2\right)^{\frac{1}{2}} du\,dv\,dw\,d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{\frac{j}{2}}\|g\|_{L^2(\mathbb{R}^3)}\int\left(\int |J_\mu|^{-1}H^2(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\right)^{\frac{1}{2}} d\theta_1\,d\theta_2 \\
&\quad\lesssim \|g\|_{L^2(\mathbb{R}^3)}\int\left(\int\prod_{i=1,2} f_{j_i}^2\big(\xi_i,\mu_i,\theta_i+\omega(\xi_i,\mu_i)\big)\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\right)^{\frac{1}{2}} d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{\frac{j_1}{2}}2^{\frac{j_2}{2}}\|g\|_{L^2(\mathbb{R}^3)}\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

It follows from (28) that the sum in (27) is $\lesssim\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}$.

⋆ *Subsubcase* (ii)

$$\left|5\left(\xi_1^4-\xi_2^4\right)-3\alpha\left(\xi_1^2-\xi_2^2\right)-\left[\left(\frac{\mu_1}{\xi_1}\right)^2-\left(\frac{\mu_2}{\xi_2}\right)^2\right]\right|\leqslant 2^j.$$

In this setting, the change of variables taken in subsubcase (i) does not work because the determinant of the Jacobian may be zero. So, we cannot help finding a new change of variables. Before doing this, we notice that the size $|\xi_1|\sim 2^m$ (for $m\geqslant 0$) can be used but also the integral in (25) may be rewritten as

$$\begin{aligned}
\sum_{j_1,m\geqslant 0}\ \sum_{2m>j,j_2\geqslant 0}\Bigg(&2^{-j(\frac{1}{2}-)}2^{-j_1(\frac{1}{2}+)}2^{-j_2(\frac{1}{2}+)}2^m\int\limits_{A_{j,j_1,j_2}}\Big[g\big(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2)\big) \\
&\times\eta_j\big(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\big) \\
&\times f_{m,j_1}\big(\xi_1,\mu_1,\theta_1+\omega(\xi_1,\mu_1)\big)f_{j_2}\big(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\mu_2)\big)\Big]\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\,d\theta_1\,d\theta_2\Bigg),
\end{aligned}\tag{32}$$

where $f_{m,j_1} = \eta_m(\xi_1)\eta_{j_1}(\sigma_1) f_1(\xi_1, \mu_1, \tau_1)$. Now, we choose the following transformation:

$$\begin{cases} u = \xi_1 + \xi_2, \\ v = \mu_1 + \mu_2, \\ w = \theta_1 + \omega(\xi_1, \mu_1) + \theta_2 + \omega(\xi_2 + \mu_2), \\ \xi_1 = \xi_1, \end{cases} \tag{33}$$

and moreover assume that $A^{(2)}_{j,j_1,j_2}$ is the image under (33) of the set of those points $(\xi_1, \mu_1, \theta_1, \xi_2, \mu_2, \theta_2) \in A_{j,j_1,j_2}$ satisfying the just-given subsubcase (ii) condition. A calculation yields that the associated Jacobian determinant of the last transformation (33) is:

$$J_\xi = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 5\xi_1^4 - 3\alpha\xi_1^2 - \frac{\mu_1^2}{\xi_1^2} & 5\xi_2^4 - 3\alpha\xi_2^2 - \frac{\mu_2^2}{\xi_2^2} & 2\frac{\mu_1}{\xi_1} & 2\frac{\mu_2}{\xi_2} \\ 1 & 0 & 0 & 0 \end{vmatrix} = 2\left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}\right). \tag{34}$$

From this formula it follows that $|J_\xi| \gtrsim |\xi_1|$. Next, we fix $\theta_1, \theta_2, \xi_2, \mu_1, \mu_2$, and estimate the interval length Δ_{ξ_1} of the free variable ξ_1. Putting

$$h(\xi) = 5\big(\xi^4 - \xi_2^4\big) - 3\alpha\big(\xi^2 - \xi_2^2\big) - \left[\left(\frac{\mu_1}{\xi}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2\right], \tag{35}$$

we compute

$$h'(\xi) = 20\xi^3 - 6\alpha\xi + 2(\mu_1/\xi)^2\xi^{-1}. \tag{36}$$

Since now $h'(\xi_1)$ has the same sign as ξ_1's, we conclude $|h'(\xi_1)| \gtrsim |\xi_1|^3$, thereby finding $\Delta_{\xi_1} \lesssim 2^{j-3m}$. Consequently, the sum in (32) is:

$$\lesssim \sum_{j_1,m\geqslant 0}\ \sum_{2m>j,j_2\geqslant 0} 2^{j(-\frac{1}{2}+)}2^m \int\limits_{A^{(2)}_{j,j_1,j_2}} \frac{g(u,v,w)}{|J_\xi|} H(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\,d\theta_1\,d\theta_2, \tag{37}$$

where $H(u,v,w,\xi_1,\theta_1,\theta_2)$ equals $\eta_j f_{m,j_1} f_{j_1}$ under the change of variables (33). Note that by the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\int\limits_{A^{(2)}_{j,j_1,j_2}} \frac{g(u,v,w)}{|J_\xi|} H(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\,d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{-\frac{3}{2}m}2^{\frac{j}{2}} \int g(u,v,w)\left(\int |J_\xi|^{-2} H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,d\xi_1\right)^{\frac{1}{2}} du\,dv\,dw\,d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{-\frac{3}{2}m}2^{\frac{j}{2}} \|g\|_{L^2(\mathbb{R}^3)} \int\left(\int |J_\xi|^{-2} H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\right)^{\frac{1}{2}} d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{-2m}2^{\frac{j}{2}} \|g\|_{L^2(\mathbb{R}^3)} \int\left(\int |J_\xi|^{-1} H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\right)^{\frac{1}{2}} d\theta_1 d\theta_2 \\
&\quad\lesssim 2^{-2m}2^{\frac{j}{2}} \|g\|_{L^2(\mathbb{R}^3)} \int\left(\int \prod_{l=1,2} f_l^2\big(\xi_l,\mu_l,\theta_l + \omega(\xi_l,\mu_l)\big)\,d\xi_l\,d\mu_l\right)^{\frac{1}{2}} d\theta_1\,d\theta_2 \\
&\quad\lesssim 2^{-2m}2^{\frac{j}{2}}2^{\frac{j_1}{2}}2^{\frac{j_2}{2}} \|g\|_{L^2(\mathbb{R}^3)}\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Thus the sum in (32) is

$$\begin{aligned}
&\lesssim \sum_{m,j_1\geqslant 0}\ \sum_{2m>j_2,j\geqslant 0} \big(2^{-j(\frac{1}{2}-)}2^{-m}2^{\frac{j}{2}}2^{-j_1((\frac{1}{2}+)-\frac{1}{2})}2^{j_2((\frac{1}{2}+)-\frac{1}{2})}\|g\|_{L^2(\mathbb{R}^3)}\|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}\big) \\
&\lesssim \|f_1\|_{L^2(\mathbb{R}^3)}\|f_2\|_{L^2(\mathbb{R}^3)}. \qquad \square
\end{aligned}$$

4. Proof of Theorem 1.1

• **Local well-posedness.** Consider the integral equation associated with (1),

$$u(t)=\psi(t)S(t)\phi-\frac{\psi_T(t)}{2}\int_0^t S(t-t')\partial_x\big(u^2(t')\big)\,dt', \tag{38}$$

where $0<T<1$, and $\psi_T(t)$ is the bump function defined in Section 2. It is clear that a solution to (38) is a fixed point of the nonlinear operator:

$$L(u)=\psi(t)S(t)\phi-\frac{\psi_T(t)}{2}\int_0^t S(t-t')\partial_x\big(u^2(t')\big)\,dt'. \tag{39}$$

Therefore we are required to verify that L is a contractive mapping from the following closed set to itself

$$B_a=\big\{u\in X_b^{s_1,s_2}\colon\ \|u\|_{X_b^{s_1,s_2}}\leqslant a=4c\|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)},\ 2^{-1}<b\big\}. \tag{40}$$

Here and hereafter $c>0$ is a time-free constant and may vary from one line to the other. By Proposition 2.1 and Theorem 3.1, there exists $\sigma>0$ such that

$$\big\|L(u)\big\|_{X_b^{s_1,s_2}}\leqslant c\|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}+cT^\sigma\|u\|^2_{X_b^{s_1,s_2}}. \tag{41}$$

Next, since $\partial_x(u^2)-\partial_x(v^2)=\partial_x[(u-v)(u+v)]$, we similarly get:

$$\big\|L(u)-L(v)\big\|_{X_b^{s_1,s_2}}\leqslant cT^\sigma\|u-v\|_{X_b^{s_1,s_2}}\big(\|u\|_{X_b^{s_1,s_2}}+\|v\|_{X_b^{s_1,s_2}}\big). \tag{42}$$

Choosing $T=T(\|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)})$ such that $8cT^\sigma\|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}<1$, we deduce from (41) and (42) that L is strictly contractive on the ball B_a. Thus there exists a unique solution $u\in X_b^{s_1,s_2}([-T,T])\subseteq C([-T,T];H^{s_1,s_2}(\mathbb{R}^2))$ (thanks to $b>1/2$) to the IVP of the fifth order KP-I equation. The smoothness of the mapping from $H^{s_1,s_2}(\mathbb{R}^2)$ to $X_b^{s_1,s_2}([-T,T])$ follows from the fixed point argument. Because the dispersive function $\omega(\xi,\mu)$ is singular at $\xi=0$, the requirement $|\xi|^{-1}\hat{\phi}\in\mathcal{S}'(\mathbb{R}^2)$ is necessary in order to have a well defined time derivative of $S(t)\phi$. So, the argument for the local well-posedness is complete.

• **Global well-posedness.** We first handle the global well-posedness of (1) in the anisotropic Sobolev space $H^{s_1,0}(\mathbb{R}^2)$ with $s_1\geqslant 0$. Suppose $\phi\in H^{s_1,0}(\mathbb{R}^2)$. Then by local well-posedness there exists a unique solution $u\in C([-T,T];H^{s_1,0}(\mathbb{R}^2))$ of (1). We claim that there exists T, depending on $\|\phi\|_{L^2(\mathbb{R}^2)}$, such that on the interval $[-T,T]$ one has:

$$\sup_{|t|\leqslant T}\big\|u(t)\big\|_{H^{s_1,0}(\mathbb{R}^2)}\leqslant c\|\phi\|_{H^{s_1,0}(\mathbb{R}^2)}. \tag{43}$$

With the help of this claim and the local well-posedness part of Theorem 1.1 with $u(T)$ and $u(-T)$ being initial values, we can extend the exit time to the positive infinity step by step in that the exist time T' depends only on,

$$\big\|u(T)\big\|_{L^2(\mathbb{R}^2)}=\big\|u(-T)\big\|_{L^2(\mathbb{R}^2)}=\|\phi\|_{L^2(\mathbb{R}^2)},$$

and

$$\max\big\{\big\|u(T)\big\|_{H^{s_1,0}(\mathbb{R}^2)},\ \big\|u(-T)\big\|_{H^{s_1,0}(\mathbb{R}^2)}\big\}\leqslant c\|\phi\|_{H^{s_1,0}(\mathbb{R}^2)}.$$

To check the claim, let $J_x^{s_1}=(I-\partial_x^2)^{s_1/2}$. Then from the definitions of the anisotropic Sobolev space and the Bourgain space it follows that

$$\big\|J_x^{s_1}u\big\|_{L^2(\mathbb{R}^2)}=\|u\|_{H^{s_1,0}(\mathbb{R}^2)}\quad\text{and}\quad\big\|J_x^{s_1}u\big\|_{X_b^{0,0}}=\|u\|_{X_b^{s_1,0}}.$$

Letting $J_x^{s_1}$ act on both sides of the integral equation (38), we derive

$$J_x^{s_1} u(t) = \psi(t) S(t) J_x^{s_1} \phi - \frac{\psi_T(t)}{2} \int_0^t S(t-t') J_x^{s_1} \partial_x \big(u^2(t')\big)\, dt'. \tag{44}$$

By Proposition 2.1, we have:

$$\big\| \psi S(t) J_x^{s_1} \phi \big\|_{X_b^{0,0}} \leqslant c \|\phi\|_{H^{s_1,0}(\mathbb{R}^2)}, \tag{45}$$

as well as

$$\left\| \psi_T(t) \int_0^t S(t-t') J_x^{s_1} \partial_x \big(u^2(t')\big)\, dt' \right\|_{X_b^{0,0}} \leqslant c T^{1-b+b'} \big\| J_x^{s_1} \partial_x \big(u^2(t')\big) \big\|_{X_{b'}^{0,0}}. \tag{46}$$

A slight modification of the argument for the bilinear estimates carried out in Section 3 can produce the following bilinear estimate:

$$\big\| J_x^{s_1} \partial_x \big(u^2\big) \big\|_{X^{0,0}_{-\frac{1}{2}+}} \leqslant c \|u\|_{X^{0,0}_{\frac{1}{2}+}} \big\| J_x^{s_1} u \big\|_{X^{0,0}_{\frac{1}{2}+}}. \tag{47}$$

Combining (45), (46) and (47), we get:

$$\big\| J_x^{s_1} u \big\|_{X_b^{0,0}} \leqslant c \|\phi\|_{H^{s_1,0}} + c T^{\sigma} \|u\|_{X_b^{0,0}} \big\| J_x^{s_1} u \big\|_{X_b^{0,0}}.$$

By (41) with $s_1 = s_2 = 0$, we can choose $T = T(\|\phi\|_{L^2(\mathbb{R}^2)})$ such that $cT^{\sigma}\|u\|_{X_b^{0,0}} < \frac{1}{2}$. Thus by (47), we have:

$$\big\| J_x^{s_1} \partial_x \big(u^2\big) \big\|_{X_b^{0,0}} \leqslant c \|\phi\|_{H^{s_1,0}(\mathbb{R}^2)} + 2^{-1} \big\| J_x^{s_1} u \big\|_{X_b^{0,0}}.$$

Since $b > \frac{1}{2}$, we obtain the fundamental embedding inequality:

$$\sup_{|t| \leqslant T} \big\| u(t) \big\|_{H^{s_1,0}(\mathbb{R}^2)} \leqslant \big\| J_x^{s_1} u \big\|_{X_b^{0,0}} \leqslant c \|\phi\|_{H^{s_1,0}(\mathbb{R}^2)},$$

as well as (43) which verifies the claim.

Similarly, the operator $J_y^{s_2} = (I - \partial_y^2)^{s_2/2}$ can act on both side of the integral equation (38). As a result, we get:

$$\big\| J_y^{s_2} \partial_x \big(u^2\big) \big\|_{X^{s_1,0}_{-\frac{1}{2}+}} \leqslant c \|u\|_{X^{s_1,0}_{\frac{1}{2}+}} \big\| J_y^{s_2} u \big\|_{X^{s_1,0}_{\frac{1}{2}+}}, \tag{48}$$

thereby obtaining the following estimate:

$$\big\| J_y^{s_2} \partial_x \big(u^2\big) \big\|_{X^{s_1,0}_{-\frac{1}{2}+}} \leqslant c \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)} + c T^{\sigma} \|u\|_{X_b^{s_1,0}} \big\| J_y^{s_2} u \big\|_{X_b^{s_1,0}}.$$

By (43), we can also choose a time T so that it depends on $\|\phi\|_{H^{s_1,0}(\mathbb{R}^2)}$ and obeys $cT^{\sigma}\|u\|_{X^{s_1,0}} < \frac{1}{2}$. Finally, we arrive at

$$\sup_{|t| \leqslant T} \big\| u(t) \big\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \sup_{|t| \leqslant T} \big\| J_y^{s_2} u(t) \big\|_{H^{s_1,0}(\mathbb{R}^2)} \leqslant \big\| J_y^{s_2} u \big\|_{X_b^{s_1,s_2}} \leqslant c \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)}.$$

Note that the previous constant $c > 0$ is time-free. So, as before we can extend the exist time to infinity step by step, and therefore finish the proof of the global well-posedness.

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