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Higher order optimality conditions for Henig efficient solutions in set-valued optimization ☆

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Abstract

In this paper, higher order generalized contingent epiderivative and higher order generalized adjacent epiderivative of set-valued maps are introduced. Necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem are given by employing the higher order generalized epiderivatives.

Keywords: Higher order generalized (contingent) adjacent epiderivative; Henig efficiency; Higher order optimality conditions; Set-valued optimization

1. Introduction

There has been an increasing interest in optimality conditions of set-valued optimization problems since many optimization problems encountered in economics, engineering and other fields involve vector-valued mappings (or set-valued mappings) as constraints and objectives (see [7,12]). Until now, various derivative-like notions have been proposed to express these optimality conditions. In [4], Corley investigated first order Fritz John type necessary and sufficient

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conditions for general set-valued optimization problems by virtue of Clarke tangent derivative and contingent derivative (see [1]). In [6], Jahn and Rauh introduced a single-valued map as a contingent epiderivative of a set-valued map and obtained an unified necessary and sufficient condition. But the existence of the contingent epiderivative of a set-valued map in a general setting is still an open question. To overcome the difficulty, Chen and Jahn [3] introduced a generalized contingent epiderivative of a set-valued map which is a set-valued map. They proved that the generalized contingent epiderivative exists under standard assumptions and obtained an unified necessary and sufficient condition. In [9], Jahn and Khan investigated optimality conditions of local proper minimizers, local weak minimizers and local strong minimizers for general set-valued optimization problems by using the generalized contingent epiderivative. In [5], Gong et al. investigated necessary and sufficient conditions for five kinds of properly efficient solutions of a set-valued optimization problem with constraints by virtue of so called contingent epiderivative, Clarke tangent epiderivative and radial epiderivative. In general, since the epigraph of a set-valued map has nicer properties than the graph of a set-valued map, it is advantageous to employ the epiderivatives in set-valued optimization.

Recently, Jahn et al. [8] introduced second-order contingent epiderivative and generalized contingent epiderivative for set-valued maps and obtained some second-order optimality conditions based on these concepts. In [10], Li et al. studied some properties of higher order tangent sets and higher order derivatives introduced in [1] and then obtained higher order necessary and sufficient optimality conditions for set-valued optimization problems in terms of the higher order derivatives. By using these concepts, they also discussed higher order Mond–Weir duality for set-valued optimization in [11].

Motivated by the work reported in [3,5,8,10], we introduce the definitions of higher order generalized contingent epiderivative and higher order generalized adjacent epiderivative. Then, we discuss their some properties under the condition that set-valued maps are cone-convex. Finally, based on the higher order generalized adjacent epiderivative and contingent epiderivative, we investigate higher order necessary and sufficient optimality conditions for Henig properly efficient solutions of a set-valued optimization problem with constraints.

The rest of the paper is organized as follows. In Section 2, we collect some concepts and recall the definitions of the higher order tangent sets and some of their properties. In Section 3, we introduce the definitions of the higher order generalized contingent epiderivative and adjacent epiderivative. Then, we discuss the existence theorem and their properties. In Section 4, we introduce a constrained set-valued optimization problem and the concept of a Henig efficient solution, and then obtain higher order necessary optimality conditions of the set-valued optimization problem. In Section 5, we establish higher order Fritz John type necessary and sufficient optimality conditions of the set-valued optimization problem.

2. Mathematical preliminaries and higher order tangent sets

Throughout this paper, let X, Y and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed convex cones $C \subset Y$ and $D \subset Z$, respectively. We also assume that Y^* is the topological dual space of Y, S is a nonempty subset of X and $F : S \to 2^Y$ and $G : S \to 2^Z$ are two given set-valued maps. The domain, the graph and the epigraph of F are defined by

$$\operatorname{dom}(F) = \{x \in S\colon\ F(x) \neq \emptyset\},$$
$$\operatorname{graph}(F) = \{(x, y) \in X \times Y\colon\ x \in S,\ y \in F(x)\},$$

$$\mathrm{epi}(F) = \big\{(x, y) \in X \times Y\colon\ x \in S,\ y \in F(x) + C\big\}.$$

The profile map $F_+ : S \to 2^Y$ is defined by $F_+(x) = F(x) + C$, for every $x \in \mathrm{dom}(F)$. Let S be convex. The map F is said to be C-convex on S, if, for any $x_1, x_2 \in S$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F\big(\lambda x_1 + (1 - \lambda)x_2\big) + C.$$

It is well known that if F is C-convex on S, then $\mathrm{epi}(F)$ is a convex subset in $X \times Y$. Let C^* be the dual cone of cone C, defined by

$$C^* = \big\{\lambda \in Y^*\colon\ \lambda(y) \geqslant 0 \text{ for all } y \in C\big\}.$$

Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp = \big\{\lambda \in Y^*\colon\ \lambda(y) > 0 \text{ for all } y \in C \setminus \{\theta\}\big\}.$$

Let M be a nonempty set in Y. Denote the closure of M by $\mathrm{cl}(M)$ and the interior of M by $\mathrm{int}(M)$. The cone hull of M is defined by

$$\mathrm{cone}(M) = \{ty\colon\ t \geqslant 0,\ y \in M\}.$$

A nonempty convex subset B of the convex cone C is called a base of C, if

$$C = \mathrm{cone}(B) \quad \text{and} \quad \theta \notin \mathrm{cl}(B).$$

It follows from [7, Lemma 3.3] that $C^\sharp \neq \emptyset$ if and only if C has a base. Suppose that C has a base B. Denote

$$C_\varepsilon(B) = \mathrm{cone}(B + \varepsilon U) \quad \text{for all } 0 < \varepsilon < \delta,$$

where $\delta = \inf\{\|b\|\colon\ b \in B\}$ and U is the closed unit ball of Y. It follows from [2] that, for $\delta > 0$, $\mathrm{cl}(\mathrm{int}\, C_\varepsilon(B))$ is a closed convex pointed cone and $C \setminus \{\theta\} \subset \mathrm{int}\, C_\varepsilon(B)$ for all $0 < \varepsilon < \delta$.

Let C be a convex cone with base B. Denote

$$C^\triangle(B) = \big\{f \in C^*\colon\ \inf\big\{f(b)\colon\ b \in B\big\} > 0\big\}.$$

By the separation theorem, $C^\triangle(B) \neq \emptyset$ (see [5]). Obviously, $C^\triangle(B) \subset C^\sharp$.

Let m be a positive integer, X be a normed space supplied with a distance d and K be a subset of X. We denote by $d(x, K) = \inf_{y \in K} d(x, y)$ the distance from x to K, where we set $d(x, \emptyset) = +\infty$. Now let us recall the definitions of the higher order tangent sets in [1].

Definition 2.1. (See [1].) Let x belong to a subset K of a normed space X and $v_1, \ldots, v_{m-1}$ be elements of X.

(i) We say that the set

$$\begin{aligned} &T_K^{(m)}(x, v_1, \ldots, v_{m-1}) \\ &\quad = \limsup_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m} \\ &\quad = \left\{ y \in X \;\middle|\; \liminf_{h \to 0^+} d\left(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}\right) = 0 \right\} \end{aligned}$$

is the mth-order contingent set of K at $(x, v_1, \ldots, v_{m-1})$.

(ii) We say that the set

$$\begin{aligned} &T_K^{\flat(m)}(x, v_1, \dots, v_{m-1}) \\ &\quad = \liminf_{h\to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m} \\ &\quad = \left\{ y \in X \;\Big|\; \lim_{h\to 0^+} d\left(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}\right) = 0 \right\} \end{aligned}$$

is the mth-order adjacent set of K at $(x, v_1, \dots, v_{m-1})$.

Remark 2.1.

(a) The following inclusion holds:

$$\begin{aligned} T_K^{\flat(m)}(x, v_1, \dots, v_{m-1}) &\subset T_K^{(m)}(x, v_1, \dots, v_{m-1}) \\ &\subset \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}\right). \end{aligned}$$

(b) [1, p. 172] They are closed subsets satisfying, for any $\lambda > 0$,

$$T_K^{(m)}\left(x, \lambda v_1, \lambda^2 v_2, \dots, \lambda^{m-1} v_{m-1}\right) = \lambda^m T_K^{(m)}(x, v_1, v_2, \dots, v_{m-1}),$$

and

$$T_K^{\flat(m)}\left(x, \lambda v_1, \lambda^2 v_2, \dots, \lambda^{m-1} v_{m-1}\right) = \lambda^m T_K^{\flat(m)}(x, v_1, v_2, \dots, v_{m-1}).$$

From [10, Propositions 3.1 and 3.2], we have the following results.

Proposition 2.1. *If K is convex, then $T_K^{\flat(m)}(x_0, v_1, \dots, v_{m-1})$ is convex.*

Proposition 2.2. *If K is a convex subset and $v_1, v_2, \dots, v_{m-1} \in K$, then*

$$\begin{aligned} &T_K^{\flat(m)}(x_0, v_1 - x_0, \dots, v_{m-1} - x_0) \\ &\quad = T_K^{(m)}(x_0, v_1 - x_0, \dots, v_{m-1} - x_0) \\ &\quad = \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right). \end{aligned}$$

Corollary 2.1. *If K is a convex subset and $v_1, v_2, \dots, v_{m-1} \in K$, then the set $T_K^{(m)}(x_0, v_1 - x_0, \dots, v_{m-1} - x_0)$ is convex.*

3. Higher order generalized epiderivatives

In [3], Chen and Jahn introduced first-order generalized contingent epiderivative of a set-valued map. Recently, Jahn et al. [8] introduced second-order epiderivatives for set-valued maps and obtained some optimality conditions. In this section, we introduce the definitions of mth-order generalized contingent epiderivative and adjacent epiderivative, and then investigate their properties under the condition that the set-valued mapping is C-convex.

Definition 3.1. Let $H \subset Y$. $\bar{y} \in H$ is said to be a minimal point of H if $H \cap (\bar{y} - C) = \{\bar{y}\}$. The set of all minimal elements of H is denoted by $\mathrm{Min}_C H$. If $H = \emptyset$, we define $\mathrm{Min}_C H = \emptyset$.

Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map.

Definition 3.2.

(i) The mth-order generalized contingent epiderivative $D_g^{(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \mathrm{graph}(F)$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &D_g^{(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \\ &\quad = \mathrm{Min}_C \big\{ y \in Y \bigm| y \in D^{(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \big\} \\ &\quad = \mathrm{Min}_C \big\{ y \in Y \bigm| (x, y) \in T^{(m)}_{\mathrm{epi}(F)}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) \big\}, \\ &\qquad x \in \mathrm{dom}\big[D^{(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\big]. \end{aligned}$$

(ii) The mth-order generalized adjacent epiderivative $D_g^{\flat(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \mathrm{graph}(F)$ for vectors $(u_1, v_1), \ldots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &D_g^{\flat(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \\ &\quad = \mathrm{Min}_C \big\{ y \in Y \bigm| y \in D^{\flat(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x) \big\} \\ &\quad = \mathrm{Min}_C \big\{ y \in Y \bigm| (x, y) \in T^{\flat(m)}_{\mathrm{epi}(F)}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1}) \big\}, \\ &\qquad x \in \mathrm{dom}\big[D^{\flat(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\big]. \end{aligned}$$

Now we give two examples to explain Definition 3.2.

Example 3.1. Let $F : [0, +\infty) \to 2^{R^2}$ be a set-valued map given by

$$F(x) = \big\{ (y_1, y_2) \in R^2 \bigm| 2x \leqslant y_1^2 + y_2 \leqslant 2(x - y_1 - y_2)^2 \big\},$$

and $C = R^2_+$. Take a point $(x^0, (y_1^0, y_2^0)) = (0, (1, -1)) \in \mathrm{graph}(F)$. Then,

$$T_{\mathrm{graph}(F)}\big(x^0, \big(y_1^0, y_2^0\big)\big) = \big\{ \big(u, (v_1, v_2)\big) \in R^3 \bigm| 2u \leqslant 2v_1 + v_2 \leqslant 0 \big\},$$

and

$$T_{\mathrm{epi}(F)}\big(x^0, \big(y_1^0, y_2^0\big)\big) = \big\{ \big(u, (v_1, v_2)\big) \in R^3 \bigm| 2u \leqslant 2v_1 + v_2 \big\}.$$

Take $(u^0, (v_1^0, v_2^0)) = (-1, (-1, 0)) \in T_{\mathrm{epi}(F)}(x^0, (y_1^0, y_2^0))$. Then,

$$\begin{aligned} T^{(2)}_{\mathrm{graph}(F)}\big(x^0, \big(y_1^0, y_2^0\big), u^0, \big(v_1^0, v_2^0\big)\big) &= T^{(2)}_{\mathrm{epi}(F)}\big(x^0, \big(y_1^0, y_2^0\big), u^0, \big(v_1^0, v_2^0\big)\big) \\ &= \big\{ \big(u, (v_1, v_2)\big) \in R^3 \bigm| 2u \leqslant 2v_1 + v_2 + 1 \big\}. \end{aligned}$$

Thus, we have

$$D_g F\big(x^0, \big(y_1^0, y_2^0\big)\big)(u) = \big\{ (v_1, v_2) \in R^2 \bigm| 2u = 2v_1 + v_2 \big\}, \quad u \in R,$$

and

$$D_g^{(2)} F\big(x^0, \big(y_1^0, y_2^0\big), u^0, \big(v_1^0, v_2^0\big)\big)(u) = \big\{ (v_1, v_2) \in R^2 \bigm| 2u = 2v_1 + v_2 + 1 \big\}, \quad u \in R.$$

Example 3.2. If $F : X \to Y$ is a single-valued map which is twice continuously differentiable around a point $x_0 \in K \subset X$, then the second-order contingent derivative of the restriction $F_+|_K$ of F_+ to K at x_0 in a direction u_1 is given by the formula

$$D^{(2)}(F_+|_K)\big(x_0, F(x_0), u_1, \nabla F(x_0)(u_1)\big)(x) = \nabla F(x_0)(x) + \frac{1}{2}\nabla^2 F(x_0)(u_1, u_1) + C,$$

whenever $x \in T_K^{(2)}(x_0, u_1)$ and $\nabla^{(m)} F(x_0)$, $(m = 1, 2)$ denotes the mth-order derivative of F at x_0. It is empty when $x \notin T_K^{(2)}(x_0, u_1)$.

The proof of this fact is similar to the proof of [1, Proposition 5.6.2]. Then, we have

$$D_g^{(2)}(F|_K)\big(x_0, F(x_0), u_1, \nabla F(x_0)(u_1)\big)(x) = \nabla F(x_0)(x) + \frac{1}{2}\nabla^2 F(x_0)(u_1, u_1),$$

whenever $x \in T_K^{(2)}(x_0, u_1)$. It is empty when $x \notin T_K^{(2)}(x_0, u_1)$.

Definition 3.3. (See [12].)

(i) The cone C is called Daniell, if any decreasing sequence in Y having a lower bound converges to its infimum.
(ii) A subset H of Y is said to be minorized, if there is a $y \in Y$ so that $H \subset \{y\} + C$.
(iii) The domination property is said to hold for a subset H of Y if $H \subset \mathrm{Min}_C H + C$.

Now we give an existence theorem of $D_g^{(m)} F$ and $D_g^{\flat(m)} F$.

Theorem 3.1. *Let C be a closed pointed convex cone and let C be Daniell.*

(i) *Suppose that for every $x \in \Omega := \mathrm{dom}[D^{(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})]$, the set $P(x) := \{y \in Y \mid (x, y) \in T_{\mathrm{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\}$ is minorized. Then for all $x \in \Omega$, $D_g^{(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ exists.*
(ii) *Suppose that for every $x \in \Omega := \mathrm{dom}[D^{\flat(m)} F_+(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})]$, the set $P(x) := \{y \in Y \mid (x, y) \in T_{\mathrm{epi}(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})\}$ is minorized. Then for all $x \in \Omega$, $D_g^{\flat(m)} F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(x)$ exists.*

Proof. From Remark 2.1(b) we know, mth-order contingent set and mth-order adjacent set are closed. Then we can prove it as the proof of Theorem 2 in [3]. □

Now we discuss a crucial proposition of mth-order generalized adjacent epiderivative and contingent epiderivative. By using the basic idea of Theorem 4.1 in [10], we have the following result.

Proposition 3.1. *Let F be C-convex on a nonempty convex subset $E \subset X$. Let $x_0 \in E$, $y_0 \in F(x_0)$, $u_1, \ldots, u_{m-1} \in E$ and $v_1 \in F(u_1) + C, \ldots, v_{m-1} \in F(u_{m-1}) + C$. If $D_g^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \neq \emptyset$ and the set $P(x - x_0) := \{y \in Y \mid (x - x_0, y) \in T_{\mathrm{epi}(F)}^{\flat(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the domination property for all $x \in E$, then for all $x \in E$,*

$$F(x) - y_0 \subset D_g^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Proof. Take any $x \in E$ and $y \in F(x)$. For arbitrary sequence $\{\lambda_n\} \subset (0,1)$ with $\lambda_n \to 0$, since E is convex and F is C-convex, we have

$$x_0 + \frac{\lambda_n^m}{2}(x - x_0) = \left(1 - \frac{\lambda_n^m}{2}\right)x_0 + \frac{\lambda_n^m}{2}x \in E,$$
$$x_0 + \lambda_n^{m-1}(u_{m-1} - x_0) = \left(1 - \lambda_n^{m-1}\right)x_0 + \lambda_n^{m-1}u_{m-1} \in E,$$

and

$$y_0 + \frac{\lambda_n^m}{2}(y - y_0) = \left(1 - \frac{\lambda_n^m}{2}\right)y_0 + \frac{\lambda_n^m}{2}y \in F\left(x_0 + \frac{\lambda_n^m}{2}(x - x_0)\right) + C,$$
$$y_0 + \lambda_n^{m-1}(v_{m-1} - y_0) = \left(1 - \lambda_n^{m-1}\right)y_0 + \lambda_n^{m-1}v_{m-1} \in F\left(x_0 + \lambda_n^{m-1}(u_{m-1} - x_0)\right) + C.$$

Consequently by the convexity again we get

$$x_0 + \frac{\lambda_n^{m-1}}{2}(u_{m-1} - x_0) + \frac{\lambda_n^m}{2^2}(x - x_0) \in E,$$

and

$$y_0 + \frac{\lambda_n^{m-1}}{2}(v_{m-1} - y_0) + \frac{\lambda_n^m}{2^2}(y - y_0)$$
$$\in F\left(x_0 + \frac{\lambda_n^{m-1}}{2}(u_{m-1} - x_0) + \frac{\lambda_n^m}{2^2}(x - x_0)\right) + C.$$

Proceed with the above process, we have the following sequence $\{(x_n, y_n)\}$ satisfying

$$x_n := x_0 + \frac{\lambda_n}{2}(u_1 - x_0) + \cdots + \frac{\lambda_n^{m-1}}{2^{m-1}}(u_{m-1} - x_0) + \frac{\lambda_n^m}{2^m}(x - x_0) \in E,$$

and

$$y_n := y_0 + \frac{\lambda_n}{2}(v_1 - y_0) + \cdots + \frac{\lambda_n^{m-1}}{2^{m-1}}(v_{m-1} - y_0) + \frac{\lambda_n^m}{2^m}(y - y_0) \in F(x_n) + C.$$

Hence, $(x_n, y_n) \in \operatorname{epi}(F)$. Moreover, we obtain

$$\frac{(x_n, y_n) - (x_0, y_0) - \lambda_n(\frac{u_1 - x_0}{2}, \frac{v_1 - y_0}{2}) - \cdots - \lambda_n^{m-1}(\frac{u_{m-1} - x_0}{2^{m-1}}, \frac{v_{m-1} - y_0}{2^{m-1}})}{\lambda_n^m}$$
$$= \frac{1}{2^m}(x - x_0, y - y_0).$$

It follows that

$$\frac{1}{2^m}(x - x_0, y - y_0)$$
$$\in T^{\flat(m)}_{\operatorname{epi}(F)}\left((x_0, y_0), \frac{1}{2}(u_1 - x_0, v_1 - y_0), \ldots, \frac{1}{2^{m-1}}(u_{m-1} - x_0, v_{m-1} - y_0)\right).$$

By Remark 2.1(b), we have

$$(x - x_0, y - y_0) \in T^{\flat(m)}_{\operatorname{epi}(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0),$$

i.e., $y - y_0 \in P(x - x_0)$. By the definition of $D_g^{\flat(m)}F$ and the domination property, we have

$$P(x - x_0) \subset D_g^{\flat(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Thus, $F(x) - y_0 \subset D_g^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C$. The proof is complete. □

Corollary 3.1. *Let F be C-convex on a nonempty convex subset $E \subset X$. Let $x_0 \in E$, $y_0 \in F(x_0)$, $u_1, \ldots, u_{m-1} \in E$ and $v_1 \in F(u_1) + C, \ldots, v_{m-1} \in F(u_{m-1}) + C$. If $D_g^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \neq \emptyset$ and the set $P(x - x_0) := \{y \in Y \mid (x - x_0, y) \in T^{(m)}_{\mathrm{epi}(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the domination property for all $x \in E$, then for all $x \in E$,*

$$F(x) - y_0 \subset D_g^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Proof. Since F is C-convex and $(u_i, v_i) \in \mathrm{epi}(F)$, $i = 1, \ldots, m-1$, by Proposition 2.2, we get that $T^{(m)}_{\mathrm{epi}(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0) = T^{\flat(m)}_{\mathrm{epi}(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)$. It follows from Proposition 3.1 that

$$F(x) - y_0 \subset D_g^{(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Thus, the proof is complete. □

4. Set-valued optimization and higher order optimality conditions

In this section, we introduce a set-valued optimization problem, and then establish higher order necessary optimality conditions of the set-valued optimization problem by virtue of the mth-order generalized adjacent epiderivative and contingent epiderivative. Consider the following constrained set-valued optimization problem (GVOP):

$$\begin{cases} \min & F(x) \\ \text{s.t.} & x \in S, \quad G(x) \cap -D \neq \emptyset. \end{cases}$$

A triple $(x, y, z) \in S \times Y \times Z$ is said to be feasible if $x \in \mathrm{dom}(F) \cap \mathrm{dom}(G)$, $y \in F(x)$ and $z \in G(x) \cap -D$.

Set

$$A = \{x \in S\colon\ G(x) \cap -D \neq \emptyset\} \text{ and } F(A) = \bigcup \{F(x)\colon\ x \in A\}.$$

The notation $(F, G)(x)$ is used to denote $F(x) \times G(x)$. The notations F_A and G_A are used to denote the restriction of F to A and G to A, respectively.

Definition 4.1. (See [2].) Suppose that C has a base B. A pair (x_0, y_0) with $x_0 \in A$ and $y_0 \in F(x_0)$ is called a Henig efficient solution of (GVOP) if for some $0 < \varepsilon < \delta$,

$$\big(F(A) - y_0\big) \cap -\mathrm{int}\, C_\varepsilon(B) = \emptyset.$$

Theorem 4.1. *Suppose that C has a base B and $\mathrm{int}\, D \neq \emptyset$. Let $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D)$, $i = 1, \ldots, m-1$. Let $(x_0, y_0) \in \mathrm{graph}(F)$ and $\delta = \inf\{\|b\|\colon\ b \in B\}$. If (x_0, y_0) is a Henig efficient solution of (GVOP), then for some $0 < \varepsilon < \delta$ and for any $z_0 \in G(x_0) \cap -D$,*

$$\begin{aligned}\big[D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0,\\ w_{m-1} - z_0)(x) + C \times D + (\theta_Y, z_0)\big] \cap -\mathrm{int}\big(C_\varepsilon(B) \times D\big) = \emptyset, \qquad (1)\end{aligned}$$

for all $x \in \Omega := \mathrm{dom}[D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)]$.

Proof. Since (x_0, y_0) is a Henig efficient solution of (GVOP), then there exists $0 < \varepsilon < \delta$ such that

$$\big(F(A) - y_0\big) \cap -\operatorname{int} C_\varepsilon(B) = \emptyset. \tag{2}$$

Assume that the result (1) does not hold. Then there exist $x \in \Omega$, $(y, z) \in Y \times Z$, $c_0 \in C$ and $d_0 \in D$ such that

$$(y, z) \in D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x), \tag{3}$$

and

$$(y, z) + (c_0, d_0) + (\theta_Y, z_0) = (y + c_0, z + d_0 + z_0) \in -\operatorname{int}\big(C_\varepsilon(B) \times D\big). \tag{4}$$

It follows from (3) and the definition of the mth-order generalized adjacent epiderivative that

$$(x, y, z) \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0).$$

So, for any sequence $\{h_n\}$ with $h_n \to 0^+$, there exists $\{(x_n, y_n, z_n)\}$ with $(x_n, y_n, z_n) \in \operatorname{epi}(F, G)$ such that

$$\frac{(x_n, y_n, z_n) - (x_0, y_0, z_0)}{h_n^m} - \frac{h_n(u_1 - x_0, v_1 - y_0, w_1 - z_0)}{h_n^m} - \cdots - \frac{h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)}{h_n^m} \to (x, y, z). \tag{5}$$

From (4) and (5), there exists sufficiently large $N > 0$ such that $h_n + \cdots + h_n^m < 1$ and

$$\frac{(y_n, z_n) - (y_0, z_0) - h_n(v_1 - y_0, w_1 - z_0) - \cdots - h_n^{m-1}(v_{m-1} - y_0, w_{m-1} - z_0)}{h_n^m} + (c_0, d_0) + (\theta_Y, z_0) \in -\operatorname{int}\big(C_\varepsilon(B) \times D\big), \quad \text{for } n \geqslant N.$$

It follows from $h_n > 0$, $C_\varepsilon(B)$ and D are cones that

$$y_n - y_0 - h_n(v_1 - y_0) - \cdots - h_n^{m-1}(v_{m-1} - y_0) + h_n^m c_0 \in -\operatorname{int} C_\varepsilon(B), \quad \text{for } n \geqslant N, \tag{6}$$

and

$$z_n - z_0 - h_n(w_1 - z_0) - \cdots - h_n^{m-1}(w_{m-1} - z_0) + h_n^m z_0 + h_n^m d_0 \in -\operatorname{int} D, \quad \text{for } n \geqslant N. \tag{7}$$

Since $v_i - y_0 \in -C$, $i = 1, \dots, m-1$, and C is a convex cone,

$$h_n(v_1 - y_0) + \cdots + h_n^{m-1}(v_{m-1} - y_0) \in -C.$$

Then, by (6), we have

$$y_n - y_0 + h_n^m c_0 \in -\operatorname{int} C_\varepsilon(B) - C = -\operatorname{int} C_\varepsilon(B),$$

and

$$y_n - y_0 \in -\operatorname{int} C_\varepsilon(B) - C = -\operatorname{int} C_\varepsilon(B), \quad \text{for all } n \geqslant N.$$

Similarly, it follows from $z_0, w_1, \ldots, w_{m-1} \in -D$ and D is a convex cone that

$$\left(1 - h_n - \cdots - h_n^{m-1} - h_n^m\right) z_0 + h_n w_1 + \cdots + h_n^{m-1} w_{m-1} \in -D.$$

Then, from (7), we get

$$z_n \in -\operatorname{int} D - D = -\operatorname{int} D, \quad \text{for all } n \geqslant N.$$

Since $(x_n, y_n, z_n) \in \operatorname{epi}(F, G)$, there exist $\bar{y}_n \in F(x_n)$ and $\bar{z}_n \in G(x_n)$ such that $y_n \in \bar{y}_n + C$ and $z_n \in \bar{z}_n + D$. Then, for all $n \geqslant N$,

$$\bar{y}_n - y_0 \in y_n - C - y_0 \subset -\operatorname{int} C_\varepsilon(B) - C = -\operatorname{int} C_\varepsilon(B),$$

and

$$\bar{z}_n \in z_n - D \subset -\operatorname{int} D - D = -\operatorname{int} D.$$

So, $(x_n, \bar{y}_n, \bar{z}_n)$ is a feasible triple for every $n \geqslant N$ and $\bar{y}_n - y_0 \in -\operatorname{int} C_\varepsilon(B)$, which contradicts (2). Thus, (1) holds and the proof is complete. □

From the proof process of Theorem 4.1, we have the following result.

Theorem 4.2. *Suppose that C has a base B and $\operatorname{int} D \neq \emptyset$. Let $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, \ldots, m-1$. Let $(x_0, y_0) \in \operatorname{graph}(F)$ and $\delta = \inf\{\|b\|: b \in B\}$. If (x_0, y_0) is a Henig efficient solution of (GVOP), then for some $0 < \varepsilon < \delta$ and for any $z_0 \in G(x_0) \cap -D$,*

$$\left[D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})(x) + C \times D + (\theta_Y, z_0)\right] \cap -\operatorname{int}\left(C_\varepsilon(B) \times D\right) = \emptyset,$$

for all $x \in \operatorname{dom}[D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})]$.

Remark 4.1. If we use mth-order generalized contingent epiderivative instead of the mth-order generalized adjacent epiderivative in Theorems 4.1 and 4.2, then the corresponding results for mth-order generalized contingent epiderivative still hold.

Now we give an example to illustrate the necessary optimality conditions for the mth-order generalized contingent epiderivative, where we only take $m = 1, 2$.

Example 4.1. Suppose that $X = Z = R$, $S = [0, 1] \subset X$, $Y = R^2$, $C = R_+^2$ and $D = R_+$. Let $F: S \to 2^Y$ be a set-valued map with

$$F(x) = \left\{(y_1, y_2) \in R^2 \;\middle|\; y_1^2 + y_2^2 \leqslant x^2\right\},$$

and $G: S \to Z$ be a constant-valued function with

$$G(x) = 0.$$

Consider the following constrained set-valued optimization problem (GVOP):

$$\begin{cases} \min & F(x), \\ \text{s.t.} & x \in S, \quad G(x) \cap -D \neq \emptyset. \end{cases}$$

Assume that $B = \{(\xi_1, \xi_2) \in R^2 \mid \xi_1 + \xi_2 - 1 = 0,\ \xi_1 \geqslant 0,\ \xi_2 \geqslant 0\}$. Obviously, B is a base of C and $\delta = \inf\{\|b\|: b \in B\} = \frac{\sqrt{2}}{2}$. Let $C_\varepsilon(B) = \operatorname{cone}(B + \varepsilon U)$ for all $0 < \varepsilon < \delta$, where U is the closed unit ball of Y. We can easily verify that $C_\varepsilon(B) \subseteq \{(\xi_1, \xi_2) \in R^2 \mid \xi_1 + \xi_2 > 0\}$ for all $0 < \varepsilon < \delta$, so that $-\operatorname{int} C_\varepsilon(B) \subset \{(\xi_1, \xi_2) \in R^2 \mid \xi_1 + \xi_2 < 0\}$.

Let $(x_0, y_0) = (1, (-\sqrt{2}/2, -\sqrt{2}/2)) \in \operatorname{graph}(F)$. It follows from the definition of A that $A = [0, 1]$. Then, $F(A) = \{(\xi_1, \xi_2) \in R^2 \mid \xi_1^2 + \xi_2^2 \leqslant 1\}$ and $F(A) - y_0 = \{(\xi_1, \xi_2) \in R^2 \mid (\xi_1 - \frac{\sqrt{2}}{2})^2 + (\xi_2 - \frac{\sqrt{2}}{2})^2 \leqslant 1\}$. Thus, $(F(A) - y_0) \cap -\operatorname{int} C_\varepsilon(B) = \emptyset$ and the point (x_0, y_0) is a Henig efficient solution of (GVOP). Take $z_0 = 0 \in G(x_0) \cap -R_+$. We have

$$\begin{aligned}
&T_{\operatorname{epi}(F,G)}(x_0, y_0, z_0)\\
&\quad = \big\{\big(u, (\xi_1, \xi_2), \xi_3\big) \in R \times R^2 \times R \;\big|\; \xi_1 + \xi_2 \geqslant -\sqrt{2}u,\ \xi_3 \geqslant 0,\ u \leqslant 0\big\},\\
&D_g(F, G)(x_0, y_0, z_0)(u)\\
&\quad = \big\{\big((\xi_1, \xi_2), \xi_3\big) \in R^2 \times R \;\big|\; \xi_1 + \xi_2 = -\sqrt{2}u,\ \xi_3 = 0\big\}, \quad u \in R_-,
\end{aligned}$$

and

$$\operatorname{dom}\big[D_g(F, G)(x_0, y_0, z_0)\big] = R_-.$$

Thus,

$$\begin{aligned}
&\big[D_g(F, G)(x_0, y_0, z_0)(u) + R_+^2 \times R_+ + \big((0, 0), 0\big)\big] \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big)\\
&\quad = \big\{\big((\xi_1, \xi_2), \xi_3\big) \in R^2 \times R \;\big|\; \xi_1 + \xi_2 \geqslant -\sqrt{2}u,\ \xi_3 \geqslant 0\big\} \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big) = \emptyset,
\end{aligned}$$

for all $u \in \operatorname{dom}[D_g(F, G)(x_0, y_0, z_0)]$ and the 1th-order necessary optimality condition holds.

Take $u_1 = 1 + \frac{\sqrt{2}}{2}$, $v_1 = (-1 - \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, and $w_1 = 0 \in -R_+$. Then, $u_1 - x_0 = \frac{\sqrt{2}}{2}$, $v_1 - y_0 = (-1, 0) \in -R_+^2$, and $w_1 - z_0 = 0$. We have

$$\begin{aligned}
&T^{(2)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)\\
&\quad = \big\{\big(u, (\xi_1, \xi_2), \xi_3\big) \in R \times R^2 \times R \;\big|\; \xi_1 + \xi_2 \geqslant -\sqrt{2}u + \sqrt{2}/4,\ \xi_3 \geqslant 0,\ u \leqslant 0\big\},\\
&D_g^{(2)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)(u)\\
&\quad = \big\{\big((\xi_1, \xi_2), \xi_3\big) \in R^2 \times R \;\big|\; \xi_1 + \xi_2 = -\sqrt{2}u + \sqrt{2}/4,\ \xi_3 = 0\big\}, \quad u \in R_-,
\end{aligned}$$

and

$$\operatorname{dom}\big[D_g^{(2)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)\big] = R_-.$$

Thus,

$$\begin{aligned}
&\big[D_g^{(2)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)(u) + R_+^2 \times R_+ + \big((0, 0), 0\big)\big]\\
&\quad \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big) = \emptyset,
\end{aligned}$$

for all $u \in \operatorname{dom}[D_g^{(2)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)]$ and the 2th-order necessary optimality condition holds.

5. Higher order Fritz John type optimality conditions

In this section, we establish higher order Fritz John type necessary and sufficient optimality conditions of the set-valued optimization problem by virtue of the mth-order generalized adjacent epiderivative and contingent epiderivative. Firstly, we state two results in [5].

Lemma 5.1. (See [5].) *For any* $\varepsilon \in (0,\delta)$, $C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\} \subset C^\triangle(B)$.

Lemma 5.2. (See [5].) *For any* $f \in C^\triangle(B)$, *there exists* $0 < \varepsilon < \delta$ *with* $f \in C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\}$.

Theorem 5.1. *Suppose that the following conditions are satisfied*:

(i) *F and G are C-convex and D-convex on the convex set S, respectively*;
(ii) $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D)$, $i = 1,\dots,m-1$ *and* $(x_0, y_0) \in \operatorname{graph}(F)$;
(iii) *C has a base B*, $\operatorname{int} D \neq \emptyset$ *and* $\delta = \inf\{\|b\|: b \in B\}$;
(iv) $P(x) := \{(y,z) \in Y \times Z \mid (x,y,z) \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ *fulfills the domination property for all* $x \in S$ *and* $(\theta_Y, \theta_Z) \in P(\theta_X)$;
(v) (x_0, y_0) *is a Henig efficient solution of* (*GVOP*).

Then, for any $z_0 \in G(x_0) \cap -D$, *there exist* $\lambda \in C^\triangle(B) \cup \{\theta_{Y^*}\}$ *and* $\mu \in D^*$, *but not both being zero functionals, such that*

$$\mu(z_0) = 0 \quad \text{and} \quad \lambda(y) + \mu(z) \geqslant 0,$$

for all $(y,z) \in D^{\flat(m)}_g(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x)$ *and* $x \in \Omega := \operatorname{dom}[D^{\flat(m)}_g(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)]$.

Proof. Let $z_0 \in G(x_0) \cap -D$. Define

$$M = \bigcup_{x \in \Omega} D^{\flat(m)}_g(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) + C \times D + (\theta_Y, z_0).$$

First we prove that M is convex by proving that $M_0 = M - (\theta_Y, z_0)$ is convex. Let $(y_i, z_i) \in M_0$, $i = 1,2$. Then there exist $x_i \in \Omega$, $(y_i', z_i') \in D^{\flat(m)}_g(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x_i)$ and $(c_i, d_i) \in C \times D$ such that

$$(y_i, z_i) = (y_i', z_i') + (c_i, d_i), \quad i = 1,2.$$

By the definition of $D^{\flat(m)}_g(F,G)$, we have

$$(x_i, y_i', z_i') \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0), \quad i = 1,2.$$

Since F and G are cone-convex, so is the map (F,G). Hence the epigraph $\operatorname{epi}(F,G)$ is a convex set. Thus by Proposition 2.1, $T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)$ is a convex set. Therefore for any $t \in [0,1]$,

$$t(x_1, y_1', z_1') + (1-t)(x_2, y_2', z_2') \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0).$$

By the domination property, we have

$$t(y_1', z_1') + (1-t)(y_2', z_2') \in D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\big(tx_1 + (1-t)x_2\big) + C \times D.$$

Then, $t(y_1, z_1) + (1-t)(y_2, z_2) = t(y_1', z_1') + (1-t)(y_2', z_2') + t(c_1, d_1) + (1-t)(c_2, d_2) \in M_0$, i.e., M_0 is convex. Thus, M is convex. By Theorem 4.1, it is clearly that

$$M \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big) = \emptyset.$$

By the separation theorem for convex sets, there exist $\lambda \in Y^*$ and $\mu \in Z^*$, not both zero functionals, and a real number γ such that

$$\lambda(\bar{y}) + \mu(\bar{z}) < \gamma \leqslant \lambda(\tilde{y}) + \mu(\tilde{z}), \quad \text{for all } \bar{y} \in -\operatorname{int} C_\varepsilon(B),\ \bar{z} \in -\operatorname{int} D,\ (\tilde{y}, \tilde{z}) \in M. \tag{8}$$

It follows from $(\bar{y}, \bar{z}) \in -\operatorname{int}(C_\varepsilon(B) \times D)$ and (8) that

$$\lambda(\bar{y}) + \mu(\bar{z}) \leqslant 0, \quad \text{for all } \bar{y} \in -\operatorname{int} C_\varepsilon(B),\ \bar{z} \in -\operatorname{int} D, \tag{9}$$

and

$$0 \leqslant \lambda(\tilde{y}) + \mu(\tilde{z}), \quad \text{for all } (\tilde{y}, \tilde{z}) \in M. \tag{10}$$

Then, by (9), we have

$$\lambda(\bar{y}) \leqslant 0, \quad \text{for all } \bar{y} \in -\operatorname{int} C_\varepsilon(B),$$

and

$$\mu(\bar{z}) \leqslant 0, \quad \text{for all } \bar{z} \in -\operatorname{int} D.$$

Thus, $\lambda \in C_\varepsilon(B)^*$ and $\mu \in D^*$. By Lemma 5.1, $\lambda \in C^\triangle(B) \cup \{\theta_{Y^*}\}$. Since $P(x)$ fulfills the domination property for all $x \in S$,

$$P(x) \subseteq D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) + C \times D, \quad \text{for all } x \in \Omega.$$

Then, we have

$$\bigcup_{x \in \Omega} P(x) \subseteq M - (\theta_Y, z_0).$$

It follows from $(\theta_Y, \theta_Z) \in P(\theta_X)$ that

$$(\theta_Y, \theta_Z) \in M - (\theta_Y, z_0),$$

i.e.,

$$(\theta_Y, z_0) \in M.$$

From (10), we have

$$\mu(z_0) \geqslant 0. \tag{11}$$

It follows from $z_0 \in -D$ and $\mu \in D^*$ that

$$\mu(z_0) \leqslant 0. \tag{12}$$

Then, by (11) and (12), we get

$$\mu(z_0) = 0.$$

Thus, it follows from (10) that

$$\lambda(y) + \mu(z) \geqslant 0,$$

for all $(y, z) \in D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x)$ and $x \in \Omega$. The proof is complete. □

Similarly as in the proof of Theorem 5.1, we have the following result.

Theorem 5.2. *Suppose that the following conditions are satisfied*:

(i) *F and G are C-convex and D-convex on the convex set S, respectively*;
(ii) $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, \ldots, m-1$ *and* $(x_0, y_0) \in \operatorname{graph}(F)$;
(iii) *C has a base B*, $\operatorname{int} D \neq \emptyset$ *and* $\delta = \inf\{\|b\|: b \in B\}$;
(iv) $P(x) := \{(y, z) \in Y \times Z \mid (x, y, z) \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})\}$ *fulfills the domination property for all $x \in S$ and* $(\theta_Y, \theta_Z) \in P(\theta_X)$;
(v) (x_0, y_0) *is a Henig efficient solution of (GVOP).*

Then, for any $z_0 \in G(x_0) \cap -D$, there exist $\lambda \in C^{\triangle}(B) \cup \{\theta_{Y^*}\}$ and $\mu \in D^*$, but not both being zero functionals, such that

$$\mu(z_0) = 0 \quad \textit{and} \quad \lambda(y) + \mu(z) \geqslant 0,$$

for all $(y, z) \in D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})(x)$ *and*

$$x \in \operatorname{dom}\big[D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1, \ldots, u_{m-1}, v_{m-1}, w_{m-1})\big].$$

Remark 5.1. If F and G are C-convex and D-convex on the convex set S, respectively, it follows from Proposition 2.2 that

$$\begin{aligned} &D_g^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0,\\ &\qquad w_{m-1} - z_0)(x)\\ &\quad = D_g^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0,\\ &\qquad\qquad w_{m-1} - z_0)(x), \end{aligned}$$

for any $x \in \Omega$. Thus, if we use mth-order generalized contingent epiderivative instead of the mth-order generalized adjacent epiderivative in Theorem 5.1, then, the corresponding result for mth-order generalized contingent epiderivative still holds.

The following example illustrates the Fritz John type necessary optimality conditions for the mth-order generalized contingent epiderivative. Here we only take $m = 1, 2$ yet.

Example 5.1. Suppose that $X = Y = Z = R$, $S = [-1, 1] \subset X$ and $C = D = R_+$. Let $F : S \to 2^Y$ with

$$F(x) = \big\{y \in R \;\big|\; x^4 \leqslant y \leqslant 1\big\},$$

and $G : S \to Z$ be a real-valued function with

$$G(x) = 2x - 1.$$

Naturally, F and G are two R_+-convex functions on the convex set $[-1,1]$, respectively. Consider the following constrained set-valued optimization problem (GVOP):

$$\begin{cases} \min & F(x), \\ \text{s.t.} & x \in S, \quad G(x) \cap -D \neq \emptyset. \end{cases}$$

By the definition of A, we have

$$A = \{x \in [-1,1] \mid G(x) \cap -R_+ \neq \emptyset\} = \{x \in [-1,1] \mid 2x - 1 \leqslant 0\} = [-1, 1/2],$$

and

$$F(A) = \bigcup_{x \in [-1,1/2]} F(x) = [0,1].$$

Obviously, $B = 1$ is a base of C and $\delta = 1$. Let $C_\varepsilon(B) = \operatorname{cone}(B + \varepsilon U)$ for all $0 < \varepsilon < \delta$, where U is the closed unit ball of R.

Let $(x_0, y_0) = (0,0) \in \operatorname{graph}(F)$. Since $(F(A) - y_0) \cap -\operatorname{int} C_\varepsilon(B) = \emptyset$, the point (x_0, y_0) is a Henig efficient solution of (GVOP). It follows from the definitions of F and G that

$$\operatorname{epi}(F,G) = \{(x,(y,z)) \in R \times R^2 \mid y \geqslant x^4,\ z \geqslant 2x - 1,\ -1 \leqslant x \leqslant 1\}.$$

Take $z_0 = -1 \in G(x_0) \cap -R_+$. Then, we have

$$T_{\operatorname{epi}(F,G)}(x_0, y_0, z_0) = \{(x,(y,z)) \in R \times R^2 \mid y \geqslant 0,\ z \geqslant 2x\},$$

and

$$D_g(F,G)(x_0,y_0,z_0)(x) = \{(y,z) \in R^2 \mid y = 0,\ z = 2x\}, \quad x \in R.$$

Let $P(x) = \{(y,z) \in R^2 \mid (x,y,z) \in T_{\operatorname{epi}(F,G)}(x_0,y_0,z_0)\}$. It is easy to verify that $P(x)$ fulfills the domination property and $(0,0) \in P(0)$. Then, by Remark 5.1, we have that the conditions of Theorem 5.1 are satisfied. Take $\lambda > 0$ and $\mu = 0$. Thus, for any $(y,z) \in D_g(F,G)(x_0,y_0,z_0)(x)$ and $x \in R$, we have

$$\lambda(y) + \mu(z) = 0 \quad \text{and} \quad \mu(z_0) = 0,$$

which shows that the 1th-order Fritz John type necessary optimality condition of Theorem 5.1 holds.

Take $u_1 = 1/4$, $v_1 = 0$, and $w_1 = -1/2 \in -R_+$. Then, $u_1 - x_0 = 1/4$, $v_1 - y_0 = 0 \in -R_+$, and $w_1 - z_0 = 1/2$. We have

$$T^{(2)}_{\operatorname{epi}(F,G)}(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0) = \{(x,(y,z)) \in R \times R^2 \mid y \geqslant 0,\ z \geqslant 2x\},$$

and

$$\begin{aligned} &D_g^{(2)}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0)(x) \\ &\quad = \{(y,z) \in R^2 \mid y = 0,\ z = 2x\} = D_g(F,G)(x_0,y_0,z_0)(x), \quad x \in R. \end{aligned} \tag{13}$$

Hence, it follows from (13) that the conditions of Theorem 5.1 are satisfied for $D_g^{(2)}(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)$. Simultaneously, take $\lambda > 0$ and $\mu = 0$. We have that the 2th-order Fritz John type necessary optimality condition of Theorem 5.1 holds.

Now we give sufficient conditions involving multiplier rule for the problem (GVOP).

Theorem 5.3. *Suppose that the following conditions are satisfied*:

(i) *F and G are C-convex and D-convex on the nonempty convex set $S \subset \operatorname{dom}(F) \cap \operatorname{dom}(G)$, respectively*;
(ii) *$A = \{x \in S\colon\ G(x) \cap -D \neq \emptyset\}$, $u_1, \dots, u_{m-1} \in A$, $v_1 \in F(u_1) + C, \dots, v_{m-1} \in F(u_{m-1}) + C$, $w_1 \in G(u_1) + D, \dots, w_{m-1} \in G(u_{m-1}) + D$, $(x_0, y_0) \in \operatorname{graph}(F)$ and B is a base of C*;
(iii) *$P(x - x_0) := \{(y, z) \in Y \times Z \mid (x - x_0, y, z) \in T^{\flat(m)}_{\operatorname{epi}(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ fulfills the domination property for all $x \in S$*;
(iv) *There exist $z_0 \in G(x_0) \cap -D$, $\lambda \in C^{\triangle}(B)$, and $\mu \in D^*$ such that*

$$\mu(z_0) = 0 \quad \text{and} \quad \lambda(y) + \mu(z) \geqslant 0,$$

for all $(y, z) \in D^{\flat(m)}_g(F_A, G_A)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x - x_0)$ and $x \in A$.

Then (x_0, y_0) is a Henig efficient solution of (GVOP).

Proof. Since $\lambda \in C^{\triangle}(B)$, by Lemma 5.2, there exists $\varepsilon \in (0, \delta)$ such that $\lambda \in C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\}$. Assume that

$$\bigl(F(A) - y_0\bigr) \cap -\operatorname{int} C_\varepsilon(B) \neq \emptyset. \tag{14}$$

Then, there exist $x' \in A$ and $y' \in F(x')$ such that $y' - y_0 \in -\operatorname{int} C_\varepsilon(B)$. Since $x' \in A$, there exists $z' \in G(x') \cap -D$. By the domination property and Proposition 3.1, we have

$$\begin{aligned}(y' - y_0, z' - z_0) &\\ \in D^{\flat(m)}_g(F_A, G_A)&(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots,\\ &u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x' - x_0) + C \times D.\end{aligned}$$

Thus,

$$\lambda(y' - y_0 - c) + \mu(z' - z_0 - d) \geqslant 0, \quad \text{for any } c \in C,\ d \in D. \tag{15}$$

Since $y' - y_0 \in -\operatorname{int} C_\varepsilon(B)$, then $y' - y_0 - c \in -\operatorname{int} C_\varepsilon(B) - C = -\operatorname{int} C_\varepsilon(B)$, and $\lambda \in C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\}$, hence $\lambda(y' - y_0 - c) < 0$. Since $z' \in G(x') \cap -D$, $\mu(z_0) = 0$ and $\mu \in D^*$, we have $\mu(z' - z_0 - d) = \mu(z') - \mu(d) \leqslant 0$. Thus,

$$\lambda(y' - y_0 - c) + \mu(z' - z_0 - d) < 0,$$

which contradicts (15). Then, (14) does not hold, namely, $(F(A) - y_0) \cap -\operatorname{int} C_\varepsilon(B) = \emptyset$. Thus, (x_0, y_0) is a Henig efficient solution of (GVOP) and the proof is complete. □

Theorem 5.4. *Suppose that the following conditions are satisfied*:

(i) *F and G are C-convex and D-convex on the nonempty convex set $S \subset \operatorname{dom}(F) \cap \operatorname{dom}(G)$, respectively*;
(ii) *$A = \{x \in S\colon\ G(x) \cap -D \neq \emptyset\}$, $u_1, \dots, u_{m-1} \in A$, $v_1 \in F(u_1) + C, \dots, v_{m-1} \in F(u_{m-1}) + C$, $w_1 \in G(u_1) + D, \dots, w_{m-1} \in G(u_{m-1}) + D$, $(x_0, y_0) \in \operatorname{graph}(F)$ and B is a base of C*;

(iii) $P(x-x_0) := \{(y,z) \in Y \times Z \mid (x-x_0, y, z) \in T^{(m)}_{\mathrm{epi}(F,G)}(x_0, y_0, z_0, u_1-x_0, v_1-y_0, w_1-z_0, \ldots, u_{m-1}-x_0, v_{m-1}-y_0, w_{m-1}-z_0)\}$ *fulfills the domination property for all* $x \in S$;

(iv) *There exist* $z_0 \in G(x_0) \cap -D$, $\lambda \in C^{\triangle}(B)$, *and* $\mu \in D^*$ *such that*

$$\mu(z_0) = 0 \quad \text{and} \quad \lambda(y) + \mu(z) \geqslant 0$$

for all $(y,z) \in D_g^{(m)}(F_A, G_A)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \ldots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x - x_0)$ *and* $x \in A$.

Then (x_0, y_0) *is a Henig efficient solution of* (*GVOP*).

Proof. The conclusion can be directly obtained similarly as in the proof of Theorem 5.3 and Corollary 3.1. □

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