An economic growth model with endogenous fertility: multiple growth paths, poverty trap and bifurcation

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Abstract

In this paper, an extended Ramsey model with endogenous fertility is present. The existence, uniqueness and multiplicity of the steady states of the model are given. When the multiple steady states exist, the steady state with low fertility has high per capita capital and per capital consumption, which can explain the persistent difference in consumption and fertility between poor and rich countries. The dynamical system undergoes a bifurcation when parameters satisfy some given conditions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The factors that cause different countries to have different economic growth paths have interested economists for several centuries. One of the most important factors is the population; see Ehrich and Lui for a comprehensive survey [2]. In recent years, the economic growth model with endogenous fertility has aroused many interesting arguments [1,3,5].

Palivos gave a Cass–Koopmans optimal growth model to allow for endogenous fertility choice [4]. The dynamic behavior of the model is decided by a pair of differential equations. The model may have multiple steady states and multiple growth paths. But the dynamic behavior of the model is not explicit and the existence of multiple steady states cannot be directly induced by the differential equations.
In this paper, an extended Ramsey model incorporated fertility with the specific utility function and production function is presented. The existence, uniqueness and multiplicity of the steady states of the model are given. The dynamical system undergoes a bifurcation when the parameters satisfy some given conditions. When the multiple steady states exist, the steady state with lower fertility has more per person capital and consumption than the one with higher fertility. Furthermore, we prove that the model has a unique optimal growth path along which the per capita capital and consumption converge to the steady states with the highest capital and consumption per person. This implies that the model includes a Malthus poverty trap and theoretically confirms the negative relationship between the level of consumption and the level of fertility.

2. The model

Consider an economy that consists of identically integrated \( N \) \((N \in (0, \infty))\) household-firms. Individuals are assumed to be infinite-lived and to have perfect foresight. Time is taken to be continuous. The fertility at time \( t \) is \( n(t) \) and the range of \( n(t) \) is \([0, b]\), \( 0 < b \leq \frac{1}{2}\), where \( b \) is the fertility limit which households can reach. \( b \) is less than \( \frac{1}{2} \) as fertility is one-sex decided. The utility function is strongly additive, i.e., \( u(c, n) = u_1(c) + u_2(n) \), where \( c \) is per capita consumption, satisfying the Inada conditions: \( \lim_{c \to 0} u'_1(c) = \lim_{n \to 0} u'_2(n) = \infty \), \( \lim_{c \to \infty} u'_1(c) = \lim_{n \to b} u'_2(n) = 0 \), \( u'_1(c) > 0, u'_2(n) > 0, u''_1(c) < 0, u''_2(n) < 0 \). The time consumed for child-rearing per person is \( x(t) = \phi(n(t)) \), where \( \phi(\cdot) : [0, b] \to [0, 1] \) is a strictly increasing function of \( n \), satisfying \( \phi(0) = 0, \phi(b) = 1, \phi'(n) > 0, \phi''(n) \geq 0 \). The total time endowment is normalized to one. So, the time per person used in production is \( l(t) = 1 - \phi(n) \).

The production function at \( t \) is \( y(t) = Ak^x(t)(1 - \phi(n(t))) \), \( 0 < x < 1 \) and the households face the constraint: \( c(t) + k(t) + n(t)k(t) = Ak^x(t)(1 - \phi(n)) \). So, the household’s optimization problem is

\[
\max \int_0^\infty \exp(-\rho t) (u_1(c) + u_2(n)) \, dt
\]

\[\text{ s.t. } \dot{k} = Ak^x(1 - \phi(n)) - c - nk,\]

where \( \rho \in (0, \infty) \) is the (constant) rate of time preference.

From current-value Hamiltonian \( H(k, c, n, q) = u_1(c) + u_2(n) + q[Ak^x(1 - \phi(n)) - c - nk] \), we obtain the first order conditions and the transversality condition

\[
\dot{k} = Ak^x(1 - \phi(n)) - c - nk, \quad \text{(1)}
\]

\[
\dot{q} = [\rho + n - xAk^{x-1}(1 - \phi(n))]q, \quad \text{(2)}
\]

\[
u'_1(c) = q, \quad \text{(3)}
\]

\[
u'_2(n) = [Ak^x\phi'(n) + k]q, \quad \text{(4)}
\]

\[
\lim_{t \to \infty} \exp(-\rho t)q(t)k(t) = 0. \quad \text{(5)}
\]
From (3) and (4), \( u'_1(c) = u'_2(n)/[Ak^2\phi'(n) + k] \). Let \( F(k,c,n) = (Ak^2\phi'(n) + k)u'_1(c) - u'_2(n) \), then \( \frac{\partial F}{\partial c} = Ak^2\phi''(n)u'_1(c) - u''_2(n) > 0, \frac{\partial F}{\partial k} = (Ak^2\phi'(n) + 1)u'_1(c) > 0, \frac{\partial F}{\partial n} = (Ak^2\phi'(n) + 1)u''_1(c) < 0 \), so \( \frac{\partial F}{\partial k} < 0, \frac{\partial F}{\partial c} > 0 \). For any given \( (k_0, c_0) \in \mathbb{R}_+^2 \) (the set of 2-tuples of positive real numbers), \( \lim_{n \to 0} F(k_0, c_0, n) = -\infty, \lim_{n \to b} F(k_0, c_0, n) = (Ak^2\phi'(b) + k_0)u'_1(c_0) > 0 \). There exists a unique \( n_0 \in (0, b) \) such that \( F(k_0, c_0, n_0) = 0 \). Hence, by the implicit function theorem, we have

**Lemma 1.** The equation \( F(k,c,n) = 0 \) uniquely determines a differentiable function \( n(k,c) \) in the positive orthant of \( k,c \) plane and \( \frac{\partial n}{\partial c} < 0, \frac{\partial n}{\partial k} > 0 \).

Substituting \( q = u'_1(c), \dot{q} = u''_1(c)c \) and \( n(k,c) \) into (1) and (2), we obtain

\[
\dot{k} = Ak^2(1 - \phi(n(k,c))) - c - n(k,c)k, \tag{6}
\]

\[
\dot{c} = [\rho + n(k,c) - \alpha Ak^{-1}(1 - \phi(n(k,c)))]u'_1(c)u''_1(c). \tag{7}
\]

3. The existence of the steady states

A point \( (k_0, c_0) \in \mathbb{R}_+^2 \) is a steady state of (6), (7) if and only if it satisfies

\[
Ak^2(1 - \phi(n(k,c))) - c - n(k,c)k = 0, \tag{8}
\]

\[
\rho + n(k,c) - \alpha Ak^{-1}(1 - \phi(n(k,c))) = 0. \tag{9}
\]

Let

\[
E_1(n) = \left[ \frac{\alpha A(1 - \phi(n))}{\rho + n} \right]^{1/(1-\alpha)},
\]

then (9), (3) and (4) derive

\[
k = E_1(n), \quad u'_1(c) = \frac{u'_2(n)}{A\phi'(n)E_1^2(n) + E_1(n)}. \]

Suppose that

\[
E_2(n) = (u'_1)^{-1}\left[ \frac{u'_2(n)}{A\phi'(n)E_1^2(n) + E_1(n)} \right]
\]

and

\[
H(n) = A(1 - \phi(n))E_1^2(n) - E_2(n) - nE_1(n), \tag{10}
\]

then \( k_0 = E_1(n_0) > 0, c_0 = E_2(n_0) > 0 \) is a zero point of (8), (9) if \( n_0 \in (0, b) \) is a zero point of \( H(n) = 0 \).

**Theorem 1.** If \( \lim_{n \to b} E_2(n) > 0 \), then the dynamical system (6),(7) at least has a nonzero steady state.
Proof. Since $\lim_{n \to 0} E_1(n) = [\alpha A/\rho]^{1/(1-\gamma)}$, $\lim_{n \to b} E_1(n) = 0$, $\lim_{n \to 0} E_2(n) = 0$ and $\lim_{n \to b} H(n) = A[\alpha A/\rho]^{1/(1-\gamma)} > 0$, $\lim_{n \to b} H(n) = -\lim_{n \to b} E_2(n) < 0$, there exists $n_0 \in (0, b)$ such that $H(n_0) = 0$, that is, the dynamical system (6), (7) has a nonzero steady state.

Note that

$$\lim_{n \to b} E_2(n) = (u_1')^{-1} \left[ \lim_{n \to b} u_2'(n) \frac{u_2'(n)}{\phi'(n)E_1'(n) + E_1(n)} \right]$$

and

$$\lim_{n \to b} A\phi'(n)E_1'(n) + E_1(n) = \frac{(\rho + b)^{2/(1-\gamma)}}{\alpha^{2(1-\gamma)}A^{1/(1-\gamma)}[\phi'(b)]^{1/(1-\gamma)}} \lim_{n \to b} \frac{u_2'(n)}{(b-n)^{2(1-\gamma)}}$$

we obtain the following corollaries:

Corollary 1. If

$$\lim_{n \to b} \frac{u_2'(n)}{(b-n)^{2(1-\gamma)}} < \infty,$$

then the dynamical system (6), (7) at least has a nonzero steady state.

If $u_2(n) = p(n)/b^{2\gamma}$, $p(n) = 2bn - n^2$, $0 < \gamma < 1$, then $u_2'(n) = \frac{2}{(b-n)^{(1-2\gamma)/(1-\gamma)}} = \begin{cases} 0, & 0 < \alpha < \frac{1}{2}, \\ \frac{2}{b^2}, & \alpha = \frac{1}{2}, \\ \infty, & \frac{1}{2} < \alpha < 1. \end{cases}$

Corollary 2. If $u_2(n) = p(n)/b^{2\gamma}$, $p(n) = 2bn - n^2$, $0 < \gamma < 1$, then the dynamical system (6), (7) at least has a nonzero steady state when $0 < \alpha \leq \frac{1}{2}$.

4. Multiple steady states and bifurcation

In order to determine the number and the type of the steady states, in the discussion below, we assume that the utility with respect to consumption is $u_1(c) = c^{1-\theta}/(1-\theta)$, $0 < \theta < 1$.

Theorem 2. Suppose that $u_1(c) = c^{1-\theta}/(1-\theta)$, $0 < \theta < 1$, $u_2(n) = p(n)/b^{2\gamma}$.

(1) If $0 < \alpha < (1+\theta)/2$, then the dynamical system (6), (7) at least has one nonzero steady state and there exists a constant $A$ such that the non-zero steady state is unique when $A > A$;

(2) If $(1+\theta)/2 < \alpha < 1$, then there exists a constant $A$ such that the dynamical system (6), (7) has more than two nonzero steady states when $A > A$;

(3) If $\alpha = (1+\theta)/2$, then dynamical system (6), (7) has at least one nonzero steady state when $A$ is big enough;

(4) If $(1+\theta)/2 < \alpha < 1$, then the dynamical system (6), (7) has no nonzero steady state when $A$ is small enough.
Proof. (1) By the definition of \( E_z(n) \) and \( z\alpha(1 - \phi(n))E_z^{z-1}(n) = \rho + n \),

\[
E_z(n) = \left[ \frac{b^{z\gamma} p^{1-\gamma}(n) (A\phi'(n)E_z^{z-1}(n) + 1)E_z^{1-\theta}(n)}{2\gamma(b - n)} \right]^{1/\theta} E_z(n),
\]

\[
H(n) = E_z(n) \left( \frac{\rho}{\alpha} + \frac{1 - \alpha}{\alpha} n \right) - \left\{ \frac{b^{z\gamma} p^{1-\gamma}(n)[\phi'(n) + \alpha(1 - \phi(n))]}{2\gamma(b - n)} \right\}^{1/\theta}.
\]

Let

\[
D(n) = \frac{b^{z\gamma} p^{1-\gamma}(n)[\phi'(n) + \alpha(1 - \phi(n))]}{2\gamma(b - n)} \left( \frac{1 - \phi(n)}{b - n} \left( \frac{\alpha}{\rho + n} \right)^{(1-\theta)/(1-\alpha)} \right),
\]

\[
H_1(n) = \frac{\rho}{\alpha} + \frac{1 - \alpha}{\alpha} n - A^{(1-\theta)/(\alpha(1-\alpha))} D^{1/\theta}(n) (1 - \phi(n))^{(2\alpha - (1+\theta))/(\theta(1-\alpha))},
\]

then \( H(n) = E_z(n)H_1(n) \) and \( n_0 \in (0, b) \) is a zero point of \( H(n) \) if and only if \( H_1(n_0) = 0 \). If \( 0 < \alpha < (1 + \theta)/2 \), then \( \lim_{n \to 0} H_1(n) = \rho/\alpha \), \( \lim_{n \to b} H_1(n) = -\infty \). Hence, there exists an \( n_0 \in (0, b) \) such that \( H_1(n_0) = 0 \).

Let \( \beta = (1 + \theta - 2\alpha)/(\theta(1 - \alpha)) \), \( B = A^{(1-\theta)/(\theta(1-\alpha))} \),

\[
\Phi_1(n) = (1 - \phi(n))^\beta \left( \frac{\rho}{\alpha} + \frac{(1 - \alpha)n}{\alpha} \right),
\]

\[
\Phi_2(n) = D^{1/\theta}(n), \quad \Phi(n) = \Phi_1(n)/\Phi_2(n) \text{ and } H_2(n) = \Phi(n) - B, \text{ then } H(n) = E_z(n)(1 - \phi(n))^{-\beta}\Phi_2(n)H_2(n)
\]

and \( n_0 \in (0, b) \) is a zero point of \( H(n) \) if and only if it is a zero point of \( H_2(n) \).

It is not difficult to verify that \( \lim_{n \to 0} \Phi(n) = \infty \), \( \lim_{n \to b} \Phi(n) = 0 \), \( \Phi_1(n)/\Phi_1(n) < (1 - \alpha)/\rho \), and

\[
\frac{\Phi_1'(n)}{\Phi_2(n)} = \frac{2(1 - \gamma)(b - n)}{\theta(2bn - n^2)} + \frac{\phi''(n) - \alpha\phi'(n)}{\theta(\phi(n) + \alpha(1 - \phi(n)))} + \frac{1 - \phi(n) - (b - n)\phi'(n)}{\theta(b - n)(1 - \phi(n))} - \frac{1 - \theta}{\theta(1 - \alpha)(\rho + n)}.\]

Note that

\[
\frac{\phi''(n) - \alpha\phi'(n)}{\phi'(n) + \alpha(1 - \phi(n))} > -\alpha > -1, \quad \frac{1 - \phi(n) - (b - n)\phi'(n)}{(b - n)(1 - \phi(n))} = \frac{\phi'(\xi) - \phi'(n)}{1 - \phi(n)} \geq 0, \quad n < \xi < b
\]

for \( \phi(b) = 1, \phi''(n) \geq 0 \) and

\[
-\frac{1 - \theta}{(1 - \alpha)(\rho + n)} > -\frac{1 - \theta}{(1 - \alpha)\rho} > -\frac{2}{\rho} \quad \text{for } 0 < \alpha < \frac{1 + \theta}{2}.
\]
Therefore, \( \lim_{n \to 0} 2(1 - \gamma)/(2bn - n^2) = \infty \) implies that there is \( \delta > 0 \) such that
\[
\frac{\Phi'(n)}{\Phi(n)} = \frac{\Phi'_1(n)}{\Phi_1(n)} - \frac{\Phi'_2(n)}{\Phi_2(n)} < 0, \quad n \in (0, \delta)
\]
and \( \Phi(n) \) is strictly decreasing on \( (0, \delta) \). On the interval \([\delta, b]\), \( \Phi(n) \) is continuous and has the maximum \( M \). So, if \( B > M \), \( H_2(n) \) has no zero point on \([\delta, b]\) and one unique zero point on \((0, \delta)\) since the function \( \Phi(n) \) strictly decreases from infinity to \( \Phi(\delta) \leq M \). Let \( \tilde{A} = M^\theta(1-\gamma)/(1-\theta) \), then \( A > \tilde{A} \) implies \( B > M \) and there is only one \( n_0 \in (0, b) \) such that \( \Phi(n_0) = B \). So, the dynamical system (6), (7) has a unique nonzero steady state.

(2) If \( (1 + \theta)/2 < \alpha < 1 \), then \( -\beta > 0 \) and \( \lim_{n \to 0} D^{1/\theta}(n) (1 - \phi(n))^{-\beta} = \lim_{n \to b} D^{1/\theta}(n) (1 - \phi(n))^{-\beta} = 0 \). Therefore, \( D^{1/\theta}(n) (1 - \phi(n))^{-\beta} \) is continuous in the interval \([0, b]\) and there exists a point \( n_0 \in (0, b) \) such that \( D^{1/\theta}(n) (1 - \phi(n))^{-\beta} \) achieves the maximum \( M > 0 \) at it. Let \( \tilde{B} \) be the constant such that \( \tilde{B}M = \rho + (1 - \alpha) b/\alpha \), then curve \( BD^{1/\theta}(n)(1 - \phi(n))^{-\beta} \) crosses the line \( (\rho + (1 - \alpha)n)/\alpha \) at least one time on the interval \( (0, n_0) \) and \( (n_0, b) \), respectively, provided \( B > \tilde{B} \). Let \( \tilde{A} = \tilde{B}^\theta(1-\gamma)/(1-\theta) \), then the dynamical system (6), (7) has at least two nonzero steady states when \( A > \tilde{A} \).

(3) If \( \alpha = (1 + \theta)/2 \), then
\[
\lim_{n \to b} H_1(n) = \frac{\rho}{\alpha} + \frac{(1 - \alpha) b}{\alpha} - A^{(1-\theta)/(\theta(1-\alpha)} \left[ \frac{b^2}{2\alpha'}[\phi'(b)]^2 \left( \frac{\alpha}{\rho + b} \right)^{(1-\theta)/(1-\gamma)} \right]^{1/\theta} < 0,
\]
when \( A \) is high enough. So, there is at least one \( n_0 \in (0, b) \) such that \( H_1(n_0) = 0 \).

(4) By the proof of (2), \( D^{1/\theta}(n) (1 - \phi(n))^{-\beta} \) is continuous on \([0, b]\) and achieves its maximum \( M \) at a point \( n_0 \in (0, b) \). Hence,
\[
A^{(1-\theta)/(\theta(1-\gamma))} D^{1/\theta}(n)(1 - \phi(n))^{-\beta} \leq M A^{(1-\theta)/(\theta(1-\gamma))} < \rho
\]
on \([0, b]\) provided \( A \) is small enough. So, the curves \( \rho/\alpha + ((1 - \alpha)/\alpha)n \) and \( A^{(1-\theta)/(\theta(1-\gamma))} D^{1/\theta}(n)(1 - \phi(n))^{-\beta} \) have no intersection point on \([0, b]\) and the function \( H_1(n) \) has no zero point on \([0, b]\).

The dynamic behavior of the dynamical system has fundamentally changed on the two sides of \( \alpha = (1 + \theta)/2 \) when \( A \) is big enough. So, \( \alpha = (1 + \theta)/2 \) is a bifurcation point of the dynamical system. We have the following theorem:

**Theorem 3.** The dynamical system (6), (7) undergoes a bifurcation when the technological level \( A \) is high enough and \( \alpha = (1 + \theta)/2 \).

When \( (1 + \theta)/2 < \alpha < 1 \), the dynamical system has no nonzero steady state with low technological level and has more than two nonzero steady states with high technological level. Denote
\[
H_1(n; A) = \frac{\rho}{\alpha} + \frac{1 - \alpha}{\alpha} n - 2A^{(1-\theta)/(\theta(1-\gamma))} D^{1/\theta}(n)(1 - \phi(n))^{-\beta}
\]
and \( \bar{A} = \sup \{ A \mid H_1(n; A) \text{ has no zero point} \} \), then \( H_1(n; A) \) has no zero point when \( A < \bar{A} \) and at least two zero points when \( A > \bar{A} \), that is, the dynamics of the dynamical system substantially changed on the two sides of \( A = \bar{A} \). So, we have the following theorem:

**Theorem 4.** If \((1 + \theta)/2 < \alpha < 1\), then there exists a technological level \( \bar{A} \) such that the dynamical system (6), (7) has no nonzero steady state when \( 0 < A < \bar{A} \) and has more than two nonzero steady states when \( A > \bar{A} \), that is, the dynamical system (6), (7) undergoes a bifurcation when \( A = \bar{A} \).

5. The dynamics of the economy

**Lemma 2.** Eq. (8) uniquely decides a differentiable curve \( c_1(k) \) satisfies \( \lim_{k \to 0} c_1(k) = 0 \), \( c_1(k) > 0 \), \( k > 0 \).

**Proof.** Given \( k_0 > 0 \), \( \lim_{c \to 0} n(k_0, c) = 0 \), \( \lim_{c \to \infty} n(k_0, c) = b \) for \( n(k_0, c) \) satisfies \( A k_0^\alpha \phi'(n(k_0, c)) \) \( u_1'(c) = u_2'(n(k_0, c)) \) and \( \frac{\partial n(k_0, c)}{\partial c} > 0 \). Let \( F_1(k, c) = A k^2 (1 - \phi(n(k, c))) - c - n(k, c) k \), then for any given \( k_0 > 0 \), \( \lim_{c \to 0} F_1(k_0, c) = A k_0^2 > 0 \), \( \lim_{c \to \infty} F_1(k_0, c) = -\infty \), hence there is a \( c_0 \) such that \( F_1(k_0, c_0) = 0 \). By the implicit function theorem, there exists a differentiable curve \( c_1(k) \) such that \( F_1(k, c_1(k)) = 0 \) on \((0, +\infty)\) for

\[
\frac{\partial F_1(k, c)}{\partial c} = -\left( A k^2 \phi'(n) + k \right) \frac{\partial n(k, c)}{\partial n} - 1 < 0, \quad c \in (0, +\infty).
\]

Since \( F_1(k, c_1(k)) = 0 \) implies \( 0 < c_1(k) = A k^2 (1 - \phi(n)) - nk < A k^2 \), \( \lim_{k \to 0} c_1(k) = 0 \). On the curve \( c_1(k), \ k = 0 \). The first quadrant of \( k, c \) plane is separated into two parts, \( \dot{k} < 0 \) above the curve \( c_1(k) \) and \( k > 0 \) below the curve \( c_1(k) \).

**Lemma 3.** Let \( k^* = E_1(0) \), then Eq. (9) uniquely decides a differentiable curve \( c_2(k) \) on \((0, k^*)\), satisfying \( \lim_{k \to k^*} c_2(k) = 0 \).

**Proof.** Since \( \lim_{n \to 0} E_1(n) = (\alpha A/\beta)^{1/(1-\alpha)} \), \( \lim_{n \to b} E_1(n) = 0 \) and \( dE_1(n)/dn < 0 \), \( k = E_1(n) \) uniquely decides a differentiable function \( n = n(k) \) on \((0, k^*)\) and \( \lim_{k \to k^*} n(k) = 0 \). Therefore, \( u_1'(c) = u_2'(n)/(Ak^2 \phi'(n) + k) \) decides a differentiable curve \( c_2(k) \), satisfying \( \lim_{k \to k^*} c_2(k) = 0 \). On the curve \( c_2(k) \), \( \dot{c} = 0 \). The first quadrant of \( k, c \) plane is separated into two parts, \( \dot{c} < 0 \) above the curve \( c_2(k) \) and \( \dot{c} > 0 \) below the curve \( c_2(k) \).

**Theorem 5.** If the nonzero steady states of the dynamical system (6), (7) are unique, then the steady state is a saddle.

**Proof.** By the assumption, the curves \( c_1(k) \), \( c_2(k) \) have only one intersection point \( k_1 \in (0, k^*) \). The point \( (k_1, c_1) \) \((c_1 = c_1(k_1) = c_2(k_1)) \) is the unique steady state since the intersections of two curves are the nonzero steady states of (6), (7). And \( c_1(k) < c_2(k) \), \( 0 < k < k_1 \), \( c_1(k) > c_2(k) \), \( k_1 < k < k^* \) for
$c_1(k)$ is over $c_2(k)$ when $k$ approximates $k^*$. The first quadrant of $k,c$ plane is separated into four regions and $(k_1,c_1)$ is saddle point by phase portrait analysis.

The phase portrait with one nonzero steady state is shown in Fig. 1.

Under the assumptions of Theorem 2, the economy has a unique growth path when the technological level is high enough and $0 < \alpha < (1 + \theta)/2$. In this case, the dynamics of the model is similar to the Ramsey model.

Now, we assume that the dynamical system has finite nonzero steady states and $(k_1,c_1)$ is the steady state on the right of $k,c$ plane, then the curve $c_1(k)$ is above the curve $c_2(k)$ on the interval $(k_1,k^*)$ and below the curve $c_2(k)$ on an interval $(k_1 - \delta,k_1)$ provided $\delta > 0$ is small enough. Using the above method, we have the following theorem:

**Theorem 6.** If the dynamical system (6), (7) has finite nonzero steady states, then the right steady state $(k_1,c_1)$ is a saddle.

With the assumptions of Theorem 2, the economy has at least two nonzero steady states when $(1 + \theta)/2 < \alpha < 1$ and the technological level is high enough. If the number of nonzero steady states is finite, then the right one is a saddle. Fig. 2 shows the case with two nonzero steady states.

6. Multiple growth paths and poverty trap

Obviously, the unique saddle path is the optimal growth path when the dynamical system has only one nonzero steady state. If the dynamic system has multiple nonzero steady states, then the right steady state is a saddle and the saddle path is the optimal growth path for the per capita capital and consumption is higher at it than at others. Correspondingly, the fertility is lower at it than at others. We confirm this result below.
Theorem 7. If \((k_i, c_i), i = 1, 2\) are two steady states of (6), (7) and \(k_1 > k_2\), then \(c_1 > c_2, n_1 < n_2\); i.e., the steady state with high per capital corresponds to high per capital consumption and low fertility.

Proof. If \(k_1 > k_2\), then \(E_1(n_1) > E_1(n_2)\) and \(n_1 < n_2\) for \(E_1(n)\) is a strictly decreasing function.

By (8), (9) and \(\alpha A(1 - \phi(n_1))k_1^{\alpha-1} = \rho + n_1 > n_1,\)

\[
c_1 - c_2 = (\alpha A(1 - \phi(n_1)))^{\alpha-1} - n_1 (k_1 - k_2) + (A\phi'(\tau)k_2^2 + k_2) (n_2 - n_1)
\]

\[
> (\alpha A(1 - \phi(n_1))k_1^{\alpha-1} - n_1) (k_1 - k_2) + (A\phi'(\tau)k_2^2 + k_2) (n_2 - n_1)
\]

\[
> 0,
\]

where \(\xi\) is a point between \(k_1\) and \(k_2\) and \(\tau\) is a point between \(n_1\) and \(n_2\). So, \(c_1 > c_2\). □

This theorem confirms the negative relationship between the fertility and per person consumption level. Furthermore, the leftmost nonzero steady state is a poverty trap which has lowest per person consumption and capital and has highest fertility.

On the \(k,c\) plane, the saddle \((k_1, c_1)\) is to the northeast of \(k,c\) plane. Similar to classical Ramsey model, one arm of the saddle path is a curve from \((0,0)\) to \((k_1, c_1)\), hence, for any given initial per capita capital \(k_0, 0 < k_0 < k_1\), there is a unique optimal per capita consumption choice \(c_0, \min\{c_1(k_0), c_2(k_0)\} > c_0 > 0\), such that \((k_0, c_0)\) is on the saddle path.

Proposition 1. For any given initial per capita capital \(k_0, 0 < k_0 < k_1\), there is a unique optimal consumption \(c_0, \min\{c_1(k_0), c_2(k_0)\} > c_0 > 0\), such that economy with the initial state \((k_0, c_0)\) grows along the optimal path.

Under the parameters \((A, \alpha, \rho, b, \theta, \gamma) = (0.2, 0.75, 0.06, 0.08, 0.4, 0.5)\), we obtain two steady states \((k_2, c_2) = (1.3118, 0.1177), (k_1, c_1) = (25.8854, 2.1015)\) and the fertility decreases along the optimal growth path with the initial state \((k_0, c_0)\), \(0 < k_0 < k_1\) by numerical analysis method.
7. Effects of the parameters

In this section, we assume that

\[ u(c,n) = \frac{c^{1-\theta}}{1-\theta} + \frac{p^*(n)}{b^{2\tau}} \quad \text{and} \quad 0 < \alpha < \frac{1+\theta}{2}. \]

From Theorem 2, when the technological level \( A \) is high enough, there exists \( \delta \) such that the function \( \Phi(n) \) is strictly decreasing on the interval \((0, \delta)\) and the dynamical system has a unique nonzero steady state \( k = E_1(n), c = E_2(n), n \in (0, \delta) \). Since the function \( \Phi(n) \) does not include the parameter \( A \) and \( \Phi(n_1) > \Phi(n_2), n_i \in (0, \delta), i = 1, 2 \) implies \( n_2 < n_1 \), we have

**Lemma 4.** If \( A_2 > A_1 > \bar{A} \) and \( n_i, i = 1, 2 \) are the solutions of the equations \( \Phi(n_i) = A_i^{(1-\theta)/(1+1-\tau))}, i = 1, 2 \), then \( n_2 < n_1 \).

**Proposition 2.** The per capita capital and per capita consumption increase and the fertility decreases at the steady state when the technological level increases.

**Proof.** Let \( A_1, A_2 \) be two technological levels satisfying \( A_2 > A_1 > \bar{A} \) and \( n_i, k_i, c_i, i = 1, 2 \) be the fertility, the per capita capital and consumption at the steady states corresponding to \( A_i, i = 1, 2 \), respectively. Lemma 2 implies \( n_2 < n_1 \), hence, \( k_2 > k_1 \) for

\[ k_i = \left( \frac{\alpha A_i(1 - \phi(n_i))}{\rho + n_i} \right)^{1/(1-\tau)}, \quad i = 1, 2. \]

Now we prove that \( c_2 > c_1 \).

Since \( \alpha A_2(1 - \phi(n_2))k_2^{\zeta-1} = \rho + n_2 > n_2 \),

\[ c_2 - c_1 > A_2k_2^\zeta(1 - \phi(n_2)) - n_2k_2 - A_2k_1^\zeta(1 - \phi(n_1)) + n_1k_1 \]

\[ = (\alpha A_2(1 - \phi(n_2))\zeta^{\zeta-1} - n_2)(k_2 - k_1) + (A_2\phi'(\tau)k_1^\zeta + k_1)(n_1 - n_2) \]

\[ > (\alpha A_2(1 - \phi(n_2))k_2^{\zeta-1} - n_2)(k_2 - k_1) + (A_2\phi'(\tau)k_1^\zeta + k_1)(n_1 - n_2) \]

\[ > 0, \]

where \( \zeta \) is a point between \( k_1 \) and \( k_2 \) and \( \tau \) is a point between \( n_1 \) and \( n_2 \).

This proposition implies that the promoted technological level increases the consumption per person and decreases the fertility of the household. This confirms the empirical fact that technological level is positively correlated with per capita consumption and negatively correlated with the fertility.

**Proposition 3.** When the technological level \( A \) is high enough and \( \rho < \alpha \), the per capita capital and consumption increase, and the fertility decreases at the steady state as the capital share increases.

**Proof.** By directly calculating, \( [\partial \ln \Phi(n; \alpha)]/\partial \alpha < 0 \) provided \( \alpha(1 - \phi(n)) > \rho + n \). When \( A \) is big enough, the proof of Theorem 2 implies that there exists a \( \delta > 0 \) such that \( \Phi(n; \alpha) \) crosses the line...
Therefore, we can wind a technological level to prove of Proposition 2, we have

\[
\frac{\partial \Phi(n; x)}{\partial x} = \frac{\partial \ln \Phi(n; x)}{\partial x} \Phi(n; x) < 0, \quad n \in (0, \delta).
\]

If \( x_2 > x_1 \), then for the high enough technological level \( A \), assuming \( n_i, i = 1, 2 \) are the fertilities at the steady states \((k_i, c_i)\), \( i = 1, 2 \) corresponding to the parameters \( x_i, i = 1, 2 \), respectively.

By \( \Phi(n_1; x_1) = \Phi(n_2; x_2) = A^{(1-\theta)/(\theta(1-x))} \), \( \Phi(n_1; x_1) = \Phi(n_2; x_2) = \Phi(n_2; x_1) \), \( n_2 < n_1 \). Similar to the proof of Proposition 2, we have \( k_2 > k_1, c_2 > c_1 \). □

**Proposition 4.** When the technological level \( A \) is high enough, the per capita capital, per capita consumption increase and the fertility decreases at the steady state as the coefficient of relative risk aversion increases.

**Proof.** Since

\[
\frac{\partial \ln \Phi(n; \theta)}{\partial \theta} = \frac{1}{\theta^2} \ln \left[ \frac{b^\gamma}{x_0^\gamma} p^{1-n}(n) \left( \phi'(n) + x(1 - \phi(n)) \right) \left( \frac{x}{\rho + n} \right)^{1/(1-x)} \frac{1}{b - n} (1 - \phi(n))^{\gamma/(1-x)} \right]
\]

and

\[
\lim_{n \to 0} \frac{b^\gamma}{x_0^\gamma} p^{1-n}(n) \left( \phi'(n) + x(1 - \phi(n)) \right) \left( \frac{x}{\rho + n} \right)^{1/(1-x)} \frac{1}{b - n} (1 - \phi(n))^{\gamma/(1-x)} = 0,
\]

there exists a \( \delta > 0 \) such that \( \partial \Phi(n; \theta)/\partial \theta < 0, n \in (0, \delta) \). When \( A \) is big enough, the unique zero point \( n_0 \) of \( H_1(n) \) is on the interval \( (0, \delta) \). Similar to the proof of Proposition 2, we obtain this proposition. □

Denote the households’ utility function with respect to fertility by \( u_2(n; \gamma) = p^{\gamma}(n)/b^2 \), then for a given \( n > 0 \), the households’ utility decrease as the parameter \( \gamma \) increases, for \( \partial u_2(n; \gamma)/\partial \gamma = u_2(n; \gamma) \ln \rho(n)/b^2 < 0 \). The households with low value of \( \gamma \) get more utility from the same reared children and the parameter \( \gamma \) describes the choice of household to have children. The following proposition shows us the effects of the choice of the household on the economic growth processes.

**Proposition 5.** If \( u_2(n; \gamma_i) = p^{\gamma_i}(n)/b^2, i = 1, 2, \gamma_1 > \gamma_2 \) are the two different household’s utility functions with respect to fertility and \( n_i, k_i, c_i, i = 1, 2 \) are the fertility, per capita capital and consumption at the steady state corresponding to \( \gamma_i, i = 1, 2 \), respectively, then \( n_1 < n_2, k_1 > k_2, c_1 > c_2 \) when \( A \) is big enough.

**Proof.** Since the function \( \Phi_i(n) \) does not include the parameter \( \gamma \),

\[
\frac{\partial \ln \Phi_i(n; \gamma)}{\partial \gamma} = -\frac{\partial \ln \Phi_i(n; \gamma)}{\partial \gamma} = \frac{1}{\gamma} \left[ \ln \frac{2bn - n^2}{b^2} + 1 \right]
\]

and \((\partial \Phi_i(n; \gamma)/\partial \gamma) < 0, \gamma \in [\gamma_2, \gamma_1] \), provided \( n \) is small enough for \( \lim_{n \to 0} \ln(2bn + n^2)/b^2 = -\infty \). Therefore, we can find a technological level \( A \) and a constant \( \delta > 0 \) such that \( \Phi_i(n; \gamma_i), i = 1, 2 \) are strictly decreasing on \( (0, \delta) \) and the zero point of \( H_1(n) \) is on the interval \( (0, \delta) \). So,
\[ \Phi(n; \gamma_2) > \Phi(n; \gamma_1), \ n \in (0, \delta) \] implies \( n_1 < n_2 \) and
\[
k_1 = \left[ \frac{\alpha A(1 - \phi(n_1))}{\rho + n_1} \right]^{1/(1-\alpha)} > \left[ \frac{\alpha A(1 - \phi(n_2))}{\rho + n_2} \right]^{1/(1-\alpha)} = k_2.
\]

Similar to the proof of Proposition 2, it can be proved that \( c_1 > c_2 \). \( \square \)

8. Summary

We have set up an extended Ramsey model in which the fertility is endogenously determined. Under different economic structures (i.e., with different parametric forms), the dynamical system appeared to have three different dynamics: no nonzero steady state, a unique nonzero steady state and multiple nonzero steady states. Correspondingly, the economics has no growth path, a unique optimal growth path and multiple growth paths in which there is an optimal growth path and at least a Malthus poverty trap. Along the optimal growth path, the economics converged to a high level of consumption and capital per person, and low level of fertility.

The dynamics of the model change substantially as the parameter varied slightly at some specified parametric value. When \((1+\theta)/2 < \alpha < 1\), there exists a technological level \( \tilde{A} \) such that the dynamical system has no nonzero steady state when \( 0 < \tilde{A} < \tilde{A} \) and at least two nonzero steady states when \( A > \tilde{A} \), which means that the model has multiple growth path and includes at least a Malthus poverty trap. Therefore, the dynamical system undergoes a bifurcation at \( A = \tilde{A} \). When the technological level is high enough, the dynamical system has a unique nonzero steady state when \( 0 < \alpha < (1 + \theta)/2 \) and at least two steady states when \( (1 + \theta)/2 < \alpha < 1 \), that is, the dynamical system undergoes a bifurcation when \( \alpha = (1 + \theta)/2 \). We particularly discussed the case with finite nonzero steady states in which the economy includes an optimal growth path and a Malthus poverty trap in Sections 5 and 6.

The next topic we focussed on was the effects of the parameters in case the economy has a unique growth path. The main results and their economic meanings have been given in Section 7.

References