Generalizations of the $q$-Morris Constant Term Identity

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The $q$-Morris constant term identity gives the constant term in the Laurent polynomial expansion of

$$
\Phi := \prod_{i=1}^{N} (w_i; q)_a (q/w_i; q)_b \prod_{1 \leq j < k \leq N} (w_j/w_i; q)_c (qw_k/w_i; q)_d.
$$

A conjecture is given for the constant term of $\Phi$ when multiplied by

$$
\prod_{j < k} (1 - q^a x_j x_k).
$$

This conjecture is proved in the cases

$$
a = b = 0 \text{ (general } N_0, N_1), \quad a = b = * \text{ (general } N_0, N_1), \quad N_1 = 2 \text{ (general } a, b, *, N_0),
$$

where $N_0 + N_1 = N$. Also, a general identity relating $q$-Morris-type constant terms and $q$-Selberg-type integrals is derived and is used to rewrite the conjecture as a $q$-Selberg-type integral evaluation.

1. INTRODUCTION

Suppose $a, b, * \in \mathbb{Z}_{\geq 0}$ and let

$$
H(x_1, ..., x_N; a, b, *) := \prod_{i=1}^{N} \left(1 - x_i\right)^a \left(1 - \frac{1}{x_i}\right)^b \prod_{j < k} \left(1 - \frac{x_j}{x_k}\right)^*. \quad (1.1)
$$

The Morris constant term identity [17] states that

$$
\text{CT}_{x_1, ..., x_N} H(x_1, ..., x_N; a, b, *) = M_N(a, b, *) \quad (1.2)
$$

where $\text{CT}_{x_1, ..., x_N}$ denotes the constant term in the Laurent polynomial expansion as a function of $\{x_1, ..., x_N\}$ and

$$
M_N(a, b, *) := \prod_{j=0}^{N-1} \frac{\Gamma(a + b + 1 + j\lambda) \Gamma(1 + j\lambda + 1)}{\Gamma(a + 1 + j\lambda) \Gamma(b + 1 + j\lambda) \Gamma(1 + \lambda)}.
$$

This identity has application in the study of some model systems in statistical and quantum physics [10, 11, 4], and these applications in turn suggest generalizations of (1.2). One such generalization is [11]

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\[
\text{CT}_{\{x_i\}} \prod_{i=1}^{N} \left(1 - \frac{y}{x_i}\right) \left(1 - \frac{x_i}{z}\right) H(x_1, ..., x_N; a, b, \lambda) \\
= M_N(a, b, \lambda) \ _2F_1\left(-N, \frac{1}{\lambda}; (a+1), \frac{1}{\lambda}(b+1); \frac{y}{z}\right)
\] (1.4)
while another is [9]
\[
\text{CT}_{\{x_i\}} \prod_{j,k=N_0+1}^{N} \left(1 - \frac{x_j}{x_k}\right) H(x_1, ..., x_N; a, b, \lambda) \\
= M_N(a, b, \lambda) \times \prod_{j=0}^{N_1-1} \frac{\Gamma((\lambda+1)j+a+b+\lambda N_0+1)}{\Gamma((\lambda+1)(j+1)+\lambda N_0+1)} \frac{\Gamma((\lambda+1)(j+1)+b+\lambda N_0+1)}{\Gamma((\lambda+1)(j+1)+b+\lambda N_0+1)}
\] (1.5)
where \( N = N_0 + N_1 \). We remark that although (1.4) has been proved in general, (1.5) has only been proved in the special cases \( a = b = 0 \), general \( \lambda, N_0, N_1 \), and \( \lambda = 1 \), general \( a, b, N_0, N_1 \).

Our interest in this paper is the \( q \)-analog of (1.5): we want to first formulate its conjectured form and then prove this conjecture in a number of special cases. Our motivation stems from the fact that (1.2), (1.4), and other constant term identities closely related to (1.4) (see [12]) have \( q \)-generalizations [7, 13–15, 19, 20], as does the theorem of Bressoud and Goulden [5, 6] used in [9] to prove (1.5) in the case \( a = b = 0 \), general \( \lambda \).

The conjectured \( q \)-generalization of (1.5) is given by (2.12) below, and this conjecture is further generalized in (2.18). In Section 3 (2.12) is proved in the special cases \( a = \lambda \) (general \( N_0, N_1, b, \lambda \)) and \( a = b = 0 \) (general \( N_0, N_1, \lambda \)) by using the \( q \) version of the theorem of Bressoud and Goulden cited above. Also, the special case \( N_1 = 2 \) (general \( a, b, \lambda, N_0 \)) is proved by adapting the method used by Zeilberger [19] to prove the \( q \)-analogue of the Morris identity (1.2). In Section 4 the conjecture (2.12) is recast as a generalization of the \( q \)-Selberg integral [3].

2. THE CONJECTURED \( q \)-GENERALIZATION

2.1. Revision of the \( q \)-Morris Identity

In the case \( a = b = 0 \) the Morris identity (1.2) reduces to the Dyson identity [8]
\[
\text{CT}_{\{x_i\}} H(x_1, ..., x_N; 0, 0, \lambda) = \frac{(\lambda N)!}{\lambda^{N^2}}
\] (2.1)
This was first q-generalized by Andrews [1]. Let

\[(a; q)_\lambda := \prod_{t=0}^{\lambda-1} (1 - a q^t), \quad \lambda \in \mathbb{Z}_{\geq 0} \tag{2.2}\]

and

\[\Gamma_q(n + 1) := \prod_{j=1}^{n} [j]_q, \quad [j]_q := \frac{1 - q^j}{1 - q}. \tag{2.3}\]

Then according to Andrews [1], (2.1) has the q-generalization

\[\text{CT}_{\{x\}} H_q(x_1, ..., x_N; 0, 0, \hat{\lambda}) = \frac{\Gamma_q(\hat{\lambda}N + 1)}{(\Gamma_q(\hat{\lambda} + 1))^N}, \quad \hat{\lambda} \in \mathbb{Z}_{\geq 0}, \tag{2.4}\]

where

\[H_q(x_1, ..., x_N; a, b, \hat{\lambda}) := \prod_{j=1}^{N} (x_j; q) \prod_{b < j < k < N} \left( q^{x_k - x_j} \right) \left( q^{x_j - x_k} \right). \tag{2.5}\]

Note the restriction \(\hat{\lambda} \in \mathbb{Z}_{\geq 0}\) in (2.4). Now, for general \(\hat{\lambda}\) we can interpret (2.5) by extending the definition (2.2):

\[(a; q)_\hat{\lambda} := \frac{(a; q)_\infty}{(aq^\hat{\lambda}; q)_\infty}. \tag{2.6}\]

As pointed out by Stembridge [18], the identity (2.4) then remains valid for general \(\hat{\lambda}\) with

\[\Gamma_q(x) = \frac{(q; q)_x}{(q^x; q)_\infty} (1 - q)^{-x} = (q; q)_{x-1} (1 - q)^{-x+1} \tag{2.7}\]

and

\[\text{CT}_{\{x\}} f(x_1, ..., x_N) = \prod_{j=1}^{N} \int_0^1 dx_j f(e^{2\pi i x_1}, ..., e^{2\pi i x_N}). \tag{2.8}\]

Similar to the q-identity (2.4), the Morris identity (1.1) for general \(a\) and \(b\) has the q-generalization [14, 13, 19]

\[\text{CT}_{\{x\}} H_q(x_1, ..., x_N; a, b, \hat{\lambda}) = M_N(a, b, \hat{\lambda}; q), \quad a, b, \hat{\lambda} \in \mathbb{Z}_{\geq 0}, \tag{2.9}\]
where $H_q$ is defined by (2.5) and 
\[
M_N(a, b, \lambda; q) := \prod_{l=0}^{N-1} \frac{F_q(a+b+1+\lambda l)}{F_q(a+1+\lambda l) \Gamma_q(b+1+\lambda l) \Gamma_q(1+\lambda l)}. \tag{2.10}
\]

For general $a, b, \lambda$ this identity remains valid with $\text{CT}_{\{x\}}$ replaced by the integral (2.8).

2.2. $q$-Generalization of (1.5)

Motivated by the $q$-Morris identity (2.9), we formulated the $q$-generalization of the l.h.s. of (1.5) to be
\[
\text{CT}_{\{x\}} \prod_{N_0+1 \leq j \leq k \leq N} \left( 1 - q^j \frac{x_j}{x_k} \right) \left( 1 - q^{j+1} \frac{x_k}{x_j} \right) H_q(x_1, \ldots, x_N; a, b, \lambda). \tag{2.11}
\]

Note that in the case $N_0 = 0$ this reduces to the l.h.s. of the $q$-Morris identity with $\lambda$ replaced by $\lambda + 1$. Similarly, on the r.h.s. the $q$-Morris identity suggests we replace the functions $\Gamma(u)$ by $\Gamma_q(u)$. But on the r.h.s. there are also factors $(1+\lambda)(j+1)$. To obtain the correct form of the $q$-generalization we obtained some exact computer generated evaluations of (2.12) for $\lambda = 1$ and 2 and various “small” values of $N_0$, $N_1$, $a$, and $b$. These data were consistent with replacing $(1+\lambda)(j+1)$ by $[(1+\lambda)(j+1)]_q$, and thus supported the following conjecture for the $q$-generalization of (1.5).

**Conjecture 2.1.** With $H_q(x_1, \ldots, x_N; a, b, \lambda)$ defined by (2.5), $M_N(a, b, \lambda; q)$ defined by (2.10), and $a, b, \lambda \in \mathbb{Z}_{\geq 0}$, we have
\[
\text{CT}_{\{x\}} \prod_{N_0+1 \leq j \leq k \leq N} \left( 1 - q^j \frac{x_j}{x_k} \right) \left( 1 - q^{j+1} \frac{x_k}{x_j} \right) H_q(x_1, \ldots, x_N; a, b, \lambda) = M_N(a, b, \lambda; q)
\times \prod_{j=0}^{N-1} \frac{F_q(\lambda+1)}{F_q(\lambda+2) \Gamma_q(\lambda+1)} \frac{F_q(\lambda+1+j+a+b+\lambda N_0+1)}{F_q(\lambda+1+j+a+b+\lambda N_0+1)}
\times \prod_{j=0}^{N-1} \frac{F_q(\lambda+1+j+a+\lambda N_0+1)}{F_q(\lambda+1+j+a+\lambda N_0+1)} \Gamma_q(\lambda+1+j+a+b+\lambda N_0+1). \tag{2.12}
\]

Note that the r.h.s. of (2.12) is manifestly a symmetric function of $a$ and $b$ whereas this symmetry is not immediate on the l.h.s. In fact, this symmetry can be made an immediate feature of the l.h.s. if we first apply a
lemma of Kadell [14, Lemma 4], which gives that for any \( f \) symmetric in 
\( y_1, \ldots, y_m \),
\[
\CT_{\{y\}} f(y_1, \ldots, y_m) H_q(y_1, \ldots, y_m; 0, 0, \lambda) = \frac{[m]_q!}{m!} \CT_{\{y\}} f(y_1, \ldots, y_m) \tilde{H}_q(y_1, \ldots, y_m; 0, 0, \lambda),
\]
(2.13)
where
\[
\tilde{H}_q(y_1, \ldots, y_m; a, b, \lambda) = \prod_{i=1}^{\lambda} (y_i; q)_a \prod_{k=1}^{\lambda-1} \left( \frac{y_k}{y_j}; q \right)_a \left( \frac{y_j}{y_k}; q \right)_a.
\]
Using (2.13), first with \( \{ y \} = \{ x_j \}_{j=1, \ldots, N_0} \), and then with \( \{ y \} = \{ x_{N_0+1}, \ldots, x_N \} \) and \( \lambda \) replaced by \( \lambda + 1 \), we see that the l.h.s. of (2.12) can be rewritten as
\[
\frac{[N_0]_q! \cdot [N_1]_q^{\lambda+1}}{N_0!} \CT_{\{x\}} \prod_{j, k = N_0+1}^{N} \left( 1 - \frac{x_j}{x_k} \right) \tilde{H}_q(x_1, \ldots, x_N; a, b, \lambda).
\]
(2.15)
Now making the replacements \( x_j \rightarrow q/x_j \) \( (j = 1, \ldots, N) \) the constant term must be unchanged, while from (2.14) we see that \( \tilde{H}_q \) transforms to the same function with \( a \) and \( b \) interchanged, thus establishing that the l.h.s. of (2.12) is indeed symmetric in \( a \) and \( b \).

2.3. The Multi-component Generalization of (1.5) and Its q-Generalization

The factor \( \prod_{1 \leq j < k \leq N} (1 - x_j/x_k) \) in (1.5) distinguishes the variables \( \{ x_1, \ldots, x_N \} \) from the variables \( \{ x_{N_0+1}, \ldots, x_N \} \). In this sense the function on the r.h.s. of (1.5) consists of two components. In Ref. [9] a \((p + 1)\)-component generalization of the Morris constant term identity (1.1), which reduces to (1.5) in the case \( p = 1 \), has been identified. This is the constant term
\[
\CT_{\{x\}} \prod_{j, k = N_0+1}^{N_0+N_1} \left( 1 - \frac{x_j}{x_k} \right) \prod_{j, k = N_0+N_1+1}^{N_0+N_1+N_2} \left( 1 - \frac{x_j}{x_k} \right)
\]
\[
\cdots \prod_{j, k = N_0+N_1+N_2+1}^{N_0+N_1+N_2+N_3} \left( 1 - \frac{x_j}{x_k} \right) H(x_1, \ldots, x_N; a, b, \lambda)
\]
(2.16)
where \( N = \sum_{q=0}^{p} N_q \). It was conjectured in [9, Eq. (4.8a)] that this constant term satisfies a recurrence formula, which implies that it can be expressed as a product of gamma functions. From the conjecture of Ref. [9] and the conjecture (2.12) above, it is straightforward to conjecture the \( q \)-generalization of the recurrence formula.

**Conjecture 2.2.** Let

\[
D_p(N_1, \ldots, N_{p-1}, N_p; N_0; a, b, \lambda; q) = C T_{\{x\}_j} \prod_{N_q + 1 \leq j < k < N_q + N_1} \left( 1 - q^j \frac{x_j}{x_k} \right)
\]

\[
\times \prod_{N_q + 1 \leq j < k < N_q + N_1 + N_2} \left( 1 - q^j \frac{x_j}{x_k} \right) \left( 1 - q^{j+1} \frac{x_j}{x_k} \right)
\]

\[
\times \cdots \times \prod_{\sum_{q=0}^{p-1} N_q + 1 \leq j < k < \sum_{q=0}^{p} N_q} \left( 1 - q^j \frac{x_j}{x_k} \right) \left( 1 - q^{j+1} \frac{x_j}{x_k} \right)
\]

\[
\times H_q(x_1, \ldots, x_N; a, b, \lambda),
\]

\(2.17\)

where \( N = \sum_{q=0}^{p} N_q \), and suppose \( N_p \geq N_j - 1 \) (\( j = 1, \ldots, p - 1 \)). As a function of \( N_p \) the function \( D_p \) satisfies the recurrence

\[
\frac{D_p(N_1, \ldots, N_{p-1}, N_p; N_0; a, b, \lambda; q)}{D_p(N_1, \ldots, N_{p-1}, N_p; N_0; a, b, \lambda; q)}
\]

\[
= \left[ (\lambda + 1)(N_p + 1) \right]_q
\]

\[
\times \frac{\sum_{\lambda=0}^{p} f_{\lambda+1}(N_p + a + b + \lambda \sum_{q=0}^{p-1} N_q + 1)}{\sum_{\lambda=0}^{p} f_{\lambda+1}(N_p + a + \lambda N_0 + 1)}
\]

\(2.18\)

In the limit \( q \to 1 \) the formula in Conjecture 2.2 is equivalent to the recurrence equation conjectured in [9, Eq. (4.8a)].

3. PROOF OF SOME SPECIAL CASES OF CONJECTURE 2.1

In this section Conjecture 2.1 will be proved in some special cases. We will use the notation (2.17) and thus denote the l.h.s. of (2.12) by

\[
D_1(N_1; N_0; a, b, \lambda; q)
\]
3.1. The Cases $a = \lambda$, General $N_0$, $N_1$, $b$, $\lambda$, and $a = b = 0$, General $N_0$, $N_1$, $\lambda$.

It was noted in Ref. [9] that the constant term identity (1.5) in the case $a = b = 0$ and general $N_0$, $N_1$, $\lambda$ follows from a theorem of Bressoud and Goulden [6]. This theorem has a $q$-counterpart, obtained by the same authors in an earlier publication.

**Proposition 3.1** [5, Proposition 2.4, with $A$ Replaced by $\bar{A}$, the Complement of $A$, to Be Consistent with the Formulation in Ref. [6]]. Let $a_1, \ldots, a_N$ be positive integers, $A$ be an arbitrary subset of $\{(i, j): 1 \leq i < j \leq n\}$, $G_A$ be the set of permutations $\sigma$ on $\{1, \ldots, n\}$ (with $\sigma(1) := \sigma_i$) whose inversions $I(\sigma) := \{(\sigma_i, \sigma_j): \sigma_i > \sigma_j \text{ and } i < j\}$ are contained in $A$,

$$G_A = \{\sigma: (i, j) \in I(\sigma), \text{then } (i, j) \in A\},$$

and let $\chi(T)$ be the characteristic function which is 1 if $T$ is true, and 0 otherwise. We have

$$CT = \prod_{1 \leq i < j \leq n} \frac{q^{X_i} - q}{X_j - q} a_{i-j} \cdot \frac{q^{X_j} - q}{X_i - q} a_{j-i} = \prod_{i=1}^{a_1} T_d(a_1) \cdots T_d(a_n) S_d\{a_j\}_{j=1, \ldots, n}: G_A,$$

where

$$S_d\{a_j\}_{j=1, \ldots, n}: G_A := \sum_{\sigma \in G_A} q^{\sum_{i<j} \sigma_i} \prod_{i=1}^{n} \frac{1-q}{1-q^{a_1+\cdots+a_n}}.$$

In this theorem, suppose $n = 1 + N_0 + N_1$, $A = \{(i, j): 1 \leq i < j \leq N_0 + 1 \text{ or } N_0 + 2 \leq i < j \leq N_0 + N_1 + 1\}$,

$$a_1 = b_\lambda, \quad a_2 = \cdots = a_{N_0 + 1} = \lambda, \quad a_{N_0 + 2} = \cdots = a_{N_0 + N_1 + 1} = \lambda + 1,$$

(3.1)

and replace $x_2, \ldots, x_{N_0} + N_1 + 1$ by $x_1 x_2, \ldots, x_1 x_{N_0} + N_1 + 1$. We see that the l.h.s. is of the form (2.14) with $a = \lambda$ and $b$, $\lambda$ arbitrary positive integers. Proposition 3.1 therefore gives

$$D_1(N_1; N_0; \lambda, b, \lambda; q) = \prod_{i=1}^{a_1} T_d(\lambda + 1) N_1 \lambda N_0 + b + 1) S_{\lambda N_0 N_1}\{a_j\}: G_A.$$

(3.2)

Our proof of Conjecture 2.1 for the evaluation of $D_1(N_1; N_0; \lambda, b, \lambda; q)$ now follows from the following evaluation of $S_{\lambda N_0 N_1}$. 

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**Proposition 3.2.** With $A, \{a_j\}$ given by (3.1) we have

$$S_{N_0, N_1}(\{a_j\}; G_A) = \frac{1}{[b]_q} \sum_{j=1}^{N_1} \frac{[\lambda + 1]_q j}{[\lambda + 1]_q} S_{N_0, N_1}(\{a_j\}; G_A).$$

**Proof.** We will establish a recurrence relation in $N_1$. To solve the recurrence we require the value of $S_{N_0, 0}$. This is obtained by noting that when $N_1 = 0$, $D_1$ is given by the $q$-Morris identity (2.9). Comparison with (3.2) then gives

$$S_{N_0, 0}(\{a_j\}; G_A) = \frac{1}{[b]_q} [\lambda]_q S_{N_0, 0}.$$  \hspace{1cm} (3.3)

The recurrence is obtained by noting that all permutations in $G_A$ are of the form $(\sigma^*, \sigma^*)$, where $\sigma^*$ is a permutation of $\{1, ..., N_0 + 1\}$, and $\sigma^*$ is a permutation of $\{N_0 + 2, ..., N_0 + N_1 + 1\}$. Thus, if $N_1$ is increased by 1 only $\sigma^*$ can be affected. Furthermore, for each $\sigma = (\sigma^*, \sigma^*)$ in $G_A$ before increasing $N_1$ by 1, there are $N_1 + 1$ permutations in $G_A$ after $N_1$ is increased by 1, which are given by $\sigma = (\sigma^*, \sigma^*_{k \rightarrow N_0 + N_1 + 2}, k)$, $k = N_0 + 2, ..., N_0 + N_1 + 2$ (for $k = N_0 + N_1 + 2$, $\sigma^*$ remains unchanged). Denote these permutations by $G_d(k)$ so that for $N_1$ increased by 1, $G_A = \bigcup_{k=N_0+2}^{N_0+N_1+2} G_d(k)$. The facts that the replacement $k \mapsto N_0 + N_1 + 2$ in $\sigma^*$ creates $N_o + N_1 + 2 - k$ new inversions, that $a_i = \lambda + 1$ for $i = N_0 + 2, ..., N_0 + N_1 + 2$, and that

$$(1 - q)/(1 - q^{a_i + ... + a_{N_0 + N_1 + 2}}) = (1 - q)/(1 - q^{a_i + ... + a_{N_0 + N_1 + 2}})$$

is a common factor in the summand now gives the recurrence

$$S_{N_0, N_1+1}(\{a_j\}; G_A(k)) = \frac{q^{(\lambda + 1)(N_0 + N_1 + 2 - k)}}{([\lambda + 1](N_1 + 1) + 2N_0 + b)_q} S_{N_0, N_1}(\{a_j\}; G_A(k)).$$

Summing over $k$ we have

$$S_{N_0, N_1+1}(\{a_j\}; G_A) = \frac{[\lambda + 1](N_1 + 1)}{([\lambda + 1](N_1 + 1) + 2N_0 + b)_q} S_{N_0, N_1}(\{a_j\}; G_A),$$

which upon iteration and use of the initial condition gives the stated result.

Substituting the result of Proposition 3.2 in (3.2) evaluates $D_1(N_1; N_0; \lambda, b, \lambda, q)$ for $\lambda$ and $b$ arbitrary positive integers; by a simple lemma of Stembridge [18, Lemma 3.2], the validity of the positive integer case
implies the validity for all (complex) \( \lambda \) and \( b \). Comparison between the resulting expression for \( D_1 \) and the expression of Conjecture 2.1 in the case \( a = \lambda \) shows, after simplification of the latter, that the two expressions are identical.

The validity of Conjecture 2.1 in the case \( a = b = 0 \), general \( N_0, N_1, \lambda \), follows from its validity in the case \( a = \lambda \), general \( N_0, N_1, b, \lambda \), by setting \( b = 0 \) in the latter. Thus from the definition we see that in general

\[
D_1(N_1; N_0; a, 0, \lambda; q) = D_1(N_1; N_0; 0, 0, \lambda; q)
\]

and this is a property of the r.h.s. of Conjecture 2.1.

3.2. The Case \( N_1 = 2 \) and General \( N_0, N_1, a, b, \lambda \).

We address the case \( N_1 = 2 \), as this is the first non-trivial case; when \( N_1 = 1 \), it is clear that

\[
D_1(1; N_0; a, b, \lambda; q) = D_0(N_0 + 1; a, b, \lambda; q)
\]

which is evaluated by the \( q \)-Morris identity. To prove Conjecture 2.1 in this particular case, we adopt the method of Stembridge [18] and Zeilberger [19]. The essence of this method, applied to the problem at hand, is to express the constant term of the two-component function, in terms of the constant terms of \( H_q \) multiplied by suitable Laurent polynomials, by means of a partial expansion of the two-component function.

It will prove useful to briefly summarize the results of Zeilberger [19] as we shall be aiming to extend his proof of the \( q \)-Morris identity.

Let \([x^0]()\) denote the coefficient of \( x^0 \) in the expression for \( F_0(x) \), and in general let \([x^0]()\) denote the coefficient of \( x^0 \) in the expansion of \( f \). For notation convenience, this can be extended so that for a general function \( g(x) = \sum_{a \geq 0} a x^a \), one writes \([g] f := \sum_{a \geq 0} a [x^a] f \).

The “reduced” \( q \)-Morris identity takes the form

\[
[x^0] F_0(x) = \prod_{j=0}^{n-1} \frac{\Gamma_q(\lambda a + b + 1) \Gamma_q(\lambda j + 1)}{\Gamma_q(\lambda j + a + 1) \Gamma_q(\lambda j + b + 1) \Gamma_q(\lambda)}
\]

where

\[
F_0(x) := \prod_{i=1}^{n} \frac{q}{x_i} \prod_{1 \leq i < j \leq n} \left( \frac{x_i}{x_j} \right) \left( \frac{q}{x_i} \right) \left( \frac{x_j}{q} \right).
\]

A lemma of Stembridge [18] gives that (3.4) is equivalent to the original \( q \)-Morris identity (2.19). Zeilberger’s proof of the “reduced” \( q \)-Morris identity relies on the function \( F_0(x) \) being almost anti-symmetric. Indeed,
F_\delta(x) = x^{-\delta} G_\delta(x) \quad \text{where} \quad \delta := (n-1, n-2, \ldots, 2, 1, 0), \quad \text{and} \quad G_\delta(x) \text{ is anti-symmetric. Thus, the constant term of the function } F_\delta(x) \text{ is just } \{x^0\} G_\delta(x).

As part of his inductive proof, he essentially uses the equation

\[
q^n \sum_{\pi} \left( l^{(\pi)} \prod_{i=1}^{n-1} \left( 1 - \frac{z x_i}{x_i} \right) \right) G_\delta(x) = t^{n-1} \left( x^{\delta+\beta} \prod_{i=1}^{n-1} \left( 1 - t^{-1} \frac{x_i}{x_i} \right) \right) G_\delta(x),
\]  

where \( u := q^n, \ s := q^n, \ t := q^n \), to relate \{x^0\} G_\delta(x) to \{x^\beta\} G_\delta(x) for various special values of \( \beta \). This is done by using the anti-symmetry of \( G_\delta(x) \) and his “Crucial Lemma.”

**Lemma 3.3 [19].** If \( G_{\delta}(x_1, \ldots, x_n) \) is an anti-symmetric Laurent polynomial, \( \gamma \in \mathbb{Z}_n \), and \( \sigma \) is a permutation then \( \{x^{\sigma(\gamma)}\} G_\delta(x) = \text{sgn } \sigma \{x^\gamma\} G_\delta(x) \). In particular, if any two components of \( \gamma \) are equal, then \( \{x^\gamma\} G_\delta(x) = 0 \).

As an example of how this is done, let us give a result we shall use subsequently.

**Lemma 3.4.** Let \( \alpha_1 = (1, 0, \ldots, 0) \) and \( \alpha_2 = (1, 0, \ldots, 0, -1) \). Then

\[
\{x^{\alpha_1+\beta}\} G_\delta(x) = \frac{(s-1)(1-t^n)}{(1-q^{n-1} t^{n-1})} \{x^\beta\} G_\delta(x),
\]

\[
\{x^{\alpha_2+\beta}\} G_\delta(x) = \frac{(t-qs)(1-t^n-1)}{(1-qst^{n-1})} \{x^\beta\} G_\delta(x) + \frac{q(n-t)(1-t^n-1)}{(1-qst^{n-1})} \{x^{\alpha_1+\beta}\} G_\delta(x).
\]

**Proof.** Bearing in mind (3.6), first look at the expansion

\[
\prod_{i=1}^{n-1} \left( 1 - \frac{z x_i}{x_i} \right) = \sum_T (-z)^{|T|} x^{-T}
\]

where the sum is over all \( T \subseteq \{1, 2, \ldots, n-1\}, \) and \( x^{-T} := \prod_{T \subseteq \{1, 2, \ldots, n-1\}} x_i^{-T} \). For each \( m \) with \( 0 \leq m \leq n-1 \), there exists a unique set \( T \) such that \(|T| = m\) and \( x^{\sigma(T)} x^{-T} \) has distinct exponents; namely, \( T = \{n-m, n-m+1, \ldots, n-1\} \). In fact, \( x^\sigma(T) x^{-T} = x^\sigma(\delta) \) where \( \sigma \) is a permutation with \( \text{sgn } \sigma = (-1)^m \). Thus

\[
\{x^\sum_{m=0}^{n-1} (-z)^m (1-z)^m \} G_\delta(x) = \prod_{i=1}^{n-1} \left( 1 - \frac{z x_i}{x_i} \right) G_\delta(x) = \frac{(1-z^n)}{(1-z)} \{x^\beta\} G_\delta(x).
\]
We must also expand
\[ x_n \prod_{i=1}^{n-1} \left( 1 - \frac{z}{x_i} \right) = \sum_T (-z)^{|T|} (x_n)^{|T|+1} x^{-T}. \] (3.11)

In this case, there is only one set \( T \) such that \( x^d x_n^{\mid T\mid} x^{-T} \) has distinct exponents, \( T = \{1, 2, ..., n-1\} \). Moreover, for this set \( T \),
\[ x^d x_n^{\mid T\mid} x^{-T} = x_n^{n-2} x_n^{n-3} \cdot x_n^{0} = x^m(x_1 + d) \]
where \( \text{sgn } \sigma = (-1)^{n-1} \).

Thus
\[ \left[ x^d x_n \prod_{i=1}^{n-1} \left( 1 - \frac{z}{x_i} \right) \right] G_0 = z^{n-1} [x^{m+\delta}] G_0. \] (3.12)

If one now uses (3.10) and (3.12) with \( z = t \), \( t^{-1} \) in (3.6), and sets \( \beta = 0 \), the stated result (3.7) follows.

To prove (3.8), note that the sets \( T \) such that \( x^2 + x_n^{\mid T\mid} x^{-T} \) has distinct exponents are of the form
\[ T = \{1, n-m+1, n-m+2, ..., n-1\} \quad 1 \leq m \leq n-1 \]
in which case
\[ x^{2+\delta} x_n^{\mid T\mid} x^{-T} = \left\{ x^{2+\delta}, x^{\mid T\mid} \right\} \quad \text{sgn } \sigma = (-1)^{m-1} \]
Thus
\[ \left[ x^{2+\delta} x_n \prod_{i=1}^{n-1} \left( 1 - \frac{z}{x_i} \right) \right] G_0 = [x^{2+\delta}] G_0 - z \left( \frac{1 - z^{n-1}}{1 - z} \right) [x^\delta] G_0. \] (3.13)

Similarly, the sets \( T \) such that \( x^{2+\delta} x_n^{\mid T\mid+1} x^{-T} \) has distinct exponents are of the form \( T = \{n-m, n-m+1, ..., n-1\} \), \( 0 \leq m \leq n-2 \), in which case
\[ x^{2+\delta} x_n^{\mid T\mid+1} x^{-T} = x^{m+\delta}, \] with \( \text{sgn } \sigma = (-1)^n \).

Hence
\[ \left[ x^{2+\delta} x_n \prod_{i=1}^{n-1} \left( 1 - \frac{z}{x_i} \right) \right] G_0 = \left[ \frac{1 - z^{n-1}}{1 - z} \right] [x^{2+\delta}] G_0 \] (3.14)
Again, using (3.13), (3.14) in (3.6) (setting \( \beta = \alpha_2 \)), with \( z = t \), \( t^{-1} \) yields (3.8). \( \blacksquare \)
Returning to the proof of the \( N_1 = 2 \) case of Conjecture 2.1, we first make the substitutions
\[
\begin{align*}
 x_i &\rightarrow w_{N_0 + 1} \ldots (i = 1, \ldots, N_0), \\
x_i + N_0 &\rightarrow z_{N_0 + 1} \ldots (i = 1, \ldots, N_1)
\end{align*}
\]
(which has no effect on the constant term) and then follow the arguments in Ref. [18], whereby we replace \((q^{z_i} z_j; q)_i \rightarrow (q^{z_i} z_j; q)_i \) and \((q^{w_i} w_j; q)_i \rightarrow (q^{w_i} w_j; q)_i \), to obtain an alternative statement of Conjecture 2.1 which reads as
\[
\begin{align*}
&C_T \prod_{i=1}^{N_0} (w_i; q)_a (q^{w_i} w_j; q)_b \\
&\times \prod_{1 \leq i, j \leq N_0} \left( \frac{q^{w_j}}{q^{w_i}} \right)^{z_i} \left( q^{w_i} w_j; q \right)_{i-1} \\
&\times \prod_{1 \leq i, j \leq N_1} \left( \frac{z_j}{z_i} q^{z_i} \right) \left( \frac{q^{z_i}}{z_j} q \right) \prod_{i=1}^{N_0} \prod_{j=i}^{N_1} \left( q^{w_i} w_j; q \right)^{z_j} \left( \frac{z_j}{z_i} q \right) \\
&= \frac{1}{T_x(N_0 + 1) T_x(N_1 + 1)} \text{(RHS of (2.12))},
\end{align*}
\]
where \( D_1(N_1; N_0; a, b, \lambda) \) is given in (2.15). In the particular case of \( N_1 = 2 \), the function appearing on the left-hand side of the above equation, call it \( F_1(x) \), say, is simply related to the \( n = N_0 + 2 \) variable function \( F_0(x) \) in (3.5). Thus, letting \( x_i = w_i, 1 \leq i \leq N_0, \) and \( x_{N_0 + 1} = z_1, x_{N_0 + 2} = z_2, \) we have
\[
F_1(x) := \left( 1 - t^{X_{N_0 + 1}} \right) \left( 1 - t^{X_{N_0 + 2}} \right) \prod_{i=1}^{N_0} \left( 1 - t^{X_i} \right) \prod_{i=1}^{N_0} \left( 1 - t^{X_{N_0}} \right) F_0(x).
\]
Using the “reduced” \( q \)-Morris identity (3.4), it suffices to prove
\[
\begin{align*}
[x^0] F_1(x) &= (1 - t^{X_{N_0 + 1}})(1 - t^{X_{N_0 + 2}})(1 - q t^{X_{N_0 + 1}})(1 - q t^{X_{N_0 + 2}}) \cdot [x^0] F_0(x).
\end{align*}
\]
Note that we can rewrite \([x^0] F_1(x)\) in the following form:
\[
\begin{align*}
[x^0] F_1(x) &= \left[ \left( 1 + t^2 \right) \left( 1 - t^{X_{N_0 + 1}} \right) \left( 1 - t^{X_{N_0 + 2}} \right) \prod_{i=1}^{N_0} \left( 1 - t^{X_{N_0}} \right) \left( 1 - t^{X_i} \right) \right]^\delta G_0(x),
\end{align*}
\]
where \( \delta = (N_0 + 1, N_0, \ldots, 1, 0) \), and \( G_0(x) \) is anti-symmetric. Let us now show that each of the terms \([A(x)] \prod_{i=1}^{N_0} (1 - t x_{N_0 + 1}/x_i) (1 - t x_{N_0 + 2}/x_i) x^\delta \) \( G_0(x) \), for
A(x) = 1 + t^2 - t x_{N_0+1} x_{N_0+2}^{-1} and -t x_{N_0+2} x_{N_0+1}^{-1} can be expressed in terms of \([x^d] G_0\) and \([x^{n+d}] G_0\) using the above techniques.

**Lemma 3.5.** We have

\[
\prod_{i=1}^{N_0} \left( 1 - \frac{X_{N_0+1}}{X_i} \right) \left( 1 - \frac{X_{N_0+2}}{X_i} \right) G_0 = B_{N_0}(t)
\]

\[
\prod_{i=1}^{N_0} \left( 1 - \frac{-X_{N_0+1}}{X_i} \right) \left( 1 - \frac{-X_{N_0+2}}{X_i} \right) G_0 = tB_{N_0}(t)
\]

where

\[
B_{N_0}(t) = \frac{1}{1-t} \left( \frac{1 - t^{N_0+1}}{1-t} - \frac{1 - t^{N_0+2}}{1-t^2} \right) \left[ x^d \right] G_0.
\]

**Proof.** We prove only the first formula, as the proof of the second is similar. First, expand

\[
\prod_{i=1}^{N_0} \left( 1 - \frac{X_{N_0+1}}{X_i} \right) \left( 1 - \frac{X_{N_0+2}}{X_i} \right) = \sum_{n,m,\gamma} (-1)^{n+m} c_{n,m,\gamma} f_{n,m,\gamma} \left( x_i^{-1} \right) x_{N_0+1}^n x_{N_0+2}^m
\]

where \(f_{n,m,\gamma}\) is the monomial \(x_{i}^{-1} \cdots x_{N_0}^{-1}\) with exponents \(\gamma_i = 0, -1, \text{ or } -2\), and \(c_{n,m,\gamma}\) is a positive integer. The only terms in this expansion which have distinct exponents when multiplied by \(x^d\) occur when \(n \geq m\). Moreover,

\[
f_{n,m,\gamma} = x_{N_0+1}^{-n} x_{N_0+2}^{-m} \cdots x_{N_0+1}^{-m-1} x_{N_0+2}^{-n},
\]

\(c_{n,m,\gamma} = 1\), and

\[
f_{n,m,\gamma} \left( x_i^{-1} \right) x_{N_0+1}^n x_{N_0+2}^m x_i^d
\]

\[
= x_{N_0+1}^{n+1} \cdots x_{N_0+1}^{n} x_{N_0+2}^{m+1} \cdots x_{N_0+1}^{m} x_{N_0+2} x_{N_0+1}^{n+1} \cdots x_{N_0+1}^{n} x_{N_0+2}^m x_{N_0+1}^{m+1} \cdots x_{N_0+1}^0 = x_{N_0}^{m} x_{N_0}^{m-1} \cdots x_{N_0}^0 = x_{N_0}^{m} x_{N_0}^{m-1} \cdots x_{N_0}^0
\]

where \(\text{sgn } \sigma = (-1)^{n+m}\). Thus,

\[
\prod_{i=1}^{N_0} \left( 1 - \frac{X_{N_0+1}}{X_i} \right) \left( 1 - \frac{X_{N_0+2}}{X_i} \right) G_0
\]

\[
= \sum_{n=0}^{N_0} \sum_{m=0}^{n} (-1)^{n+m} \left( -t \right)^{n+m} \left[ x^d \right] G_0
\]

which yields the result upon carrying out the summation.
Lemma 3.6. We have

\[
- \frac{t^{X_{N_0}+1}}{X_{N_0^*+2}} \prod_{i=1}^{N_0} \left( 1 - \frac{t^{X_{N_0}+1}}{X_{i}} \right) \left( 1 - \frac{t^{X_{N_0}+2}}{X_{i}} \right) x^d \nabla G_d(x) \\
= - t^{N_0^*} \left[ x^{\sigma + \delta} \right] G_d(x) - \frac{t^2}{1-t^2} \left( 1 - t^{N_0} \frac{1-t^{N_0}}{1-t} \right) \left[ x^d \right] G_d(x).
\]

(3.18)

Proof. Again, expand

\[
- \frac{t^{X_{N_0}+1}}{X_{N_0^*+2}} \prod_{i=1}^{N_0} \left( 1 - \frac{t^{X_{N_0}+1}}{X_{i}} \right) \left( 1 - \frac{t^{X_{N_0}+2}}{X_{i}} \right) \\
= \sum_{n,m \gamma} (-1)^{n+m+1} \epsilon_{n,m,\gamma} f_{n,m,\gamma} (x^{n-1} x_{N_0}^{m+1} x_{N_0}^{m+2}).
\]

Once more, \( f_{n,m,\gamma} \) is a monomial in \( x_{1}^{-1}, \ldots, x_{N_0}^{-1} \) with exponents no greater than \( -2 \) and \( \epsilon_{n,m,\gamma} \) is a positive integer. The only terms in this expansion which, when multiplied by \( x^d \), have distinct exponents occur when either \( n \leq m \leq N_0 \) and \( m-1 \leq n \leq N_0 - 1 \) or \( n = N_0 \), \( m = 0 \).

In the latter case, \( \epsilon_{N_0,0,\gamma} = 1, f_{N_0,0,\gamma} = (x_1 x_2 \cdots x_{N_0})^{-1} \) and

\[
f_{N_0,0,\gamma} x_{N_0}^{N_0+1} x_{N_0} x_{N_0}^{N_0+2} x^d = x^d \sigma = (-1)^{N_0}
\]

(3.19)

In the former case the monomials \( f_{n,m,\gamma} \) take one of the \( n-m+1 \) possible forms

\[
f_{n,m,\gamma} = \begin{cases} 
\left( x_{N_0}^{-2} x_{N_0-m}^{n} x_{N_0-m+1} x_{N_0-m+2} \cdots x_{N_0}^{N_0-m+2} \cdots x_{N_0}^{N_0-m+2} \right) \\
\vdots \\
\left( x_{N_0}^{-1} x_{N_0}^{n} x_{N_0-m}^{n-1} x_{N_0-m+1} x_{N_0-m+2} \cdots x_{N_0}^{N_0-m+2} \cdots x_{N_0}^{N_0-m+2} \right)
\end{cases}
\]

(3.20)

as well as the additional form

\[
f_{n,m,\gamma} = x_{N_0}^{-1} x_{N_0}^{-1} \cdots x_{N_0}^{-1} x_{N_0-m}^{n} x_{N_0-m+1} x_{N_0-m+2} \cdots x_{N_0}^{N_0-m+2} \cdots x_{N_0}^{N_0-m+2}
\]

(3.21)

For the monomials (3.20) the corresponding \( \epsilon_{n,m,\gamma} = 1 \), but for the monomial (3.21), \( \epsilon_{n,m,\gamma} = n-m+2 \). Moreover, in the former case for each \( f_{n,m,\gamma} \), we have \( f_{n,m,\gamma} (x^{n-1} x_{N_0}^{m+1} x_{N_0}^{m+2} x^d = x^d \sigma \), where \( \sigma = (-1)^{n+m+1} \), while for the latter, \( \sigma = (-1)^{n+m} \). Hence, combining contributions from (3.20), (3.21), and (3.19), we get
LHS of (3.18) = \sum_{m=1}^{N_0} \sum_{n=m-1}^{N_0-1} (-t)^{n+m+1} \\
\times ((n-m+1)(-1)^{n+m+1} + (n-m+2)(-1)^{n+m}) [x^d] G_0 \\
- t^{N_0+1} [x^{z_2+\delta}] G_0 \\
which produces the required result after summation.

Lemmas 3.5 and 3.6 show that everything on the right hand side of (3.17) can be expressed in terms of \([x^d] G_0\) and \([x^{z_2+\delta}] G_0\). However, by eliminating \([x^{z_2+\delta}] G_0\) from (3.7) and (3.8), we have

\[
[x^d] G_0 = \frac{1}{(1-qst^{N_0+1})} \left( \frac{1-t^{N_0+1}}{1-t} \right) \\
\times \left\{ (t-qx) + q(u-t) \frac{(x-1)(1-t^{N_0+2})}{(1-quq^{N_0+1})(1-t)} \right\} [x^d] G_0.
\]

Thus, from (3.17)

\[
\left[ \frac{x^d}{x^z} \right] F_{11} \left[ \frac{x^z}{x^d} \right] G_0 = (1+t+t^2) \frac{1}{1-t} \left( \frac{1-t^{N_0+1}}{(1-t)(1-t^2)} - t \frac{(1-t^{2N_0+2})}{(1-t^2)} \right) \\
- t \frac{1}{1-t} \left( \frac{1-t^{N_0+1}}{(1-t^2)} - t^{N_0+1} \frac{(1-t^{N_0})}{(1-t)} \right) \\
- t \frac{1}{1-qst^{N_0+1}} \left( \frac{1-t^{N_0+1}}{1-t} \right) \left( (t-qx) + q(u-t) \frac{(x-1)(1-t^{N_0+2})}{(1-quq^{N_0+1})(1-t)} \right).
\]

Simplification of this expression yields the desired result (3.16).

4. AN EQUIVALENT q-SELBERG-TYPE INTEGRAL

It is known [14, 13] that the q-Morris identity (2.9) is equivalent to the q-Selberg integral [3]

\[
\int_0^1 \int_0^1 \cdots \int_0^1 d_{x1} \cdots d_{xN} t_{x1} \cdots t_{xN} \prod_{j=1}^N t_{j-1}^{(t_j q; q)} \prod_{1 \leq j < k \leq N} t_{j-1}^{(q^{1-t_k} t_k/t_j - 2)} \\
= q^{\sum_{j=1}^N (x_j + y_j)} \prod_{j=0}^{N-1} F_j(x + y + \lambda(N+1+j)) F_j(x + y + \lambda((j+1))) (4.1)
\]
where
\[ \int_0^1 (1 - q) \sum_{j=0}^{\infty} f(q^j) q^j. \] (4.2)

(This requires that \( \lambda \in \mathbb{Z}_{\geq 0} \); for the \( q \)-Selberg integral with general \( \lambda \) see [2, 16].) Here we will show that there is a general equality between \( q \)-Morris type constant term identities and \( q \)-Selberg type integrals, and we will use the general equality to rewrite the conjecture (2.12) as the evaluation of a \( q \)-Selberg type integral.

**Proposition 4.1.** For a general Laurent polynomial \( f(t_1, ..., t_N) \) we have
\[ \left( \frac{\Gamma_q(x + y)}{\Gamma_q(x) \Gamma_q(y)} \right)^N \prod_{j=1}^{N} \int_0^1 (qt_j; q)_{\infty} d_q t_j = \left( \frac{q^a; q}{q^b; q} \right)^{a+b} \left( \frac{q^a; q}{q^b; q} \right)^{a+b} + \left( \frac{q^a; q}{q^b; q} \right)^{a+b} f(q^{-b+1}t_1, ..., q^{-b+1}t_N) \] (4.3)
provided \( x = -b \) and \( y = a + b - 1 \).

**Proof.** By linearity of the \( q \)-integral and constant term, and their factorization into products of one-dimensional \( q \)-integrals and constant terms respectively for \( f \) a monomial, it suffices to consider the one dimensional case with \( f(t) = t^r \).

We have the \( q \)-beta integral evaluation
\[ \int_0^1 t^{x-1} (q; q)_\infty d_q t = \left( \frac{q^x; q}{q^y; q} \right)^{x+y} \] (4.4)
From the recurrence property \( \Gamma_q(x + 1) = [x]_q \Gamma_q(x) \) of the \( q \)-gamma function we see that
\[ \Gamma_q(x + p) = \left( \frac{q^p; q}{1 - q^p} \right)^{x+y} \] (4.5)
and so
\[ \int_0^1 t^{x+p-1} (q; q)_\infty d_q t = \left( \frac{q^p; q}{q^{x+y}; q} \right)^{x+y} \] (4.6)
We also have the constant term identity [14, Eq. (7.8)]

\[
\text{CT}_t (t; q)_n \left( q t^r q \right) \frac{1}{(q; q)_{n+p+1}} (q; q)_{n+p} (q; q)_{n-p}. \tag{4.7}
\]

Noting that

\[
\frac{1}{(q; q)_{n+p}} = \frac{1}{(q; q)_n (q^{*+1}; q)_n}
\text{ and } \frac{1}{(q; q)_{n-p}} = \frac{(q^{-p+1}; q)_p}{(q; q)_n}, \tag{4.8}
\]

and using the formula

\[
(q; q)_n = (-x)^n q^{n(n-1)/2} \left( \frac{x; q}{q} \right)_n \tag{4.9}
\]

in the denominator of the second equation in (4.8) we see that (4.7) can be rewritten to read

\[
\text{CT}_t (t; q)_n \left( q t^r q \right) \frac{1}{(q; q)_{n+p} (q; q)_{n-p}}. \tag{4.10}
\]

Comparison of (4.10) and (4.6) demonstrates (4.3) in the case \( N = 1 \).

The \( q \)-Selberg integral evaluation (4.1) results from the \( q \)-Morris identity (2.9) by choosing

\[
f = f_N := \prod_{1 \leq j < k \leq N} \left( \frac{q t_j}{t_k} q \right)^{\lambda j} \left( \frac{t_j}{t_k} q \right)^{\lambda j}, \tag{4.11}
\]

in (4.3). The constant term is then precisely that occurring in the \( q \)-Morris identity (2.9), and so is evaluated by (2.9). Also, the \( q \)-integral is simply related to the \( q \)-Selberg integral in (4.1) with \( x \) replaced by \( x - \lambda (N-1) \). This is seen by noting that we can rewrite (4.11) to read

\[
f_N = (-1)^{\lambda N(N-1)/2} \left( \prod_{j=1}^N q^{N(N-1)/2 \lambda j} \right) \left( q^{N(N-1)/2 \lambda j} \right)^{\lambda j} \prod_{j < k} \left( \frac{q^{1-N} t_j}{t_k} q \right)^{\lambda j}. \tag{4.12}
\]

Manipulation of the factor \( \Gamma_{\lambda} (b+1 + \lambda l) \) in (2.10) then allows the evaluation (4.1) to be obtained from (4.3).
The $q$-Selberg type integral corresponding to the conjectured constant term (2.12) is obtained by choosing
\begin{equation}
  f = f_{N_0,N} := \prod_{N_0+1 \leq j < k \leq N} \left( 1 - q^j t_j \right) \left( 1 - q^{j+1} t_k \right) f_N \tag{4.13}
\end{equation}
in (4.3). We then have the conjectured $q$-Selberg type integral evaluation
\begin{align}
  &\left( \frac{T_f(x+y)}{T_f(x) T_f(y)} \right)^N \prod_{j=1}^{N_0} \int_0^1 d q_j \left( q_j \right)_a \left( q_{j+N} \right)_b \\
  &\times \left( q \right)_a \left( q \right)_b \left( q_{t_j} \right)_a \left( q_{t_j} \right)_b \left( q_{t_k} \right)_a \left( q_{t_k} \right)_b \\
  &= \left( \frac{q}{q} \right)_a \left( q \right)_b \left( q \right)_a \left( q \right)_b
\end{align}
(4.14)
where $b = -x$ and $a = x + y - 1$.

REFERENCES

7. S. Cooper, Proof of a $q$-extension of a conjecture of Forrester, preprint.
15. J. Kaneko, Constant term identities of Forrester–Zeilberger–Cooper, preprint.