JOURNAL OF COMBINATORIAL THEORY, Series A 45, 207–225 (1987)

Uniformly Resolvable Pairwise Balanced Designs with Blocksizes Two and Three

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Communicated by the Managing Editors

Received May 16, 1986

A uniformly resolvable pairwise balanced design is a pairwise balanced design whose blocks can be resolved into parallel classes in such a way that all blocks in a given parallel class have the same size. We are concerned here with designs in which each block has size two or three, and we prove that the obvious necessary conditions on the existence of such designs are also sufficient, with two exceptions, corresponding to the non-existence of Nearly Kirkman Triple Systems of orders 6 and 12.

1. INTRODUCTION

A *pairwise balanced design* (PBD) (of index 1) is a set X of elements called *treatments*, together with a collection B of subsets of X called *blocks*, such that each pair of treatments is contained in exactly one block. A *parallel class* of blocks is a subset $B_1 \subseteq B$ which partitions the set X. A PBD is *resolvable* if its blocks can be partitioned into parallel classes. The *size* of a block $b \in B$ is the number $|b|$. The *replication number* of a resolvable PBD is the number of parallel classes contained in any resolution of its blocks or, equivalently, the number of blocks containing any fixed treatment.

A uniformly resolvable pairwise balanced design (URD) is a pairwise balanced design whose blocks can be resolved into parallel classes in such a way that all blocks in a given parallel class have the same size.

A URD(p, k) is a uniformly resolvable PBD on p treatments, with replication number k, in which each block has size two or three. When convenient we may think of this as a resolution of the complete graph K_p into t 1-factors and $k-t$ Δ-factors, where a Δ-factor is a 2-factor consisting of

* This work forms a part of the author's doctoral dissertation.

0097-3165/87 $3.00

triangles. (Note that by counting the treatments contained in the pencil of blocks through any fixed treatment we have $t+2(k-t)=p-1$, or $2k-t=p-1$; i.e., each of k, t will uniquely determine the other.) In particular a URD(p, $p-1$) is merely a 1-factorization of the complete graph K_p; these exist if and only if p is even. At the other end of the spectrum a URD(p, $(p-1)/2$) is a resolution of K_p into Δ-factors. These are referred to as Kirkman Triple Systems KTS(p), and are known to exist if and only if p is an odd multiple of three [7].

The purpose of this report is to investigate the existence problem for values of k between the above extremes. Since both block sizes are present we clearly must have $p \equiv 0 \pmod 6$ and $p/2 \leqslant k \leqslant p-2$. The case $k=p/2$ corresponds to a Nearly Kirkman Triple System NKTS(p), i.e., one 1-factor and $p/2-1$ Δ-factors. These have been extensively studied, and exist if and only if $p \geqslant 18$ (see [1, 2, 8, 6]). In particular, URD(6, 3) and URD(12, 6) do not exist. We prove that these are the only exceptions:

THEOREM. *Let* $p \equiv 0 \pmod 6$ *and* $p/2+1 \leqslant k \leqslant p-2$. *Then there exists a* URD(p, k).

These designs form a special type of *restricted resolvable designs*, which arise when considering the problem of determining the smallest number of blocks required to build a PBD given only the number of treatments and the size of the largest block. A restricted resolvable design (see [9]) is a resolvable PBD of index one in which all blocks have size l or $l+1$ for some integer l.

In another direction, uniformly resolvable designs can arise naturally in certain embedding problems. Thus for example a resolution of K_p into t 1-factors and $k-t$ Δ-factors is simultaneously a PRP $2-(2, 3, p; t)$ and a PRP $2-(3, 2, p; k-t)$ (PRP = partially resolvable partition; see [5]); our main theorem (together with the NKTS results) immediately yields the following result (which is a variant of a theorem of Doyen and Wilson [3]): for any integers v, w with $v \equiv w \equiv 1$ or $3 \pmod 6$ and $v > 2w+1$, $(v, w) \neq (7, 1)$ or $(13, 1)$ there exists a Steiner Triple System STS(v) containing a sub-STS(w) in such a way that the triples that miss the subsystem can be arranged into Δ-factors (i.e., Δ-factors on K_{v-w}).

Before proceeding we will need some more terminology. A *resolvable balanced incomplete block design* RBIBD(v, s, 1) is a resolvable PBD of index 1 on v treatments in which each block has size s. Thus an RBIBD(v, 3, 1) is a KTS(v). We will make use of the fact that an RBIBD(v, 4, 1) exists if and only if $v \equiv 4 \pmod{12}$; see [4]. A *group divisible design* GDD(S, G; v) consists of a set X of treatments which has been partitioned into subsets $X_1, X_2, \ldots, X_r$ (called *groups*), together with a collection B of subsets of X (called *blocks*) with the following properties:

(i) $|X| = v$,

(ii) each pair of treatments is contained in exactly one block or exactly one group (but not both), and

(iii) each group X_i has size $g_i \in G$, each block B_i has size $s_i \in S$.

A group divisible design is said to be *resolvable* if its blocks can be partitioned into parallel classes. A resolvable GDD(S, G; v) will be denoted RGD(S, G; v).

A *frame* (see [10]) is a group divisible design whose blocks can be partitioned into partial parallel classes, i.e., each partial parallel class C is a partition of $X - X_i$ for some group X_i (we will say that X_i *corresponds* to C when C partitions $X - X_i$). The groups in a frame are referred to as *holes.* By an Fr(S, G; v) we will mean a frame obtained from a GDD(S, G; v). In [10], Stinson proves that an Fr($\{3\}$, $\{g\}$, gt) exists if and only if g is even, $t \geqslant 4$ and $g(t-1) \equiv 0 \pmod 3$.

Finally, we point out that some of the material contained herein appears in an unpublished manuscript by the author entitled, The spectrum of uniformly resolvable PBDs with blocksizes two and three.

2. Preliminary Results

The bulk of the main theorem (i.e., the cases where $p \not\equiv 12 \pmod{18}$) relies on the existence of NKTSs and KTSs, together with the first construction in this section (Theorem 2.1). The constructions for the $p \equiv 12 \pmod{18}$ designs make essential use of frames together with RBIBD(v, 4, 1)s, in a manner similar to a construction in [8] (see Corollary 2.4 of that paper). Additionally, some small starting designs are needed, and these are constructed in Theorems 2.2 through 2.4.

Let n be an even integer and define a resolvable group divisible design on the treatment set $Z_n \times \{a, b, c\}$ as follows:

Groups $Z_n \times \{a\}, Z_n \times \{b\}, Z_n \times \{c\}$

Blocks
$$S_i = \{\{(x, a), (x+i, b), (x+2i, c)\} : x \in Z_n\},$$
$$0 \leqslant i \leqslant n/2 - 1,$$
$$S_i = \{\{(x, a), (x+i, b), (x+2i+1, c)\} : x \in Z_n\},$$
$$n/2 \leqslant i \leqslant n-2,$$
$$M_1 = \{\{(x, a), (x-1, b)\}, \{(x+n/2, a), (x+n/2-1, c)\},$$
$$\{(x+n/2-1, b), (x-1, c)\} : 0 \leqslant x \leqslant n/2 - 1\},$$
$$M_2 = \{\{(x-1, b), (x+n/2-1, c)\}, \{(x+n/2, a), (x+n/2-1, b)\}$$
$$\{(x, a), (x-1, c)\} : 0 \leqslant x \leqslant n/2 - 1\}.$$

Each S_i is a parallel class of blocks of size three, while M_1 and M_2 are parallel classes of blocks of size two. Define a $T(n,r)$ to be an RGD($\{2,3\}$, $\{n\}$; $3n$) whose blocks can be resolved into $2r-2n$ parallel classes of blocks of size two and $2n-r$ parallel classes of blocks of size three, i.e., there are r parallel classes in all. Thus our design above is a $T(n, n+1)$.

THEOREM 2.1. *Let n be a positive even integer. There exists a $T(n,r)$ if and only if $n \leqslant r \leqslant 2n$, with the exceptions $n=r=2$ or $n=r=6$.*

Proof. The condition $n \leqslant r \leqslant 2n$ is clearly necessary. A $T(n,n)$ is a transversal design, corresponding to the existence of two orthogonal latin squares of order n (see, e.g., [6]). Thus a $T(n,n)$ exists if and only if $n \neq 2$ or 6. We now assume $r > n$. We refer to the $T(n, n+1)$ constructed above, making the following two observations:

(A) Consider the classes M_1 and $S_{n/2-1} = \{\{(x,a), (x+n/2-1, b), (x-2, c)\} : x \in Z_n\}$. Partition M_1 into the three following subclasses:

$$M_1^1 = \{\{(x,a),(x-1,b)\} : 0 \leqslant x \leqslant n/2-1\}$$

$$M_1^2 = \{\{(x,a),(x-1,c)\} : n/2 \leqslant x \leqslant n-1 \text{ and } x \text{ even}\}$$
$$\cup \{\{(x+n/2-1,b),(x-1,c)\} : 0 \leqslant x \leqslant n/2-1 \text{ and } x \text{ even}\}$$

$$M_1^3 = M_1 - M_1^1 - M_1^2.$$

For each block b of $S_{n/2-1}$, b intersects exactly one block of M_1^i, $i=1,2,3$; let $h^i(b)$ be the intersection of b and M_1^i. By breaking up each block of $S_{n/2-1}$ into its three 2-subsets and defining for each $i=1,2,3$

$$A^i = M_1^i \cup \bigcup_{b \in S_{n/2-1}} \{b - h^i(b)\}$$

we can replace the classes M_1 and $S_{n/2-1}$ by the three parallel classes A^1, A^2, A^3 of blocks of size two. This has the effect of increasing by one the number of blocks on which each treatment lies. (A more general application of this construction appears in [9].)

(B) Let $0 \leqslant i \leqslant n/2-2$. Consider the classes S_i and $S_{i+n/2}$. Since $0 \leqslant i \leqslant n/2-2$ we have

$$S_i = \{\{(x,a),(x+i,b),(x+2i,c)\} : x \in Z_n\}$$

and

$$S_{i+n/2} = \{\{(x,a),(x+i+n/2,b),(x+2i+1,c)\} : x \in Z_n\}.$$

Break up each block of these two classes into its three 2-subsets and define four new classes of blocks of size two as follows:

$$E_1 = \{\{(x, a), (x+i, b)\}, \{(x+i+n/2, b), (x+2i+1, c)\},$$
$$\{(x+n/2, a), (x+n/2+2i+1, c)\}: 0 \leqslant x \leqslant n/2-1\}$$
$$E_2 = \{\{(x, a), (x+i+n/2, b)\}, \{(x+i, b), (x+2i, c)\},$$
$$\{(x+n/2, a), (x+n/2+2i, c)\}: 0 \leqslant x \leqslant n/2-1\}$$

E_3 (resp. E_4) is identical to E_1 (resp. E_2) except that the range $n/2 \leqslant x \leqslant n-1$ is used instead of $0 \leqslant x \leqslant n/2-1$.

Replacing S_i and $S_{i+n/2}$ by E_1, E_2, E_3, E_4 has the effect of increasing by two the number of blocks on which each treatment lies.

As pointed out previously our original design is a $T(n, n+1)$. To obtain a $T(n, n+2)$ we apply (A) to our $T(n, n+1)$. Now let $n+3 \leqslant r \leqslant 2n$. We construct a $T(n, r)$ as follows:

(i) if $r-n$ is odd, apply (B) to our $T(n, n+1)$ using the pairs S_i, $S_{i+n/2}$ for $0 \leqslant i \leqslant (r-n-3)/2$.

(ii) if $r-n$ is even, apply (B) to our $T(n, n+1)$ using the pairs S_i, $S_{i+n/2}$ for $0 \leqslant i \leqslant (r-n-4)/2$. Then apply (A).

This completes the proof of Theorem 2.1. ▮

THEOREM 2.2. *Let H_1, H_2, H_3, H_4 be four disjoint sets of six elements each. For each $j = 0, 1, 2, 3, 4$ there exists an* Fr({2, 3}, {6}; 24) *with holes H_i such that the holes H_i, $i \leqslant j$ correspond to one partial parallel class of blocks of size three and four partial parallel classes of blocks of size two, while the holes H_i, $i > j$, correspond to three partial parallel classes of blocks of size three.*

Proof. The case $j = 0$ is a Kirkman frame, in the sense of Stinson [10]. We display one of these designs below (it is obtained as in [10]; remove a point from the projective plane of order 3 and replace each block of size four by an Fr({3}, {2}; 8)). The hole H_i contains the treatments $6(i-1)+k$, $0 \leqslant k \leqslant 5$. The partial parallel classes are written vertically.

$H_1 = \{0, 1, 2, 3, 4, 5\}$	6, 12, 18	6, 14, 23	6, 16, 21
	7, 13, 19	7, 15, 22	7, 17, 20
	8, 14, 21	8, 16, 19	8, 13, 22
	9, 15, 20	9, 17, 18	9, 12, 23
	10, 16, 23	10, 13, 20	10, 14, 19
	11, 17, 22	11, 12, 21	11, 15, 18

$H_2 = \{6, 7, 8, 9, 10, 11\}$	0, 12, 19	0, 14, 20	1, 16, 22
	1, 13, 18	1, 15, 21	0, 17, 23
	2, 14, 22	3, 16, 18	3, 12, 20
	3, 15, 23	2, 17, 19	2, 13, 21
	5, 16, 20	5, 12, 22	5, 14, 18
	4, 17, 21	4, 13, 23	4, 15, 19
$H_3 = \{12, 13, 14, 15, 16, 17\}$	0, 7, 18	1, 8, 20	0, 10, 22
	1, 6, 19	0, 9, 21	1, 11, 23
	2, 11, 20	3, 6, 22	2, 8, 18
	3, 10, 21	2, 7, 23	3, 9, 19
	4, 9, 22	4, 10, 18	4, 6, 20
	5, 8, 23	5, 11, 19	5, 7, 21
$H_4 = \{18, 19, 20, 21, 22, 23\}$	0, 6, 13	0, 8, 15	0, 11, 16
	1, 7, 12	1, 9, 14	1, 10, 17
	2, 9, 16	2, 10, 12	2, 6, 15
	3, 8, 17	3, 11, 13	3, 7, 14
	4, 11, 14	4, 7, 16	4, 8, 12
	5, 10, 15	5, 6, 17	5, 9, 13

Consider now the first two partial parallel classes corresponding to H_1. Break each block into its three 2-subsets and define four partial parallel classes of blocks of size two as follows:

(i)	12, 18	11, 21	9, 17	16, 23	6, 14	8, 19	15, 20	7, 22	10, 13
(ii)	13, 19	10, 20	8, 16	14, 21	6, 23	11, 12	17, 22	9, 18	7, 15
(iii)	14, 23	8, 21	10, 16	17, 18	6, 12	11, 22	13, 20	7, 19	9, 15
(iv)	16, 19	10, 23	7, 13	15, 22	9, 20	11, 17	12, 21	6, 18	8, 14

In similar fashion the first two partial parallel classes corresponding to each of H_2, H_3, H_4 can be so decomposed:

H_2

(i)	12, 19	5, 22	2, 17	16, 20	0, 14	3, 18	15, 23	1, 21	4, 13
(ii)	13, 18	4, 23	3, 16	14, 22	0, 20	5, 12	17, 21	2, 19	1, 15
(iii)	14, 20	2, 22	5, 16	13, 23	1, 18	3, 15	17, 19	4, 21	0, 12
(iv)	15, 21	3, 23	4, 17	16, 18	5, 20	1, 13	12, 22	0, 19	2, 14

H_3

(i)	0, 7	9, 21	2, 23	5, 8	11, 19	1, 20	3, 10	6, 22	4, 18
(ii)	1, 6	8, 20	3, 22	2, 11	7, 23	5, 19	4, 9	10, 18	0, 21
(iii)	1, 8	6, 19	5, 23	2, 7	11, 20	0, 18	4, 10	9, 22	3, 21
(iv)	0, 9	7, 18	4, 22	3, 6	10, 21	1, 19	5, 11	8, 23	2, 20

H_4

(i)	0, 6	8, 15	5, 17	2, 9	10, 12	1, 14	4, 11	7, 16	3, 13
(ii)	1, 7	9, 14	4, 16	3, 8	11, 13	0, 15	5, 10	6, 17	2, 12
(iii)	0, 8	6, 13	3, 17	4, 7	11, 14	1, 12	2, 10	9, 16	5, 15
(iv)	1, 9	7, 12	2, 16	3, 11	8, 17	4, 14	5, 6	10, 15	0, 13

This completes the proof of Theorem 2.2. ∎

THEOREM 2.3. *Let H_1, H_2, H_3, H_4, H_5 be five disjoint sets of six elements each. For each $j = 0, 1, 2, 3, 4, 5$ there exists an* Fr({2, 3}, {6}; 30) *with holes H_i such that the holes H_i, $i \leqslant j$ correspond to one partial parallel clas of blocks of size three and four partial parallel classes of blocks of size two, while the holes H_i, $i > j$ correspond to three partial parallel classes of blocks of size three.*

Proof. We proceed as in Theorem 2.2. The $j = 0$ case can be constructed as in [10] by removing a point from the affine plane of order 4 and replacing each block of size 4 by an Fr({3}, {2}; 8). We consider the following such design.

$H_1 = \{0, 1, 2, 3, 4, 5\}$	12, 26, 22	18, 14, 28	24, 20, 16
	13, 27, 23	19, 15, 29	25, 21, 17
	6, 20, 28	24, 8, 22	18, 26, 10
	7, 21, 29	25, 9, 23	19, 27, 11
	24, 14, 10	6, 26, 16	12, 8, 28
	25, 15, 11	7, 27, 17	13, 9, 29
	18, 8, 16	12, 20, 10	6, 14, 22
	19, 9, 17	13, 21, 11	7, 15, 23

$H_2 = \{6, 7, 8, 9, 10, 11\}$	12, 18, 24	26, 14, 20	22, 28, 16
	13, 19, 25	27, 15, 21	23, 29, 17
	0, 29, 20	22, 2, 25	27, 18, 4
	1, 28, 21	23, 3, 24	26, 19, 5
	27, 2, 16	12, 29, 4	0, 14, 25
	26, 3, 17	13, 28, 5	1, 15, 24
	23, 14, 4	0, 19, 16	12, 2, 21
	22, 15, 5	1, 18, 17	13, 3, 20

$H_3 = \{12, 13. 14, 15, 16, 17\}$

6, 25, 18	29, 2, 18	20, 25, 4
7, 24, 19	28, 3, 19	21, 24, 5
0, 22, 27	20, 8, 27	29, 22, 10
1, 23, 26	21, 9, 26	28, 23, 11
28, 9, 4	0, 24, 11	7, 2, 26
29, 8, 5	1, 25, 10	6, 3, 27
20, 2, 11	7, 22, 4	0, 9, 18
21, 3, 10	6, 23, 5	1, 8, 19

$H_4 = \{18, 19, 20, 21, 22, 23\}$

15, 2, 28	24, 17, 4	0, 26, 13
14, 3, 29	25, 16, 5	1, 27, 12
24, 6, 13	15, 26, 8	10, 17, 28
25, 7, 12	14, 27, 9	11, 16, 29
11, 26, 4	0, 7, 28	24, 2, 9
10, 27, 5	1, 6, 29	25, 3, 8
0, 17, 8	10, 2, 13	15, 6, 4
1, 16, 9	11, 3, 12	14, 7, 5

$H_5 = \{24, 25, 26, 27, 28, 29\}$

16, 21, 4	8, 21, 14	19, 2, 14
17, 20, 5	9, 20, 15	18, 3, 15
19, 12, 6	0, 12, 23	16, 10, 23
18, 13, 7	1, 13, 22	17, 11, 22
8, 2, 23	19, 10, 4	0, 21, 6
9, 3, 22	18, 11, 5	1, 20, 7
0, 10, 15	17, 2, 6	8, 13, 4
1, 11, 14	16, 3, 7	9, 12, 5

The first two partial parallel classes of each of the H_i can be decomposed into four partial parallel classes of blocks of size two:

H_1

(i)	26, 22	6, 16	24, 8	27, 23	7, 17	25, 9
	20, 28	12, 10	18, 14	21, 29	13, 11	19, 15
(ii)	8, 16	24, 22	6, 26	9, 17	25, 23	7, 27
	14, 10	18, 28	12, 20	15, 11	19, 29	13, 21
(iii)	14, 28	24, 10	6, 20	15, 29	25, 11	7, 21
	8, 22	18, 16	12, 26	9, 23	19, 17	13, 27
(iv)	10, 20	6, 28	24, 14	21, 11	7, 29	25, 15
	26, 16	12, 22	18, 8	27, 17	13, 23	19, 9

H_2

(i)	18, 24	1, 17	23, 3	19, 25	0, 16	22, 2
	29, 20	12, 4	26, 14	28, 21	13, 5	27, 15
(ii)	3, 17	23, 24	1, 18	2, 16	22, 25	0, 19
	14, 4	26, 20	12, 29	15, 5	27, 21	13, 28
(iii)	14, 20	23, 4	0, 29	15, 21	22, 5	1, 28
	2, 25	27, 16	13, 19	3, 24	26, 17	12, 18
(iv)	29, 4	0, 20	23, 14	28, 5	1, 21	22, 15
	19, 16	13, 25	27, 2	18, 17	12, 24	26, 3

H_3

(i)	25, 18	1, 10	29, 2	24, 19	0, 11	28, 3
	22, 27	7, 4	20, 8	23, 26	6, 5	21, 9
(ii)	9, 4	21, 26	7, 22	8, 5	20, 27	6, 23
	2, 11	29, 18	0, 24	3, 10	28, 19	1, 25
(iii)	2, 18	20, 11	6, 25	3, 19	21, 10	7, 24
	8, 27	29, 5	0, 22	9, 26	28, 4	1, 23
(iv)	24, 11	7, 19	20, 2	25, 10	6, 18	21, 3
	22, 4	0, 27	28, 9	23, 5	1, 26	29, 8

H_4

(i)	2, 28	10, 13	0, 7	3, 29	11, 12	1, 6
	26, 4	24, 17	15, 8	27, 5	14, 9	25, 16
(ii)	6, 13	1, 29	10, 2	7, 12	0, 28	11, 3
	17, 8	24, 4	15, 26	16, 9	25, 5	14, 27
(iii)	17, 4	0, 8	11, 26	16, 5	1, 9	10, 27
	7, 28	25, 12	15, 2	6, 29	24, 13	14, 3
(iv)	26, 8	11, 4	0, 17	27, 9	10, 5	1, 16
	2, 13	15, 28	24, 6	3, 12	14, 29	25, 7

H_5

(i)	21, 4	8, 14	19, 10	20, 5	9, 15	18, 11
	12, 6	0, 23	17, 2	13, 7	1, 22	16, 3
(ii)	10, 15	19, 4	9, 20	11, 14	18, 5	8, 21
	2, 23	17, 6	0, 12	3, 22	16, 7	1, 13
(iii)	21, 14	16, 4	1, 11	20, 15	17, 5	0, 10
	12, 23	19, 6	8, 2	13, 22	18, 7	9, 3
(iv)	10, 4	0, 15	16, 21	11, 5	1, 14	17, 20
	2, 6	8, 23	19, 12	3, 7	9, 22	18, 13

This completes the proof of Theorem 2.3. ∎

THEOREM 2.4. *For each* $j=0, 1, 2, 3, 4$ *there exists an* RGD$(\{2,3\}, \{6\}; 30)$ *whose blocks can be resolved into* $4j$ *parallel classes of blocks of size two and* $12-2j$ *parallel classes of blocks of size three* (*i.e.*, $12+2j$ *classes in all*).

Proof.

$j=0$. See Lemma 3.10 of [8].

$j=1$. Take the treatment set $Z_{10}\times\{a,b,c\}$. Take the groups $g_x=\{(1+x,a),\ (6+x,a),\ (x,b),\ (5+x,b),\ (x,c),\ (5+x,c)\}$ for $x=0,1,2,3,4$. Take the initial parallel class of blocks

$(0,a),(2,a),(2,b)$	$(3,a),(9,a),(5,b)$
$(1,b),(5,c),(9,c)$	$(4,b),(6,c),(8,c)$
$(4,a),(8,a),(1,a)$	$(6,a),(5,a),(7,a)$
$(0,b),(6,b),(3,b)$	$(8,b),(7,b),(9,b)$
$(0,c),(4,c),(7,c)$	$(1,c),(2,c),(3,c)$

Four more classes are obtained from the above class by adding $2i$ to the first coordinate of each treatment, for $i=1,2,3,4$. The remaining five classes of blocks of size three are given below:

$$\begin{aligned}
C_6 &= \{\{(x,a),(x+6,b),(x+3,c)\}: x \text{ even}\}\\
&\quad \cup \{\{(x,a),(x,b),(x+3,c)\}: x \text{ odd}\},\\
C_7 &= \{\{(x,a),(x+3,b),(x,c)\}: x \text{ even}\}\\
&\quad \cup \{\{(x,a),(x+7,b),(x,c)\}: x \text{ odd}\},\\
C_8 &= \{\{(x,a),(x+1,b),(x+2,c)\}: x\in Z_{10}\},\\
C_9 &= \{\{(x,a),(x+5,b),(x+1,c)\}: x\in Z_{10}\},\\
C_{10} &= \{\{(x,a),(x+8,b),(x+7,c)\}: x\in Z_{10}\}.
\end{aligned}$$

The four classes of blocks of size two go as follows:

$$\begin{aligned}
D_1 = {} & (7,a),(0,b) \quad (2,b),(4,b) \quad (6,b),(8,b)\\
& (3,b),(5,c) \quad (5,b),(9,b) \quad (1,b),(7,b)
\end{aligned}$$

together with $\{\{(x,a),(x+8,c)\}: x\neq 7\}$,

$$\begin{aligned}
D_2 = {} & (5,a),(8,b) \quad (0,b),(2,b) \quad (4,b),(6,b)\\
& (9,b),(1,c) \quad (3,b),(7,b) \quad (1,b),(5,b)
\end{aligned}$$

together with $\{\{(x,a),(x+6,c)\}: x\neq 5\}$,

$$
\begin{array}{llll}
D_3 = & (0, a), (5, c) & (7, a), (2, c) & (4, b), (1, a) \\
& (6, c), (8, b) & (0, b), (8, c) & (3, a), (6, b) \\
& (4, c), (9, a) & (2, b), (0, c) & (5, a), (1, c) \\
& (6, a), (3, b) & (9, b), (2, a) & (7, c), (5, b) \\
& (8, a), (3, c) & (1, b), (4, a) & (9, c), (7, b),
\end{array}
$$

and

$$
\begin{array}{llll}
D_4 = & (5, c), (7, a) & (2, c), (4, b) & (1, a), (6, c) \\
& (8, b), (0, b) & (8, c), (3, a) & (6, b), (4, c) \\
& (9, a), (2, b) & (0, c), (5, a) & (1, c), (6, a) \\
& (3, b), (9, b) & (2, a), (7, c) & (5, b), (8, a) \\
& (3, c), (1, b) & (4, a), (9, c) & (7, b), (0, a).
\end{array}
$$

$j = 2$. Take the $j = 1$ design, and decompose the blocks of C_8, C_{10} into the following four parallel classes of blocks of size two:

$$
\begin{aligned}
E_1 &= \{\{(x, a), (x+8, b)\}: x \text{ odd}\} \cup \{\{(x, a), (x+2, c)\}: x \text{ even}\} \\
&\qquad \cup \{\{(x, b), (x+1, c)\}: x \text{ even}\}, \\
E_2 &= \{\{(x, a), (x+8, b)\}: x \text{ even}\} \cup \{\{(x, a), (x+2, c)\}: x \text{ odd}\} \\
&\qquad \cup \{\{(x, b), (x+1, c)\}: x \text{ odd}\}, \\
E_3 &= \{\{(x, a), (x+1, b)\}: x \text{ odd}\} \cup \{\{(x, a), (x+7, c)\}: x \text{ even}\} \\
&\qquad \cup \{\{(x, b), (x+9, c)\}: x \text{ odd}\}, \\
E_4 &= \{\{(x, a), (x+1, b)\}: x \text{ even}\} \cup \{\{(x, a), (x+7, c)\}: x \text{ odd}\} \\
&\qquad \cup \{\{(x, b), (x+9, c)\}: x \text{ even}\}.
\end{aligned}
$$

$j = 3$. Start with a KTS(15), replacing each treatment by two new ones. Replace each block in six of the Δ-factors by a $T(2, 3)$ (Theorem 2.1); each block of the seventh Δ-factor becomes a group with six treatments. This yields twelve classes of blocks of size two and six classes of blocks of size three, as desired (this is an example of a well known recursive technique sometimes referred to as "inflation;" see, e.g., [6] or [10]).

$j = 4$. Proceed as in the $j = 3$ case; this time each block in four of the Δ-classes is replaced by a $T(2, 3)$ while each block in two of the Δ-classes is replaced by a $T(2, 4)$. Again each block in the seventh Δ-class becomes a group.

This completes the proof of Theorem 2.4. ∎

3. THE MAIN THEOREM

THEOREM 3.1. *Let* $p \equiv 0 \pmod 6$ *and* $p/2 + 1 \leqslant k \leqslant p - 2$. *Then there exists a* URD(p, k).

We prove Theorem 3.1 by proving a sequence of lemmas which deal with the various cases that arise.

LEMMA 3.1. *If* $p \equiv 0 \pmod 6$ *and* $k \geqslant \frac{2}{3}p$ *there exists a* URD(p, k).

Proof. Let $n = p/3$ and $r = k - (n-1)$. Since $\frac{2}{3}p \leqslant k \leqslant p-2$ we have $n+1 \leqslant r \leqslant 2n-1$. Replace each group of a $T(n, r)$ (Theorem 2.1) by a K_n which has been equipped with a one-factorization. This yields a URD(p, k) as desired. ∎

LEMMA 3.2. *If* $p \equiv 0 \pmod{18}$ *and* $p/2 + 1 \leqslant k \leqslant p-2$ *there exists a* URD(p, k).

Proof. Let $n = p/3$. From Lemma 3.1 we may assume that $k < \frac{2}{3}p$. Assume first that $n \neq 6$ or 12. Let $r = k - n/2$; then $n+1 \leqslant r < 3n/2$. Replace each group of a $T(n, r)$ by an NKTS(n).

There remain $(p, k) = (36, 19)$, $(36, 20)$, $(36, 21)$, $(36, 22)$, $(36, 23)$, $(18, 10)$, $(18, 11)$. Begin by constructing a URD(12, 7), obtainable by replacing each group of a $T(4, 4)$ by a K_4 which has been equipped with a one factorization. The cases for $p = 36$ are then settled by substituting our URD(12, 7) for the groups in a $T(12, 12)$, $T(12, 13)$, $T(12, 14)$, $T(12, 15)$ and $T(12, 16)$, respectively. A URD(18, 11) is obtained by replacing each group of a $T(6, 7)$ by a URD(6, 4). A URD(18, 10) is given below (it was obtained by applying a construction similar to that used in Theorem 2.1A) to the NKTS(18) of Kotzig and Rosa [6]).

1, 5, 9	1, 2, 6′	1, 3, 5′	1, 6, 3′	1, 8, 7′	8, 9, 1′
2, 6, 7	4, 5, 9′	4, 6, 8′	2, 9, 8′	3, 5, 2′	2, 3, 4′
3, 4, 8	7, 8, 3′	7, 9, 2′	5, 7, 4′	4, 9, 6′	5, 6, 7′
1′, 5′, 9′	1′, 2′, 6	1′, 3′, 5	1′, 6′, 3	1′, 8′, 7	8′, 9′, 1
2′, 6′, 7′	4′, 5′, 9	4′, 6′, 8	2′, 9′, 8	3′, 5′, 2	2′, 3′, 4
3′, 4′, 8′	7′, 8′, 3	7′, 9′, 2	5′, 7′, 4	4′, 9′, 6	5′, 6′, 7

	1, 1′	4, 4′	7, 7′
	2, 2′	5, 5′	8, 8′
2, 4, 1′	3, 3′	6, 6′	9, 9′
6, 8, 5′	4, 7	1, 7	1, 4
3, 7, 9′	4′, 7′	1′, 7′	1′, 4′
2′, 4′, 1	5, 8	2, 8	2, 5
6′, 8′, 5	5′, 8′	2′, 8′	2′, 5′
3′, 7′, 9	6, 9	3, 9	3, 6
	6′, 9′	3′, 9′	3′, 6′

This completes the proof of Lemma 3.2. ∎

LEMMA 3.3. *If* $p \equiv 6 \pmod{18}$ *and* $p/2+1 \leqslant k \leqslant p-2$ *there exists a* URD(p, k).

Proof. From Lemma 3.1 we may assume that $k < \frac{2}{3}p$ (whence $p \geqslant 24$). Let $n = p/3$, and set $r = k - n/2$; then $n+1 \leqslant r < 3n/2$. Take a $T(n, r)$; since $r < 3n/2$ there will be a parallel class of blocks of size three. Relabel the treatments of the $T(n, r)$ so that this class is of the form $\{\{(x, a), (x, b), (x, c)\} \colon x \in Z_n\}$. Let D_n be an $\mathrm{Fr}(\{3\}, \{2\}; n)$ (D_n can be obtained by removing a treatment from a KTS$(n+1)$). For each $w = a, b, c$ replace the group $Z_n \times \{w\}$ in the $T(n, r)$ by a copy of D_n in such a way that the holes of D_n take the form $\{\{(x, w), (x+1, w)\} \colon x = 0, 2, 4, \ldots, n-2\}$. We denote that copy of D_n replacing $Z_n \times \{w\}$ by $D(w)$ and its holes by $H(w, x)$, $x = 0, 2, 4, \ldots, n-2$. To each hole $H(w, x)$ of $D(w)$ there corresponds one partial parallel class $P(w, x)$ of blocks of size three in $D(w)$. For each $x = 0, 2, \ldots, n-2$ the set

$$B_x = P(a, x) \cup P(b, x) \cup P(c, x) \cup \{\{(x, a), (x, b), (x, c)\}, \{(x+1, a), (x+1, b), (x+1, c)\}\}$$

is a parallel class of blocks of size three, while the collection of holes

$$H = \{H(a, x), H(b, x), H(c, x) \colon x = 0, 2, \ldots, n-2\}$$

forms a parallel class of blocks of size two. In this way we form a uniformly resolvable design on p treatments with replication number $n/2 + 1 + r - 1 = k$, as desired. ∎

Only the class $p \equiv 12 \pmod{18}$ remains; this seems to be the most difficult one, and we consider two subcases separately.

LEMMA 3.4. *If* $p \equiv 12 \pmod{36}$ *and* $p/2+1 \leqslant k \leqslant p-2$ *there exists a* URD(p, k).

Proof. If $k = p/2 + 1$ we take an RGD$(\{3\}, \{4\}; p)$ (see [8, Lemma 3.8] or [10, Theorem 6.6]) and replace each group with a K_4 which has been equipped with a one-factorization. From Lemma 3.1 we may now assume that $p/2+2 \leqslant k \leqslant \frac{2}{3}p - 1$ (whence $p \geqslant 48$). Let $n = p/3$ and $r = k - (n/2+1)$; then $n+1 \leqslant r \leqslant 3n/2 - 2$. We start as before with a $T(n, r)$. Now $r \leqslant 3n/2 - 2$, whence our $T(n, r)$ will have the parallel classes

$$S_{n/4-2} = \{\{(x, a), (x + n/4 - 2, b), (x + n/2 - 4, c)\} \colon x \in Z_n\}$$

and

$$S_{n/2-2} = \{\{(x, a), (x + n/2 - 2, b), (x - 4, c)\} \colon x \in Z_n\}$$

of blocks of size three (see the proof of Theorem 2.1). Let D_n be an $\mathrm{Fr}(\{3\}, \{4\}; n)$ (see Theorem 4.4 of [10]). We proceed as in Lemma 3.3, replacing the group $Z_n \times \{w\}$ in the $T(n, r)$ by a copy of D_n (denoted $D(w)$) in such a way that its holes take the form

$$H(w, x) = \{(x, w), (x + n/4, w), (x + n/2, w), (x + 3n/4, w)\}$$

for $x = 0, 1, \ldots, n/4 - 1$. To each hole $H(w, x)$ of $D(w)$ there correspond two partial parallel classes $P^1(w, x)$ and $P^2(w, x)$ of blocks of size three in $D(w)$ (see Theorem 1.2 of [10]); for each $x = 0, 1, \ldots, n/4 - 1$ the sets

$$\begin{aligned} B_x^1 = {} & P^1(a, x) \cup P^1(b, x-2) \cup P^1(c, x-4) \\ & \cup \{\{(x+i, a), (x+i+n/4-2, b), (x+i+n/2-4, c)\}: \\ & i = 0, n/4, n/2, 3n/4\} \end{aligned}$$

and

$$\begin{aligned} B_x^2 = {} & P^2(a, x) \cup P^2(b, x-2) \cup P^2(c, x-4) \\ & \cup \{\{(x+i, a), (x+i+n/2-2, b), (x+i-4, c)\}: \\ & i = 0, n/4, n/2, 3n/4\} \end{aligned}$$

are parallel classes of blocks of size three, where the expressions $x - 2$ and $x - 4$ appearing in the "P" terms are to be evaluated mod $n/4$. By replacing each hole by a K_4 which has been equipped with a one-factorization, we can describe three parallel classes of blocks of size two. This yields a uniformly resolvable design on p treatments with replication number $2(n/4) + 3 + r - 2 = n/2 + 1 + r = k$, as desired. ∎

LEMMA 3.5. *If* $p \equiv 30 \pmod{36}$ *and* $p \neq 66$; 138 *then for each* k *with* $p/2 + 1 \leqslant k \leqslant p - 2$ *there exists a* URD(p, k).

Proof. Let $p = 6t$. Then $t \equiv 5 \pmod 6$.

Case 1. $t \equiv 5 \pmod{12}$. Let $q = \lfloor (k - (p/2 + 1))/2 \rfloor$. Take an RBIBD$(t - 1, 4, 1)$ (from [4]) and fix a subset Q of "distinguished" treatments in this design with $|Q| = q$. Note that by Lemma 3.1 we may assume that $k < (2/3)p$ whence $q < t/2 < t - 1$. Regarding this resolvable design as a GDD$(\{4\}, \{4\}; t - 1)$ (our "starter" GDD) we apply a construction analogous to one found in [8], see Theorem 2.2 and Corollary 2.4. Replace each treatment x_i of the GDD by a set X_i of six treatments, and add a set X_∞ of six additional treatments.

Let $G_l = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ be a group in the GDD and let $j(l) = |Q \cap G_l|$. From Theorem 2.4 there exists an RGD$(\{2, 3\}, \{6\}; 30)$ consisting of $4j(l)$ parallel classes of blocks of size two and $12 - 2j(l)$ parallel classes of blocks of size three. Construct such a design using the groups $X_\infty, X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}$. Do this for each group G_l.

Let $B_m = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ be a block in the GDD and let $j(m) = |Q \cap B_m|$. From Theorem 2.2 we can construct a frame $\mathrm{Fr}(\{2, 3\}, \{6\}; 24)$ using holes $X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}$ such that hole X_{i_n} corresponds to one partial parallel class of blocks of size three and four partial parallel classes of blocks of size two when $x_{i_n} \in Q$, while hole X_{i_n} corresponds to three partial parallel classes of blocks of size three when $x_{i_n} \notin Q$, $n = 1, 2, 3, 4$. We do this for each block B_m.

In this way we can construct an $\mathrm{RGD}(\{2, 3\}, \{6\}; p)$ in which each x_i in the starter GDD gives rise to one parallel class of blocks of size three and four parallel classes of blocks of size two, or to three parallel classes of blocks of size three, depending on whether x_i is in Q or not. In particular, each parallel class of blocks in the $\mathrm{RGD}(\{2, 3\}, \{6\}; p)$ consists of blocks of the same size, and there are $5|Q| + 3(t - 1 - |Q|) = 2q + 3t - 3$ classes Recalling the definitions of q and t this is equal to $k - 4$ when $k - (p/2 + 1)$ is even, and $k - 5$ when $k - (p/2 + 1)$ is odd. In the former case replace each group of the $\mathrm{RGD}(\{2, 3\}, \{6\}; p)$ by a URD(6, 4) to obtain a $\mathrm{URD}(p, k)$; in the latter case replace each group by a URD(6, 5).

Case 2. $t \equiv 11 \pmod{12}$. This case proceeds exactly as does Case 1, except that our starter GDD is obtained by adjoining a group at infinity of size 6 to an $\mathrm{RBIBD}(t - 7, 4, 1)$. This can be done since $p \neq 66$, 138, i.e., $t \neq 11$, 23. Thus our starter is a $\mathrm{GDD}(\{4, 5\}, \{4, 6\}; t - 1)$. Note that since this GDD contains blocks of size 5 we will need Theorem 2.3 in a manner analogous to that for which Theorem 2.2 was used in Case 1. Furthermore, since there is a group of size 6 we will need the existence of an $\mathrm{RGD}(\{3\}, \{6\}; 42)$. This is established in [8]. (The set Q of distinguished treatments in the starter GDD is chosen so that it is disjoint from the group of size 6, and $|Q| = \lfloor (k - (p/2 + 1))/2 \rfloor$.)

This completes the proof of Lemma 3.5. ∎

LEMMA 3.6. *There exists* $\mathrm{URD}(66, k)$ *for* $34 \leqslant k \leqslant 64$.

Proof. $k = 34, 35$. We start by constructing the following PBD on the treatment set $Z_{12} \cup \{a, b, c, d, e, f\}$ (it is obtained by applying a construction similar to that in Theorem 2.1A) to Brouwer's "ingredient c" [2]). Start with the block $\{a, b, c, d, e, f\}$. The remaining blocks are arranged into the following six parallel classes:

(i)	$a, 0, 11$	$b, 1, 3$	$c, 4, 8$	$d, 6, 10$	$e, 2, 7$	$f, 5, 9$
(ii)	$a, 7, 9$	$b, 0, 10$	$c, 1, 5$	$d, 2, 4$	$e, 8, 11$	$f, 3, 6$
(iii)	$a, 4, 6$	$b, 2, 11$	$c, 0, 9$	$d, 5, 7$	$e, 3, 10$	$f, 1, 8$
(iv)	$a, 1, 10$	$b, 5, 6$	$c, 2, 3$	$d, 0, 8$	$e, 4, 9$	$f, 7, 11$
(v)	$a, 3, 8$	$b, 4, 7$	$c, 6, 11$	$d, 1, 9$	$e, 0, 5$	$f, 2, 10$
(vi)	$a, 2, 5$	$b, 8, 9$	$c, 7, 10$	$d, 3, 11$	$e, 1, 6$	$f, 0, 4$

and the following four classes on Z_{12}:

(vii)	0, 3, 7	1, 4, 11	2, 6, 9	5, 8, 10		
(viii)	0, 6	3, 9	1, 2	4, 5	7, 8	10, 11
(ix)	1, 7	4, 10	0, 2	3, 5	6, 8	9, 11
(x)	2, 8	5, 11	0, 1	3, 4	6, 7	9, 10.

Call the above design B_1. Construct a second design B_2 by replacing the classes (vii) and (viii) in B_1 by the three classes

(i)′	0, 6	10, 11	3, 7	1, 4	2, 9	5, 8
(ii)′	3, 9	4, 5	0, 7	1, 11	2, 6	8, 10
(iii)′	1, 2	7, 8	0, 3	4, 11	6, 9	5, 10.

We now apply Brouwer's construction for NKTS(66) except that "ingredient *a*" and "ingredient *c*" are replaced by a URD(18, 10) and B_1 (respectively a URD(18, 11) and B_2) to obtain a URD(66, 34) (respectively URD(66, 35)).

$k \geqslant 36$. We start by constructing a URD(66, 36). Take the treatment set $Z_{22} \times Z_3$, and take the initial parallel class of blocks

$$(0,0),(11,1),(0,2) \qquad (11,0),(0,1),(11,2)$$
$$(16,0),(17,1),(17,2) \qquad (17,0),(16,1),(16,2)$$

and

$$(2,0),(4,0),(8,0) \qquad (3,0),(19,0),(21,0)$$
$$(6,0),(9,0),(18,0) \qquad (1,0),(13,0),(14,0) \bmod(-,3)$$
$$(5,0),(12,0),(20,0) \qquad (7,0),(10,0),(15,0).$$

Ten more classes are obtained from the above class by adding $2i$ to the first coordinate of each treatment, for $i = 1, 2, \ldots, 10$. Take the seven classes:

I. $\{\{(x,0),(x+11,0)\},\{(x,1),(x+11,1)\},\{(x,2),(x+11,2)\}:$ $0 \leqslant x \leqslant 10\}$,

II. $\{\{(x,0),(x+1,0)\},\{(x,1),(x+1,1)\},\{(x,2),(x+1,2)\}:$ even $x \in Z_{22}\}$,

III. $\{\{(x,0),(x+17,0)\},\{(x,1),(x+17,1)\},\{(x,2),(x+17,2)\}:$ even $x \in Z_{22}\}$,

IV. $\{\{(x,1),(x+10,2)\},\{(x,0),(x-1,2)\},\{(x+1,0),(x+1,1)\}:$ even $x \in Z_{22}\}$,

V. $\{\{(x+1,0),(x-1,2)\},\{(x+1,1),(x,2)\},\{(x,0),(x,1)\}$: even $x \in Z_{22}\}$,

VI. $\{\{(x,1),(x-1,2)\},\{(x-1,0),(x,2)\},\{(x,0),(x-1,1)\}$: even $x \in Z_{22}\}$,

VII. $\{\{(x,0),(x-2,2)\},\{(x-1,0),(x,1)\},\{(x+11,1),(x-1,2)\}$: even $x \in Z_{22}\}$.

Finish with the eighteen classes:

$$S_i = \{\{(x,0),(x+i,1),(x+2i-1,2)\}:\ x \in Z_{22}\}, \qquad 2 \leqslant i \leqslant 10;$$

$$S_i = \{\{(x,0),(x+i,1),(x+2i,2)\}:\ x \in Z_{22}\}, \qquad 12 \leqslant i \leqslant 20.$$

By applying a construction similar to that of Theorem 2.1A) on classes I and S_{10} (in this case split class I into the three subclasses $\{\{(x,0),(x+11,0)\}: 0 \leqslant x \leqslant 10\}$, $\{\{(x,1),(x+11,1)\}: 0 \leqslant x \leqslant 10\}$ and $\{\{(x,2),(x+11,2)\}: 0 \leqslant x \leqslant 10\}$) and one similar to construction 2.1B) on the pairs S_i, S_{i+11}, $2 \leqslant i \leqslant 9$, we can now construct URD$(66, k)$ for $37 \leqslant k \leqslant 53$. The larger k values are settled by Lemma 3.1.

This completes the proof of Lemma 3.6. ▮

LEMMA 3.7. *There exist* URD$(138, k)$ *for* $70 \leqslant k \leqslant 136$.

Proof. $k = 70, 71$. The proof is exactly the same as the first case of Lemma 3.6, applying Brouwer's construction for an NKTS(138).

$k \geqslant 72$. Here the construction proceeds along the lines of the second case of Lemma 3.6, starting with the following URD(138, 72). Its treatment set is $Z_{46} \times Z_3$. Take the initial parallel class

(0, 0), (23, 1), (0, 2)	(23, 0), (0, 1), (23, 2)
(34, 0), (35, 1), (35, 2)	(35, 0), (34, 1), (34, 2)

and

(8, 0), (18, 0), (38, 0)	(2, 0), (6, 0), (14, 0)
(13, 0), (22, 0), (28, 0)	(10, 0), (12, 0), (25, 0)
(4, 0), (29, 0), (32, 0)	(15, 0), (20, 0), (42, 0)
(16, 0), (30, 0), (33, 0)	(7, 0), (27, 0), (43, 0) mod(−, 3)
(3, 0), (37, 0), (41, 0)	(11, 0), (17, 0), (24, 0)
(1, 0), (36, 0), (45, 0)	(21, 0), (39, 0), (40, 0)
(9, 0), (26, 0), (31, 0)	(5, 0), (19, 0), (44, 0)

Twenty-two more classes are obtained by adding $2i$ to the first coordinate of each treatment, for $i=1, 2,\ldots, 22$. Take the seven classes:

I. $\{\{(x,0),(x+23,0)\},\{(x,1),(x+23,1)\},\{(x,2),(x+23,2)\}:$ $0\leqslant x\leqslant 22\}$,

II. $\{\{(x,0),(x+1,0)\},\{(x,1),(x+1,1)\},\{(x,2),(x+1,2)\}:$ even $x\in Z_{46}\}$,

III. $\{\{(x,0),(x+35,0)\},\{(x,1),(x+35,1)\},\{(x,2),(x+35,2)\}:$ even $x\in Z_{46}\}$,

IV. $\{\{(x,1),(x+22,2)\},\{(x,0),(x-1,2)\},\{(x+1,0),(x+1,1)\}:$ even $x\in Z_{46}\}$,

V. $\{\{(x+1,0),(x-1,2)\},\{(x,2),(x+1,1)\},\{(x,0),(x,1)\}:$ even $x\in Z_{46}\}$,

VI. $\{\{(x,1),(x-1,2)\},\{(x-1,0),(x,2)\},\{(x,0),(x-1,1)\}:$ even $x\in Z_{46}\}$,

VII. $\{\{(x,0),(x-2,2)\},\{(x-1,0),(x,1)\},\{(x+23,1),(x-1,2)\}:$ even $x\in Z_{46}\}$.

Finish with the 42 classes:

$$S_i=\{\{(x,0),(x+i,1),(x+2i-1,2)\}:\ x\in Z_{46}\},\qquad 2\leqslant i\leqslant 22;$$

$$S_i=\{\{(x,0),(x+i,1),(x+2i,2)\}:\ x\in Z_{46}\},\qquad 24\leqslant i\leqslant 44. \ \blacksquare$$

The main theorem now follows from Lemmas 3.2 through 3.7.

Acknowledgment

This research was supported in part by a Natural Sciences and Engineering Research Council post-graduate scholarship.

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