Some graphs related to the small Mathieu groups

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ABSTRACT

Two different constructions are given of a rank 8 arc-transitive graph with 165 vertices and valency 8, whose automorphism group is $M_{11}$. One involves 3-subsets of an 11-set while the other involves 4-subsets of a 12-set, and the constructions are linked with the Witt designs on 11, 12 and 24 points. Four different constructions are given of a rank 9 arc-transitive graph with 55 vertices and valency 6 whose automorphism group is $\operatorname{PSL}(2, 11)$. This graph occurs as a subgraph of the $M_{11}$ graph, and two of the constructions involve 2-subsets of an 11-set while the remaining two involve 3-subsets of an 11-set. The $\operatorname{PSL}(2, 11)$ and $M_{11}$ graphs occur as the second and third members of a tower of graphs defined on a conjugacy class of involutions of the simple groups $A_5$, $\operatorname{PSL}(2, 11)$, $M_{11}$ and $M_{12}$ with two involutions adjacent if they generate a special $S_3$. The first graph in the tower is the line graph of the Petersen graph while the fourth is the Johnson graph $J(12, 4)$. The graphs also arise as collineation graphs of rank 2 truncations of various residues of certain $F$-geometries.

1. Introduction

When studying transitive decompositions of Johnson graphs [6] we came across two interesting graphs: one, $\Gamma$, associated with $M_{11}$ and the other, $\Pi$, associated with $\operatorname{PSL}(2, 11)$. The graphs arise as collinearity graphs of known partial linear spaces related to the Petersen graph but are interesting in their own right as they have several alternative definitions related to geometrical structures preserved by the group. This paper outlines these various constructions as well as showing how the graphs can be placed in a uniform framework by considering involutions which generate an $S_3$. 
1.1. Designs and geometries

A t-(v, k, λ) design is a collection of k-subsets (called blocks) of a v-set such that each t-subset is contained in λ blocks. The usual designs associated with the Mathieu groups are the Witt designs. The largest of these is a 5-(24, 8, 1) design \( W_{24} \) whose blocks are referred to as octads and which has full automorphism group \( M_{24} \). The Witt design \( W_{12} \) associated with \( M_{12} \) is a 5-(12, 6, 1) design (blocks referred to as hexads) and the Witt design \( W_{11} \) associated with \( M_{11} \) is a 4-(11, 5, 1) design (blocks referred to as pentads).

The group \( M_{11} \) has a 3-transitive action on a set \( Y \) of size 12 and each involution fixes four points. The collection of 4-subsets which are fixed point sets of involutions forms a 3-(12, 4, 3) design \( \mathcal{B} \) (see for example [10]). Let \( y \in Y \) and let \( \overline{\mathcal{B}} \) be the set of 3-subsets of \( Y \setminus \{y\} \) whose union with \( \{y\} \) is a block of \( \mathcal{B} \). Then \( \overline{\mathcal{B}} \) is a 2-(11, 3, 3) design known as the Petersen design and the blocks are the sets of fixed points of involutions of \( PSL(2, 11) \) in its 2-transitive action on 11 points. Repeating this contraction process again we obtain a 1-(10, 2, 3) design whose point set is the vertex set of the Petersen graph and blocks are the edges. These three designs give rise to diagram geometries for the groups \( M_{11} \), \( PSL(2, 11) \) and \( A_5 \), respectively. The elements of the first geometry are the points, 2-subsets, 3-subsets and 4-subsets in \( \mathcal{B} \) of a 12-set admitting the 3-transitive action of \( M_{11} \). The elements of the second geometry are the points, 2-subsets and 3-subsets in \( \overline{\mathcal{B}} \) of an 11-set admitting the 2-transitive action of \( PSL(2, 11) \). Finally the geometry for \( A_5 \) has as elements the vertices and edges of the Petersen graph. For each geometry, incidence is given by inclusion and each geometry occurs as a residue of the preceding geometry in the sequence. These geometries are listed as numbers 89, 88 and 84 of [1] and were characterized in [12]. Their diagrams are given in Fig. 1.

1.2. Graphs

The graph \( \Gamma \) can be defined as having as vertices the 165 blocks of the 3-(12, 4, 3) design \( \mathcal{B} \) mentioned above, and two vertices are adjacent if their intersection is a 3-subset. This graph is then the collinearity graph of the partial linear space given by the rank 2 truncation of the \( M_{11} \) geometry onto elements of the last two types with points taken to be the elements of the last type (geometry 2.6 of [5] and geometry 2.4 for \( M_{11} \) in [13]). Properties of \( \Gamma \) are recorded in Theorem 2.5. The graph \( \Pi \) is the graph with vertices the 55 blocks of \( \mathcal{B} \) such that two vertices are adjacent if their intersection is a 2-subset. Again, this is the collinearity graph of the partial linear space given by the rank 2 truncation of the \( PSL(2, 11) \) geometry onto elements of the last two types (geometry 6.1.2 of [2]). The properties of \( \Pi \) are recorded in Theorems 3.2 and 3.13. The collinearity graph for the partial linear space arising from the Petersen graph is the line graph of the Petersen graph.

Since 165 = \( \binom{n}{3} \) and the stabiliser in \( M_{11} \) of a vertex in \( \Gamma \) is the stabiliser in \( M_{11} \) of a 3-subset of an 11-set, it is natural to ask whether \( \Gamma \) has a definition in terms of 3-subsets. Similarly, since 55 = \( \binom{n}{2} \) and the stabiliser in \( PSL(2, 11) \) of a vertex in \( \Pi \) is the stabiliser in \( PSL(2, 11) \) of a 2-subset of an 11-set, it would appear that there should also be a 2-subset definition of \( \Pi \). Indeed in both cases there is and we have the following theorems.

Theorem 1.1. The graph \( \Gamma \) can be defined in the following ways.
- \( V \Gamma \) is the set of blocks of \( \mathcal{B} \) such that two vertices are adjacent if and only if they intersect in a 3-set.
- \( V \Gamma \) is the set of 3-subsets of an 11-set forming the point set of \( W_{11} \) such that two vertices are adjacent if and only if they are disjoint and the complement of their union is a pentad.

Theorem 1.2. The graph \( \Pi \) can be identified in the following four ways.

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Adjacency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks of the Petersen design</td>
<td>Intersection has size 2</td>
</tr>
<tr>
<td>Blocks of the Petersen design</td>
<td>Disjoint and complement of union is a pentad</td>
</tr>
<tr>
<td>2-subsets of an 11-set</td>
<td>Meet in one point and union is a block of the Petersen design</td>
</tr>
<tr>
<td>2-subsets of an 11-set</td>
<td>Disjoint and union contains no blocks of the Petersen design</td>
</tr>
</tbody>
</table>
The analysis for Theorem 1.1 is in Section 2 while the analysis for Theorem 1.2 is in Section 3.

1.3. Involutions

As already noted the blocks of $\mathcal{B}$ are the sets of fixed points of involutions of $M_{11}$ while the blocks of $\overline{\mathcal{B}}$ are the sets of fixed points of involutions of $\text{PSL}(2, 11)$. This suggests a purely group theoretic definition of $\Gamma$ and $\Pi$ independent of the permutation representation. Such a definition gives rise to a tower of four graphs with the smallest being the line graph of the Petersen graph and the largest being the Johnson graph $J(12, 4)$.

**Theorem 1.3.** There is a tower of graphs defined on a conjugacy class of involutions of $A_5$, $\text{PSL}(2, 11)$, $M_{11}$ and $M_{12}$ with two involutions adjacent if they generate an $S_3$ from a certain conjugacy class such that the graphs are the line graph of the Petersen graph, $\Pi$, $\Gamma$ and $J(12, 4)$.

We define the tower and prove Theorem 1.3 in Section 4. The groups $\text{PSL}(2, 11)$, $M_{11}$ and $M_{12}$ contain more than one class of subgroups $S_3$ and we specify the appropriate class in Construction 4.5 (for $\Pi$), 4.3 (for $\Gamma$) and 4.1 (for $J(12, 4)$).

The definition of the graphs in the tower is in the spirit of the investigations by Fischer [9] of groups generated by a conjugacy class of 3-transpositions, that is a conjugacy class of involutions such that any two either commute or generate an $S_3$.

It was noted in Section 1.2 that $\Gamma$ and $\Pi$ were collinearity graphs of certain partial linear spaces. The $S_3$-involution definition of these graphs then gives rise to an alternative definition of the partial linear spaces, that is, the partial linear space with points the involutions of $M_{11}$ and $\text{PSL}(2, 11)$ respectively and the lines the sets of three involutions which are the involutions of an $S_3$ in the relevant conjugacy class.

2. The $M_{11}$ graph

We begin by discussing some of the properties of the Witt design $\mathcal{W}_{24}$ which can be found for example in [7, Section 6]. First we give some properties of the octads.

(a) Two distinct octads meet in zero, two, or four points.
(b) Every 4-set is contained in five octads. Moreover, the symmetric difference $O_1 \ominus O_2$ of any two octads $O_1$, $O_2$ intersecting in a 4-subset is also an octad [7, Lemma 6.8A].
(c) If $O_1$ and $O_2$ are two disjoint octads, then the complement of $O_1 \cup O_2$ is also an octad [11, p 78].

The symmetric difference of two octads which intersect in a 2-subset is called a dodecad. The complement of a dodecad is also a dodecad.

(d) An octad intersects a dodecad in exactly two, four, or six points (consequence of [7, Lemma 6.8C]).

Let $D$ be a dodecad. We obtain the 12-point Witt design $\mathcal{W}_{12}$ on $D$ by taking as hexads the intersections of size 6 of octads with $D$. The stabiliser in $M_{24}$ of $D$ is isomorphic to $M_{12}$.

(e) Complements of hexads are hexads and their corresponding octads have two points in common (consequence of [7, Lemma 6.8C(ii)]).
Let $\alpha \in D$. Then we obtain $\mathcal{W}_1$ on $D \setminus \{\alpha\}$ by taking as pentads the 5-sets which together with $\alpha$ form a hexad of $\mathcal{W}_2$. Moreover, $(M_{24})_{D,\alpha} \cong M_{11}$ and has the usual action of $M_{11}$ on $D \setminus \{\alpha\}$ while acting 3-transitively on the complement $D^*$ of $D$.

The octads of $\mathcal{W}_{24}$ are the fixed point sets of involutions in the conjugacy class $2A$ of $M_{24}$. For each octad $O$ of $\mathcal{W}_{24}$ which intersects $D$ in four points, there is a unique involution of $M_{12}$ whose fixed points are the elements of $O$. The group $M_{11}$ has a unique class of involutions and it follows that they fix three points in $D \setminus \{\alpha\}$ and four points of $D^*$.

We have the following lemma.

**Lemma 2.1.** Let $D$ and $D^*$ be two complementary dodecads in $\mathcal{W}_{24}$ and let $\alpha$ be a point in $D$. Then for all 3-subsets $w$ of $D \setminus \{\alpha\}$, there exists a unique octad $O_w$ such that $O_w \cap D = w \cup \{\alpha\}$.

**Proof.** Let $w$ be a 3-subset in $D \setminus \{\alpha\}$. By property (b), there are five octads containing $w$ and $\alpha$. Since an octad intersects $D$ in exactly two, four, or six points (property (d)), and since there is a unique octad through any five points, there are four octads containing $w$ and $\alpha$, and intersecting $D$ in six points. Thus there is a unique octad, say $O_w$, which intersects $D$ in four points. \(\square\)

If follows from **Lemma 2.1** that

$$\mathcal{B} = \{O_w \cap D^* \mid w \text{ a 3-subset of } D \setminus \{\alpha\}\}$$

is the set of fixed point subsets of involutions in $M_{11}$ acting on the 12-set $D^*$ and is the set of blocks of the 3-(12, 4, 3) design mentioned in the introduction.

We now define two graphs.

**Definition 2.2.** Let $\Gamma_1$ be the graph whose vertices are the blocks of $\mathcal{B}$ with two blocks being adjacent if and only if they meet in a 3-subset.

**Definition 2.3.** Let $\Gamma_2$ be the graph whose vertices are the 3-subsets of an 11-set forming the point set of $\mathcal{W}_{11}$ two 3-sets being adjacent if and only if the complement of their union is a pentad.

To show that $\Gamma_1$ and $\Gamma_2$ are isomorphic we first need to set up a framework.

Let $D$ and $D^*$ be two complementary dodecads in $\mathcal{W}_{24}$ and let $\alpha$ be a point in $D$. Set $X = D \setminus \{\alpha\}$ and $Y = D^*$. For a set $W$ and integer $k \leq |W|$ define $\binom{W}{k}$ to be the set of all $k$-subsets of $W$. Define

$$i: \binom{X}{3} \rightarrow \mathcal{B}$$

$$w \mapsto Y \cap O_w. \quad (1)$$

**Theorem 2.4.** The map $i$ defined in (1) induces an isomorphism from $\Gamma_2$ onto $\Gamma_1$.

**Proof.** By **Lemma 2.1**, the map $i$ is well-defined. Let $w_1, w_2 \in \binom{X}{3}$ and suppose $Y \cap O_{w_1} = Y \cap O_{w_2}$. Then $O_{w_1}$ and $O_{w_2}$ have five points in common and hence are equal. Thus $i$ is one-to-one and by the definition of $\mathcal{B}$ is onto.

Let $w_1$ and $w_2$ be two 3-subsets of $X$ corresponding to adjacent vertices of $\Gamma_2$, that is $X \setminus (w_1 \cup w_2)$ is a pentad; in other words, $w_1$ and $w_2$ are disjoint and, by (e) and (h), $w_1 \cup w_2$ is a hexad of $D$, contained in an octad $O$ meeting $D$ in six points. The octads $O_{w_1}$ and $O$ are distinct and meet in at least the 3-subset $w_1$. Thus by property (a) they meet in four points, which means that $i(w_1)$ contains exactly one point of $O$. Moreover by (b), the symmetric difference $O \ominus O_{w_1}$ is an octad containing $w_2$ and $\alpha$ and intersecting $Y$ in four points. By **Lemma 2.1**, $O \ominus O_{w_1}$ is equal to $O_{w_2}$. Thus $i(w_1)$ and $i(w_2)$ meet in three points, and so they are adjacent vertices of $\Gamma_1$.

Conversely, let $v_1$ and $v_2$ be two 4-subsets of $Y$ corresponding to adjacent vertices of $\Gamma_1$, that is, they are elements of $\mathcal{B}$ which meet in a 3-subset. Then $O_1 = v_1 \cup \{\alpha\} \cup i^{-1}(v_1)$ and $O_2 = v_2 \cup \{\alpha\} \cup i^{-1}(v_2)$ are octads containing the 4-subset $\{\alpha\} \cup (v_1 \cap v_2)$. Since $v_1 \neq v_2$, the octads $O_1, O_2$ are distinct and so by property (a) we have $|O_1 \cap O_2| = 4$. Hence $i^{-1}(v_1)$ and $i^{-1}(v_2)$ are disjoint. Moreover, by (b)
Table 1
Orbit descriptions for \( \Gamma_2 \).

| \( w \in |w \cap w_0| \) | Extra condition |
|----------------------|------------------|
| A 0                  | \( X \setminus (w \cup w_0) \) is a pentad. |
| B 1                  | \( w = \{x, a, b\} \) with \( x \in w_0 \), the pentads \( P_a \) and \( P_b \) containing respectively \( w_0 \cup \{a\} \) and \( w_0 \cup \{b\} \) are distinct, and \( (P_a \cup P_b) \setminus (w_0 \setminus \{x\}) \) is not a pentad. |
| C 0                  | \( w \cup w_0 \) contains a pentad with three points in \( w_0 \). |
| D 2                  | \( w \cup w_0 \) is a pentad. |
| E 0                  | \( w \cup w_0 \) contains a pentad with three points in \( w \). |
| F 1                  | \( w = \{x, a, b\} \) with \( x \in w_0 \), the pentads \( P_a \) and \( P_b \) containing respectively \( w_0 \cup \{a\} \) and \( w_0 \cup \{b\} \) are distinct, and \( (P_a \cup P_b) \setminus (w_0 \setminus \{x\}) \) is a pentad. |

Fig. 2. Distance diagram of \( \Gamma' \).

the symmetric difference of \( O_1 \) and \( O_2 \) is an octad, which intersects \( D \) in \( i^{-1}(v_1) \cup i^{-1}(v_2) \). Hence \( i^{-1}(v_1) \cup i^{-1}(v_2) \) is a hexad in \( D \), and so by properties (e) and (h), its complement in \( X \) is a pentad. Thus \( i^{-1}(v_1) \) and \( i^{-1}(v_2) \) are adjacent vertices of \( \Gamma_2 \). □

Theorem 2.4 proves Theorem 1.1 as the two graphs are \( \Gamma_1 \) and \( \Gamma_2 \).

Fig. 2 gives the distance diagram of the graph \( \Gamma \cong \Gamma_2 \) according to the orbits of a vertex stabiliser as determined using MAGMA [3]. Each orbit of \( G_{w_0} \) on \( V \setminus \{w_0\} \) is denoted by a circle containing the number of vertices in the orbit. An edge from an orbit \( S \) to an orbit \( T \) with number \( a \) attached at the end connected to \( S \) means that each vertex in \( S \) is adjacent to \( a \) vertices in \( T \). The remaining number next to the orbit \( S \) is the number of vertices of \( S \) adjacent to a fixed vertex of \( S \). If this number is zero then we use a \( - \). Table 1 describes the different orbits in terms of the fixed 3-subset \( w_0 \) of the graph \( \Gamma_2 \).

There is one other graph on 165 vertices with automorphism group \( M_{11} \) in the literature [8]. This is a half-arc-transitive graph of valency 48 where the set of neighbours of \( s_0 \) is \( C \cup E \). It can be determined via MAGMA [3] that two vertices are adjacent in \( \Gamma \) if and only if in the valency 48 graph they are at distance 2 and there are precisely nine vertices adjacent to both.

We now determine some properties of \( \Gamma \).

Theorem 2.5. The graph \( \Gamma \) has 165 vertices, valency 8 and diameter 4. Its maximal cliques are of size 3 and two cliques meet in at most one vertex. Its full automorphism group is \( M_{11} \), which is arc-transitive and vertex-primitive, with vertex stabiliser \( M_8 \rtimes S_3 \cong 2 \cdot S_4 \), and arc stabiliser \( S_3 \). Moreover, it is a rank 8 graph.

Proof. The number of vertices is \( \binom{11}{5} = 165 \). Using the 4-subset definition of \( \Gamma \), the vertices of \( \Gamma \) are the blocks of the 3-(12, 4, 3) design \( B \). A vertex \( v \) contains four 3-subsets and each 3-subset is
contained in two blocks other than \( v \). Hence \( \Gamma \) has valency 8. It can be seen from the diagram of Fig. 2 that each neighbour of a vertex \( w_0 \) is adjacent to exactly one other neighbour of \( w_0 \). Hence the maximal cliques have size 3 and two such cliques meet in at most one vertex. It also follows from the diagram that \( \Gamma \) has diameter 4.

From the distance diagram for \( \Gamma \) given in Fig. 2, it follows that if an automorphism \( g \) of \( \Gamma \) fixes \( w_0 \) then it also fixes setwise \( A, B \) and \( G \) as these are the vertices at distance 1, 2 and 4 respectively. The automorphism \( g \) must also fix \( F \) setwise as these are the only vertices not adjacent to a vertex in \( G \), and fix \( E \) setwise as these are the only vertices adjacent to two vertices in \( G \). Since the vertices of \( D \) are the only vertices in \( D \cup C \) which are adjacent to vertices in \( E \), it also follows that \( g \) fixes \( D \) setwise. Hence the full automorphism group has rank 9 in its action on vertices. As \( D \) is the set of 3-subsets meeting \( w_0 \) in a 2-subset it follows that \( \text{Aut}(\Gamma) \leq \text{Aut}(f((11, 3))) = S_{11} \). Moreover, as \( g \) fixes \( A \) setwise it follows that \( \text{Aut}(\Gamma) \) preserves the Witt design \( \mathcal{W}_{11} \) and so \( \text{Aut}(\Gamma) = M_{11} \).

Using the 3-subset definition again, the stabiliser of a vertex \( w \) in \( A \) is \( M_8 \rtimes S_3 \cong 2 \cdot S_4 \), the stabiliser in \( M_{11} \) of a 3-subset. By [4, p 18], the stabiliser in \( M_{11} \) of a pentad is \( S_5 \) and this acts 3-transitively (as \( \text{PSL}(2, 5) \)) on the six points not in the pentad. Hence the stabiliser in \( A \) of \( w \) acts transitively on the set of eight pentads having an empty intersection with the 3-subset \( w \). Thus \( A_w \) acts transitively on the set of neighbours of \( w \) and so \( A \) is arc-transitive on \( \Gamma_2 \) with arc stabiliser \( S_3 \).

3. The \( \text{PSL}(2, 11) \) graph

Let \( D \) and \( D^* \) be complementary dodecads in \( \mathcal{W}_{24} \) and let \( \alpha \in D \) and \( \beta \in D^* \). We recall that the stabiliser in \( M_{24} \) of \( D \) and \( \alpha \) is \( M_{11} \) and acts 3-transitively on \( D^* \). Furthermore, the stabiliser in \( M_{11} \) of \( \beta \) is \( \text{PSL}(2, 11) \) which acts 2-transitively on \( D \setminus \{ \alpha \} \) and \( D^* \setminus \{ \beta \} \).

Let \( (D^*, \mathcal{B}) \) be the 3-(12, 4, 3) design given by the sets of fixed points of involutions of \( M_{11} \). Let \( \overline{\mathcal{B}} \) be the set of 3-subsets \( v \) of \( D^* \setminus \{ \beta \} \) such that \( v \cup \{ \beta \} \) is a block of \( \mathcal{B} \). Then \( |\overline{\mathcal{B}}| = 55 \) and \( (D^* \setminus \{ \beta \}, \overline{\mathcal{B}}) \) is a 2-(11, 3, 3) design. This is the Petersen design mentioned in the introduction. Obviously, \( \overline{\mathcal{B}} \) is the set of fixed point subsets of involutions in \( \text{PSL}(2, 11) \) acting on the 11-set \( D^* \setminus \{ \beta \} \). We note that \( \overline{\mathcal{B}} \) is the unique orbit of length 55 of \( \text{PSL}(2, 11) \) on the set of 3-subsets of \( D^* \setminus \{ \beta \} \). The other orbit has length 110.

We now give our first definition of the graph \( \Pi \).

**Definition 3.1.** Let \( X \) be a set of size 11 forming the point set of a 2-(11, 3, 3) Petersen design. Let \( \Pi_1 \) be the graph with vertex set the set of blocks of the design such that two blocks are adjacent if they have two points in common.

The graph \( \Pi_1 \) is the collinearity graph of the geometry 6.1.2 of [2], from which we have reproduced the distance diagram in Fig. 3. Note that regarding \( \overline{\mathcal{B}} \) as a subset of \( \mathcal{B} \), we see that \( \Pi_1 \) is a subgraph of \( \Gamma_1 \).

**Theorem 3.2.** The graph \( \Pi_1 \) has 55 vertices, valency 6 and diameter 3. Its maximal cliques are of size 3 and two cliques meet in at most one vertex. The group \( L = \text{PSL}(2, 11) \) is an automorphism group of \( \Pi_1 \), which is arc-transitive and vertex-primitive with vertex stabiliser \( D_{12} \) and arc stabiliser \( C_2 \). Moreover, \( \Pi_1 \) is a Cayley graph for the group \( C_{11} \rtimes C_5 \).

**Proof.** Let \( v \) be a vertex of \( \Pi_1 \). Then \( v \) contains three 2-subsets and since the vertex set of \( \Pi_1 \) is the block set of a 2-(11, 3, 3) design, each 2-subset of \( v \) is contained in three blocks. Hence \( v \) is adjacent to six other vertices. Moreover, we see from Fig. 3 that \( \Pi_1 \) has diameter 3, and the maximal cliques have size 3 and meet in at most one vertex. Since \( L = \text{PSL}(2, 11) \) preserves the design, \( L \leq \text{Aut}(\Pi_1) \).

If \( v \) is a vertex, then \( L_v = D_{12} \). Let \( w \) be a vertex adjacent to \( v \). Let \( \overline{v} = v \cup \{ \beta \} \) and \( \overline{w} = w \cup \{ \beta \} \) and regard \( \Pi_1 \) as a subgraph of \( \Gamma_1 \) with the vertices \( \overline{v}, \overline{w} \) of \( \Gamma_1 \) corresponding to the vertices \( v, w \), respectively, of \( \Pi_1 \). By Theorem 2.5, \( (M_{11})_{\text{top}} = S_5 \) and since in the action of \( M_{11} \) on twelve points, elements of order 3 only have three fixed points, \( (M_{11})_{\text{top}} \) induces \( S_3 \) on the three points of \( \overline{v} \cap \overline{w} = \{ \beta \} \cup (v \cap w) \). Since \( L \) is the stabiliser in \( M_{11} \) of \( \beta \), it follows that \( L_{vw} = C_2 \) and hence \( L \) is arc-transitive on \( \Pi_1 \).
Fig. 3. Distance diagram of $\Pi_1$.

Note that $\text{PSL}(2,11)$ has a subgroup $C_{11} \times C_5$ which intersects trivially with $D_{12}$ and so by comparing orders we have $\text{PSL}(2,11) = D_{12}(C_{11} \times C_5)$. Hence $C_{11} \times C_5$ acts regularly on the vertex set of $\Pi_1$ and so $\Pi_1$ is a Cayley graph of $C_{11} \times C_5$. $\square$

The map $i$ defined in (1) defines an isomorphism from $\Gamma_2$ to $\Gamma_1$; hence the preimage of $\Pi_1$ in $\Gamma_2$ is isomorphic to $\Pi_1$. Note that the preimages of the vertices of $\Pi_1$ are 3-sets $w$ in $D \setminus \{\alpha\}$ such that $\beta \in i(w)$. If we take these 3-sets as blocks, we find an isomorphic 2-$(11,3,3)$ design, since this block set and $\overline{B}$ are interchanged by elements of $\text{PGL}(2,11) \setminus \text{PSL}(2,11)$. This gives the following definition of a graph isomorphic to $\Pi_1$.

**Definition 3.3.** Let $X$ be an 11-set forming the point set of a 2-$(11,3,3)$ Petersen design preserved by a group $L = \text{PSL}(2,11)$ and also forming the point set of $W_{11}$ preserved by $L$. Let $\Pi_2$ be the graph with vertex set the set of blocks of the Petersen design such that blocks $v_1$ and $v_2$ are joined by an edge if $v_1 \cap v_2 = \emptyset$ and $X \setminus (v_1 \cup v_2)$ is a pentad.

**Theorem 3.4.** $\Pi_1 \cong \Pi_2$.

**Proof.** The isomorphism is given by the map $i$ from (1). $\square$

By Theorem 3.2, the stabiliser of a vertex in $L = \text{PSL}(2,11)$ is $D_{12}$ which has orbits of size 2, 3 and 6 on the 11-set. We will now describe the link between the blocks of the Petersen design and the pairs of an 11-set.

Consider $(D^*, B)$, a 3-$(12,4,3)$ design given by the sets of fixed points of involutions of $M_{11}$, where $X$ has size 12. Let $\beta \in X$. We recall that the set of 3-subsets $v$ of $D^* \setminus \{\beta\}$ such that $v \cup \{\beta\}$ is a block of $B$ yields the block set $\overline{B}$ of the Petersen design $(D^* \setminus \{\beta\}, \overline{B})$, which is a 2-$(11,3,3)$ design. Let $v \in \overline{B}$. Since $(D^*, B)$ is a 3-$(12, 4, 3)$ design there are three blocks of $B$ containing $v$, one of which is $v \cup \{\beta\}$. Let $\epsilon_1$ and $\epsilon_2$ be the two points such that $v \cup \{\epsilon_i\}$ are also blocks of $B$. This allows us to define a map

$$j: \overline{B} \rightarrow \binom{D^* \setminus \{\beta\}}{2}$$

$$v \mapsto \{\epsilon_1, \epsilon_2\}. \quad (2)$$

Since $L$ preserves $B$, we must have that the pair stabilised by $L_v$ is $j(v)$. 


Let $p$ be a pair in the 11-set $D^* \setminus \{\beta\}$. Then $p$ is contained in three blocks $s_1$, $s_2$ and $s_3$ of the Petersen design, and $(s_1 \cup s_2 \cup s_3) \setminus p$ is a 3-set of $D^* \setminus \{\beta\}$. Since $L$ preserves $\overline{B}$, the block of the Petersen design stabilised by $L_p$ must be this 3-set. We define the map

$$b : \binom{\{D^* \setminus \{\beta\}\}}{2} \rightarrow \overline{B}$$

$$b(p) \leftrightarrow (s_1 \cup s_2 \cup s_3) \setminus p. \quad (3)$$

Since $b(p)$ is the block stabilised by $L_p$ and $j(b(p))$ is the 2-set stabilised by $L_{b(p)}$, it follows that $b$ is the inverse of $j$.

The correspondence between 2-subsets and blocks suggests a 2-subset definition of $\Pi_1$.

**Definition 3.5.** Let $X$ be an 11-set equipped with a 2-(11, 3, 3) Petersen design. Let $\Pi_3$ be the graph with vertex set the set of all 2-subsets of $X$ such that two vertices $x_1$, $x_2$ are adjacent if they are disjoint and $x_1 \cup x_2$ contains no blocks of the Petersen design.

We will need the following lemma.

**Lemma 3.6.** A block of $\mathcal{B}$ does not contain two distinct blocks of $\overline{\mathcal{B}}$.

**Proof.** Suppose a block $w$ of $\mathcal{B}$ contains distinct blocks $v_1$ and $v_2$ of $\overline{\mathcal{B}}$. Then $|v_1 \cap v_2| = 2$ and the three blocks $w$, $v_1 \cup \{\beta\}$ and $v_2 \cup \{\beta\}$ of $\mathcal{B}$ form a triangle of $\Gamma_1$. On the other hand, the third block of $\mathcal{B}$ containing $v_1$ also forms a triangle with $w$ and $v_1 \cup \{\beta\}$. Since $v_2 \cup \{\beta\}$ does not contain $v_1$ this is a second triangle of $\Gamma^*$ sharing an edge with the first, contradicting Theorem 2.5. □

We have the following isomorphism.

**Lemma 3.7.** The map $j$ defined in (2) induces an isomorphism from $\Pi_1$ to $\Pi_3$.

**Proof.** We have already proved that $j$ is a bijection. By Theorem 3.2, $\Pi_1$ has valency 6. We claim that $\Pi_3$ also has valency 6. Let $p = \{\gamma, \delta\}$ be a vertex of $\Pi_3$ and let $\{\xi_1, \xi_2, \xi_3\} = b(p)$. Among the 36 pairs disjoint from $p$, the pairs that are non-adjacent to $p$ are those that either form a block of $\mathcal{B}$ with $\gamma$ or $\delta$, or contain a point in $b(p)$. There are 21 pairs in the second case, 3 contained in $b(p)$ and 18 meeting it in a single point. Let us count the pairs forming a block with $\gamma$ or $\delta$. It is easy to count that $\gamma$ is contained in 15 blocks, among which 3 also contain $\delta$, and similarly interchanging $\gamma$ and $\delta$. So there are 24 blocks containing either $\gamma$ or $\delta$ but not both. Let $p_1 = j(\{\gamma, \delta, \xi_1\})$, $p_2 = j(\{\gamma, \delta, \xi_2\})$, $p_3 = j(\{\gamma, \delta, \xi_3\})$. These three pairs form a block with $\gamma$ and a block with $\delta$, as $j$ is the inverse of $b$. Hence there are 21 pairs in the first case. Finally we count the number of pairs which are in the intersection, that is, which form a block of $\mathcal{B}$ with $\gamma$ or with $\delta$ and contain $\xi_1, \xi_2$ or $\xi_3$. By Lemma 3.6, such a pair cannot be contained in $b(p)$, as otherwise the block $\{\xi_1, \xi_2, \xi_3, \gamma\}$ (or $\{\xi_1, \xi_2, \xi_3, \delta\}$) of $\overline{\mathcal{B}}$ contains two blocks of $\mathcal{B}$. Therefore such a pair meets $\{\xi_1, \xi_2, \xi_3\}$ in exactly one point. There are two blocks containing a given point of $p$ and a given point of $b(p)$ but not containing $p$, and so there are 23.2 = 12 pairs in both the first and second cases. Thus there are $21 + 21 - 12 = 30$ vertices non-adjacent to $p$, and so there are six vertices adjacent to $p$.

It remains to show that $j$ maps adjacent vertices of $\Pi_1$ to adjacent vertices of $\Pi_3$. Let $v_1, v_2 \in \overline{\mathcal{B}}$ such that $|v_1 \cap v_2| = 2$, that is, $v_1, v_2$ are adjacent vertices in $\Pi_1$. Suppose that $j(v_1)$ and $j(v_2)$ have a point $\epsilon$ in common. Then $v_1 \cup \{\beta\}$, $v_2 \cup \{\beta\}$, $v_2 \cup \{\epsilon\}$ and $v_1 \cup \{\epsilon\}$ are blocks of $\mathcal{B}$ forming a 4-cycle in $\Gamma_1$. However, looking at the distance diagram for $\Gamma_1$ in Fig. 2, we see that $\Gamma_1$ does not contain 4-cycles. Thus $j(v_1) \cap j(v_2) = \emptyset$. The points which together with $j(v_1)$ form a block of $\overline{\mathcal{B}}$ are those in $b(j(v_1)) = v_1$. Suppose $j(v_2) \cap v_1$ is not empty. Then this intersection has size 1 (because $j(v_2) \cap v_2 = \emptyset$), and $v_1 \cup v_2 \in \mathcal{B}$, which contradicts Lemma 3.6. Therefore, $j(v_2) \cap v_1 = \emptyset$, and so no block of $\overline{\mathcal{B}}$ contains $j(v_1)$ and one point of $j(v_2)$. Similarly no block of $\overline{\mathcal{B}}$ contains $j(v_2)$ and one point of $j(v_1)$. Thus $v_1 \cup v_2$ contains no block of $\overline{\mathcal{B}}$, and hence $j(v_1)$ and $j(v_2)$ are adjacent vertices of $\Pi_3$. □

We also have the following definition of a graph on the set of 2-subsets of an 11-set.
**Definition 3.8.** Let $X$ be an 11-set forming the point set of a 2-$(11, 3, 3)$ Petersen design. Let $\Pi_4$ be the graph with vertex set the set of all 2-subsets of $X$ such that two vertices are adjacent if they have one point in common and their union is a block of the Petersen design.

Next we show that $\Pi_4$ is isomorphic to $\Pi_1$. To do this we first recall the following setup. Let $D$ and $D^*$ be two complementary dodecads in $S(5, 8, 24)$ and let $\alpha \in D$ and $\beta \in D^*$ let $L \cong PSL(2, 11)$ be the stabiliser in $M_{24}$ of $D, \alpha$ and $\beta$. Recall from (1) the map $i$ from the set of 3-subsets of $D \setminus \{\alpha\}$ into the set of 4-subsets of $D^*$ such that the image $\mathcal{B}$ of $i$ is a 3-$(12, 4, 3)$ design and the set of blocks of $\mathcal{B}$ containing $\beta$ yields a 2-$(11, 3, 3)$ Petersen design on $D^* \setminus \{\beta\}$ with blocks $\mathcal{B}$.

Let $v \in \mathcal{B}$ and $j(v) = (\epsilon_1, \epsilon_2)$. Since $\cup \{\beta\}, v \cup \{\epsilon_1\}, v \cup \{\epsilon_2\}$ are mutually adjacent in $\Gamma_1$, Theorem 2.4 implies that we can partition $D$ as $\{\alpha, \delta, \mu\} \cup w_1 \cup w_2 \cup w_3$ such that $i(w_1) = v \cup \{\beta\}$, $i(w_2) = v \cup \{\epsilon_1\}$ and $i(w_3) = v \cup \{\epsilon_2\}$, and for $i = 1, 2, 3$ the set $\{\delta, \mu\} \cup w_i$ is a pentad. Thus we can also define the map

$$k : \mathcal{B} \rightarrow \left(\frac{D \setminus \{\alpha\}}{2}\right),$$

$$v \mapsto \{\delta, \mu\}.$$  

(4)

The maps $j$ and $k$ are linked in the following way.

**Lemma 3.9.** Let $v \in \mathcal{B}$ and let $j, k$ be the maps defined in (2), (4) respectively. Then $\{\beta\} \cup v \cup j(v) \cup k(v)$ is an octad.

**Proof.** Let $j(v) = (\epsilon_1, \epsilon_2)$. By the definition of the map $i$ in (1) we have three octads $O_{w_1} = \{\alpha, \beta\} \cup v \cup w_1, O_{w_2} = \{\alpha, \epsilon_1\} \cup v \cup w_2$ and $O_{w_3} = \{\alpha, \epsilon_2\} \cup v \cup w_3$. By property (b), $O_{w_2} \cap O_{w_3} = \{\epsilon_1, \epsilon_2\} \cup w_2 \cup w_3$ is an octad disjoint to $O_{w_1}$. Thus by property (c), $k(v) \cup (D^* \setminus \{\beta, \epsilon_1, \epsilon_2\} \cup v))$ is also an octad and so by property (e), $k(v) \cup v \cup j(v) \cup \{\beta\}$ is an octad. □

We now show that $\Pi_1$ and $\Pi_4$ are isomorphic.

**Theorem 3.10.** The map $k$ defined in (4) induces an isomorphism from $\Pi_1$ to $\Pi_4$.

**Proof.** We first construct $\Pi_1$ by using the Petersen design $(D^* \setminus \{\beta\}, \mathcal{B})$. Thus the vertices of $\Pi_1$ are the blocks of $\mathcal{B}$ and two vertices are adjacent if they intersect in a 2-set. Now $L = PSL(2, 11)$ preserves the set of octads containing $\{\alpha, \beta\}$ and meeting $D$ in four points. There are 55 such octads and so the set $\mathcal{C} = \{\text{set } w \subset D \setminus \{\alpha\} | \beta \in O_w\}$ is an orbit of length 55 of $L$ on 3-subsets of $D \setminus \{\alpha\}$. Thus $(D \setminus \{\alpha\}, \mathcal{C})$ is a 2-$(11, 3, 3)$ Petersen design and we can construct $\Pi_4$ from this design: that is the vertices of $\Pi_4$ are the 2-subsets of $D \setminus \{\alpha\}$ and two vertices are adjacent if they have one point in common and their union is a block of $\mathcal{C}$.

Since $L$ preserves $\mathcal{B}$ and $\mathcal{C}$, it also preserves the set of images of $k$. Then as $L$ acts 2-transitively on $D \setminus \{\alpha\}$, it follows that $k$ is onto. Moreover, as $|\mathcal{B}| = |\mathcal{C}| = 55 = \left(\frac{11}{2}\right)$ it follows that $k$ is a bijection.

We saw in Theorem 3.2 that $\Pi_1$ is of valency 6. Moreover, since each 2-set is contained in three blocks of $\mathcal{C}$, it follows that $\Pi_4$ also has valency 6. Thus we only need to show that if $v_1$ and $v_2$ are adjacent in $\Pi_1$ then $k(v_1)$ and $k(v_2)$ are adjacent in $\Pi_4$.

Let $v_1, v_2 \in \mathcal{B}$ such that $v_1 \cap v_2 = 2$, that is, $v_1, v_2$ are adjacent vertices in $\Pi_1$. We have shown in the proof of Lemma 3.7 that $j(v_1) \cap j(v_2) = \emptyset$. Moreover, by Lemma 3.9, $O_1 = v_1 \cup j(v_1) \cup k(v_1) \cup \{\beta\}$ and $O_2 = v_2 \cup j(v_2) \cup k(v_2) \cup \{\beta\}$ are octads containing the three points $(v_1 \cap v_2) \cup \{\beta\}$. Hence by property (a) (there are four points in common and so $k(v_1) \cap k(v_2) = 1$.

Let $k(v_1) = \{\delta_1, \mu\}$ and $j(v_1) = (\epsilon_1, \epsilon_2)$, and let $w_1, w_2, w_3$ be 3-subsets of $D \setminus \{\alpha\}$ such that $O_{w_1} = \{\alpha, \beta\} \cup w_1 \cup v_1, O_{w_2} = \{\alpha, \epsilon_1\} \cup w_2 \cup v_1$ and $O_{w_3} = \{\alpha, \epsilon_2\} \cup w_3 \cup v_1$ and $w_1, w_2, w_3$ are octads. Similarly, let $k(v_2) = \{\delta_2, \mu\}$ and $j(v_2) = (\epsilon'_1, \epsilon'_2)$, and let $w'_1, w'_2, w'_3$ be 3-subsets of $D \setminus \{\alpha\}$ such that $O'_{w'_1} = \{\alpha, \beta\} \cup w'_1 \cup v_2, O'_{w'_2} = \{\alpha, \epsilon'_1\} \cup w'_2 \cup v_2$ and $O'_{w'_3} = \{\alpha, \epsilon'_2\} \cup w'_3 \cup v_2$ are octads. Recall the octads $O_1 = v_1 \cup j(v_1) \cup k(v_1) \cup \{\beta\}$ and $O_2 = v_2 \cup j(v_2) \cup k(v_2) \cup \{\beta\}$, which have the four points $(v_1 \cap v_2) \cup \{\beta, \mu\}$ in common. Hence by property (b),

$$O_1 \cap O_2 = \{\delta_1, \delta_2\} \cup (v_1 \cap v_2) \cup \{\epsilon_1, \epsilon_2, \epsilon'_1, \epsilon'_2\}$$
is an octad. Since \( O_1 \cup O_2 \) contains one point of \( v_2 \) it has at least one point in common with \( O_{w'_1} \) and so by property (a) it follows that \( \delta_1 \in w'_1 \). Similarly, comparing \( O_1 \cup O_2 \) with \( O_{w_1} \) we see that \( \delta_2 \in w_1 \). Thus \( O_1 \cup O_{w_1} = w_1 \cup \{\alpha, \delta_1, \mu, \epsilon_1, \epsilon_2\} \) and \( O_2 \cup O_{w'_1} = w'_1 \cup \{\alpha, \delta_2, \mu, \epsilon'_1, \epsilon'_2\} \) are octads with the 4-set \( w = \{\delta_1, \delta_2, \mu, \alpha\} \) in common. By property (b), the 4-set \( w \) is contained in five octads, which by Lemma 2.1 are \( O_{(\delta_1, \delta_2, \mu)} \) and four octads \( R_1, R_2, R_3, R_4 \) meeting \( D \) in six points. The octads \( R_1 := O_1 \cup O_{w_1} \) and \( R_2 := O_2 \cup O_{w'_1} \) are two of these four octads. Moreover, the 2-sets \((R_i \cap D) \setminus w\) for \( i = 1, 2, 3, 4 \), partition the set \( D \setminus w \) of eight points. The octads \( R_1, R_2 \) account for the four points of \((w_1 \cup w'_1) \setminus \{\delta_1, \delta_2\}\) and so for \( i = 3 \) or \( 4 \), the two points of \((R_i \cap D) \setminus w\) are in \( w_2 \cup w_3 \) but not in \( w'_1 \). If \( \beta \in R_i \) for \( i = 3, 4 \), then \( R_i \) will have five points in common with either \( O_1 \cup O_{w_2} = w_2 \cup \{\alpha, \beta, \delta_1, \mu, \epsilon_2\} \) or \( O_1 \cup O_{w_3} = w_3 \cup \{\alpha, \beta, \delta_1, \mu, \epsilon_1\} \). Since these two octads do not contain \( \delta_2 \), it follows that neither of them are \( R_i \), and so by property (a), \( \beta \) is not in any of the \( R_i \). Now \( R_1, R_2, R_3, R_4 \) define eight points of \( D^* \) and \( i(\delta_1, \delta_2, \mu) \) is the set of four points of \( D^* \) not in any \( R_i \). For \( i \neq j \) the sets \( R_i \cap D^* \) and \( R_j \cap D^* \) are disjoint and have size 2. Hence \( R_1, \ldots, R_4 \) provide us with eight points of \( D^* \setminus \{\beta\} \). Thus \( \beta \in i(\delta_1, \delta_2, \mu) \) and so \( \{\delta_1, \delta_2, \mu\} \subset \subset E \). Hence \( \{\delta_1, \delta_2, \mu\} \) is a block of the Petersen design \((D \setminus \{\alpha\}, E)\) preserved by \( \text{PSL}(2,11) \). Thus \( k(v_1) \) is adjacent to \( k(v_2) \) in \( \Pi_4 \) and so \( \Pi_1 \cong \Pi_4 \). \( \Box \)

**Corollary 3.11.** \( \Pi_1 \cong \Pi_2 \cong \Pi_4 \cong \Pi_3 \cong \Pi \).

This completes the proof of Theorem 1.2.

We also have that \( \Pi \) is self-dual in the following sense:

**Proposition 3.12.** \( \Pi \) is isomorphic to the graph whose vertices are the triangles of \( \Pi \) and two triangles are adjacent if they have a common vertex.

**Proof.** We will show that \( \Pi_4 \) is isomorphic to the graph whose vertices are the triangles of \( \Pi_1 \) and two triangles are adjacent if they have a common vertex. Since both \( \Pi_1 \) and \( \Pi_4 \) are isomorphic to \( \Pi \), this will yield the lemma.

Since the Petersen design is a 2-(11, 3, 3) design, each 2-subset is contained in three blocks and these three blocks form a triangle in \( \Pi_1 \). Since no edge is contained in two triangles by Theorem 3.2, each triangle in \( \Pi_1 \) is described by a unique 2-subset, this being the 2-subset in common with each of the three blocks which are vertices in the triangle. Thus there is a bijection between the vertices of \( \Pi_4 \) and the triangles of \( \Pi_1 \). Moreover, two triangles of \( \Pi_1 \) have a vertex in common if and only if the union of the two 2-sets corresponding to the two triangles is the block of the Petersen design given by the common vertex. Hence the result follows. \( \Box \)

We finish this section by giving the full automorphism of \( \Pi \).

**Theorem 3.13.** The full automorphism group of \( \Pi \) is \( \text{PSL}(2,11) \) and \( \Pi \) is a rank 9 graph.

**Proof.** It follows from Corollary 3.11 that Fig. 3 is the distance diagram for \( \Pi_4 \). It can be checked that \( A, B, \ldots, H \) are the orbits of a vertex stabiliser in \( L \) Table 2 (found with the help of \textsc{Magma} [3]) describes the different orbits in terms of the fixed 2-subset \( v_0 \) of the graph \( \Pi_4 \).
Let $g$ be an automorphism of $\Pi$ fixing $v_0$. Then $g$ clearly fixes $A$ setwise and must also fix $D$ setwise as these are the only vertices at distance 3 from $v_0$ adjacent to four vertices at distance 2 and to two vertices at distance 3 which are themselves adjacent to four vertices at distance 2. Since the vertices in $B$ are the only vertices at distance 2 adjacent to a vertex of $D$, it follows that $g$ fixes $B$ and hence also $C$ setwise. As $A \cup C$ is the set of 2-subsets meeting $v_0$ in a 1-subset it follows that $\Aut(\Pi) \leq \Aut(J(11, 2)) = S_{11}$. Moreover, as $g$ fixes $A$ setwise it follows that $\Aut(\Pi)$ preserves the Petersen design, whose automorphism group is $\PSL(2, 11)$ by [2, p182] and so $\Aut(\Pi) = \PSL(2, 11)$. It follows that $\Pi$ is a rank 9 graph. \hfill $\Box$

4. A tower of graphs

The Johnson graph $J(12,4)$ can be defined in terms of one of the two conjugacy classes of involutions of $M_{12}$.

**Construction 4.1.** Let $G = M_{12}$. Let $\Delta_1$ be the graph with vertex set the set of involutions of $G$ from the conjugacy class $2B$ with four fixed points (in an $M_{12}$ action of degree 12) [4, p 33] such that two involutions are joined by an edge if and only if they generate an $S_3$ which has three fixed points.

**Lemma 4.2.** $\Delta_1 \cong J(12,4)$.

**Proof.** Let $Y$ be a set of size 12 such that $G$ acts 5-transitively on $Y$. Since the class $2B$ has $495 = \left(\binom{12}{4}\right)$ elements, it follows that the map $\phi$ from the vertices of $\Delta_1$ to the vertices of $J(12,4)$ which maps each involution $g$ to $\phi(g)$ is a bijection.

Let $g_1$ and $g_2$ be two adjacent vertices of $\Delta_1$. Then $(g_1, g_2) \cong S_3$ fixes three points. Thus $|\Fix(g_1) \cap \Fix(g_2)| = 3$ and so $\phi(g_1)$ and $\phi(g_2)$ are adjacent in $J(12,4)$.

Conversely, let $v_1$ and $v_2$ be two adjacent vertices of $J(12,4)$, that is, 4-subsets of $Y$ intersecting in a 3-set. The pointwise stabiliser in $G$ of this 3-set is $L \cong M_9 \cong C_4^2 \times Q_8$, which has a unique Sylow 3-subgroup $S$ and a unique conjugacy class of nine subgroups isomorphic to $Q_8$ (the Sylow 2-subgroups). Let $g \in L$ be an involution. Then $g$ is the unique involution of some $Q_8$ and so inverts each nontrivial element of $S$. Moreover, the nine involutions of $L$ are the elements $sg$, for $s \in S$. The two involutions fixing respectively $v_1$ and $v_2$ (that is $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$) lie in $L$, so are $s_1g$ and $s_2g$, for some $s_1, s_2 \in S$. Now $(s_1g)(s_2g) = s_1s_2^{-1}$, which has order 3, and so $(s_1g, s_2g) \cong S_3$ and is contained in $L$, and hence fixes three points. Thus $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$ are adjacent in $\Delta_1$ and the proof is complete. \hfill $\Box$

Let $Y$ be a set of size 12 on which the group $G = M_{12}$ acts 5-transitively. Now $G$ contains two classes of subgroups isomorphic to $M_{11}$ and these are interchanged by an outer automorphism with one class being the stabilisers in $G$ of a point of $Y$. Involution in $G$ lie in one of two conjugacy classes: fixed point free involutions and the class $2B$ where they each fix four points [4, p 33]. Since the class $2B$ is fixed setwise by outer automorphisms of $M_{12}$ it follows that the involutions in any $M_{11}$ lie in the class $2B$. Now $M_{11}$ has 165 involutions and they form a single conjugacy class. Let $H$ be a subgroup of $G$ isomorphic to $M_{11}$ which is not the stabiliser of a point, that is, is 3-transitive on $Y$. Then the stabiliser in $H$ of a 3-subset of $Y$ is isomorphic to $S_3$, and as $M_{11}$ is 3-transitive on $Y$, the set of subgroups of $H$ isomorphic to $S_3$ which fix three points forms a single conjugacy class. Since the normaliser of such an $S_3$ fixes the 3-set of fixed points setwise, each such $S_3$ subgroup has normaliser $S_3 \times S_3$.

This suggests a connection with $\Gamma_1$.

**Construction 4.3.** Let $H = M_{11}$. Let $\Delta_2$ be the graph with vertex set the set of involutions of $H$ such that two involutions are adjacent if and only if they generate an $S_3$ with normaliser $S_3 \times S_3$.

**Lemma 4.4.** $\Delta_2 \cong \Gamma_1$.

**Proof.** Since $\Delta_2$ is the subgraph of $\Delta_1$ induced by the involutions of $H$ and $\Gamma_1$ is the image of $\Delta_2$ under the isomorphism between $\Delta_1$ and $J(12,4)$ it follows that $\Delta_2 \cong \Gamma_1$. \hfill $\Box$
We have just exhibited an embedding of the graph $\Delta_2$ in $J(12, 4)$. It is proved in [6] that in fact $J(12, 4)$ decomposes into 12 pairwise disjoint copies of $\Delta_2$ and these 12 copies are transitively permuted by $M_{12}$.

Next we look at a subgroup $K = \text{PSL}(2, 11)$ of $H = M_{11}$. This occurs as the stabiliser in $H$ of a point $\beta \in Y$. Hence $K$ contains a class of subgroups $S_3$ which fix three points of $Y$, one of which is $\beta$. Since $K$ is 2-transitive on the eleven points of $Y \setminus \{\beta\}$ it follows that all such subgroups $S_3$ are conjugate in $K$. Thus we can use this conjugacy class of $S_3$ subgroups to define a graph on the set of involutions of $K$, and this graph will be a subgraph of $\Delta_2$.

**Construction 4.5.** Let $\Delta_3$ be the graph whose vertex set is the set of involutions of $\text{PSL}(2, 11)$ such that two involutions are adjacent if and only if they generate an $S_3$ which fixes two points in the action on eleven points.

**Lemma 4.6.** $\Delta_3 \cong \Pi_1$

**Proof.** The graph $\Delta_3$ is the subgraph of $\Delta_2$ induced on the set of involutions of $\text{PSL}(2, 11)$. Then the isomorphism $\phi : V \Delta_3 \leftrightarrow J(12, 4)$ which maps an involution to its set of four fixed points in $Y$ induces an isomorphism from $\Delta_3$ to a subgraph of $\Pi_1$. Since all involutions in $\text{PSL}(2, 11)$ fix the point $\beta$ and the blocks of the Petersen design are the sets of fixed points of the involutions of $\text{PSL}(2, 11)$ on $Y \setminus \{\beta\}$ this provides an isomorphism from $\Delta_3$ to $\Pi_1$. \(\square\)

Note that $\text{PSL}(2, 11)$ contains two classes of subgroups $S_3$ and these are fused in $\text{PGL}(2, 11)$. Hence, the graph formed on the set of involutions of $\text{PSL}(2, 11)$, with two involutions joined by an edge if they generate an $S_3$ in the class other than the one used to define $\Delta_3$, is isomorphic to $\Delta_3$.

Since the stabiliser in $\text{PSL}(2, 11)$ of an arc of $\Delta_3$ is $C_2$ (Theorem 3.2) while the stabiliser of an arc of $\Delta_2$ in $M_{11}$ is $S_3$ (Theorem 2.5), it follows that if we spin the subgraph $\Delta_3$ of $\Delta_2$ under $M_{11}$ we do not obtain a decomposition of $\Delta_3$. Instead we obtain a cover of $\Delta_2$ with each edge occurring in precisely three of the twelve subgraphs isomorphic to $\Delta_3$, that is, we obtain a transitive 3-cover.

Inside $K$ we can choose a subgroup $L \cong A_5$. Note that there are two conjugacy classes of such subgroups that are interchanged by $\text{PGL}(2, 11)$. Each such subgroup $L$ contains 15 involutions and these form a single $L$-conjugacy class.

**Construction 4.7.** Let $\Delta_4$ be the graph with vertex set the 15 involutions of $A_5$ such that two involutions are adjacent if and only if they generate an $S_3$.

**Lemma 4.8.** $\Delta_4$ is the line graph of the Petersen graph.

**Proof.** When written in the usual permutation representation of $A_5$ on five points, the involutions of $A_5$ are of the form $(a, b)(c, d)$. Let $\tau$ be the map which takes each involution $(a, b)(c, d)$ in $A_5$ to the edge $\{(a, b), (c, d)\}$ of the Petersen graph. This is clearly a bijection. There are precisely four involutions of $A_5$ which generate an $S_3$ with $(a, b)(c, d)$. If $e$ is the fifth point of the set then these involutions are $(a, b)(c, e), (a, b)(d, e), (a, e)(c, d)$ and $(b, e)(c, d)$. Under $\tau$, these are mapped to the four edges incident with the edge $\{(a, b), (c, d)\}$ of the Petersen graph. Hence $\tau$ is an isomorphism. \(\square\)

Since the stabiliser in $A_5$ of an arc of $\Delta_4$ is $C_2$ and this is the stabiliser in $\text{PSL}(2, 11)$ of an arc in $\Delta_3$, it follows that $\Delta_3$ decomposes into eleven copies of $\Delta_4$ and these copies are transitively permuted by $\text{PSL}(2, 11)$.

**References**


