Some Formulas for Spin Models on Distance-Regular Graphs

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A spin model is a square matrix $W$ satisfying certain conditions which ensure that it yields an invariant of knots and links via a statistical mechanical construction of V. F. R. Jones. Recently F. Jaeger gave a topological construction for each spin model $W$ of an association scheme which contains $W$ in its Bose-Mesner algebra. Shortly thereafter, K. Nomura gave a simple algebraic construction of such a Bose-Mesner algebra $N(W)$. In this paper we study the case $W \in N(W)$, where $\mathcal{A}$ is the Bose-Mesner algebra of a distance-regular graph. We show the following results. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d > 1$ such that the Bose-Mesner algebra $\mathcal{A}$ of $\Gamma$ satisfies $W \in N(W)$ for some spin model $W$ on $X$. Write $W = \sum_{i=0}^{d} \alpha_i A_i$, where $A_i$ denotes the $i$th adjacency matrix. Set $x_i = \alpha_i^{-1} A_i$ and $p = x_1^{-1} x_2$. Then $x_i = p^{i+1} x_1$ holds for all $i$. Moreover, the eigenvalues and the intersection numbers of $\Gamma$ are rational functions of $x_1$ and $p$.

Key Words: spin model; distance-regular graph; association scheme; Bose-Mesner algebra.

1. INTRODUCTION

A spin model is a square matrix $W$ satisfying certain conditions which ensure that it yields an invariant of knots and links via a statistical mechanical construction of V. F. R. Jones [23]. Since their introduction, spin models have received a great deal of attention. Jones' spin models were
generalized to nonsymmetric spin models [24], and further generalized to four-weight spin models [3].

Recently F. Jaeger [20] used a topological construction to show that every (symmetric) spin model lies in the Bose–Mesner algebra of an association scheme. Immediately after Jaeger announced this result, the second author [30] constructed a Bose–Mesner algebra $N(W)$ containing $W$ for each spin model $W$ which gave a simple method for studying spin models. The algebra $N(W)$ was generalized to nonsymmetric spin models and studied precisely in [22].

Association schemes, then, provide a framework in which to study spin models. In looking for examples of spin models, it has often proved convenient to look at special classes of association schemes, such as those on few points [7], abelian group schemes [1, 2, 5, 8], and on distance-regular graphs. Known examples in the distance-regular graph case include certain strongly regular graphs [18] (see also [14]), Hadamard graphs [26] (see also [19, 21]), 2-homogeneous (almost) bipartite distance-regular graphs [27, 28, 29], and Hamming graphs [4]. (Section 9 contains a list of all examples of spin models, known to the authors, which lie in the Bose–Mesner algebra of a distance-regular graph.) Our purpose here is to unify the examples on distance-regular graphs.

In Sections 2 and 3, we review some background material for association schemes, distance-regular graphs, and spin models. In Section 4, we develop several equations which relate the spin model structure to the distance-regular graph structure. Finally, we apply these equations to prove the following result. This result does not require that $N(W)$ be the Bose–Mesner algebra of a distance-regular graph, only that $W \in \mathcal{A} \subseteq N(W)$, where $\mathcal{A}$ is the Bose–Mesner algebra of a distance-regular graph. See Sections 2 and 3 for terminology and notation.

**Theorem 1.1.** Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d \geq 2$ with intersection numbers $c_i, a_i, b_i$ ($0 \leq i \leq d$). Let $A_0, A_1, \ldots, A_d$ denote the adjacency matrices and let $\mathcal{A}$ denote the Bose–Mesner algebra of $\Gamma$. Let $W$ be a spin model such that $W \in \mathcal{A} \subseteq N(W)$. Then $\mathcal{A}$ is self-dual. Write $W = \sum_{i=0}^{d} t_i A_i$, and set $x_i = t_i t_{i-1}^{-1}$ ($1 \leq i \leq d$), $x = x_1$, and $p = x_1^{-1} x_2$. Then the following (i)–(iii) hold.

(i) $x_i = p^{i-1} x$ ($1 \leq i \leq d$).

(ii) Suppose $a_1 = 0$. Then $a_i = 0$ ($1 \leq i < d$). Moreover, if $p^2 \neq 1$, then either $p^d x = 1$ or $p^{d-1} x^2 = -1$.

(iii) Suppose $x^2 \neq 1$. Then the eigenvalues $\theta_i$ ($0 \leq i \leq d$) (in the standard ordering) and the intersection numbers $c_i$ ($0 < i < d$), $e_d$, $b_0$, and $b_i$ ($0 < i < d$) of $\Gamma$ are given by
\[
\theta_i = \frac{px^2 - 1}{x(p^{d-1}x + 1)(1 - p^{d-1}x)} \\
\times \left( (p^{d+i-1}x^3 + 1) \begin{bmatrix} d-i \\ 1 \end{bmatrix} + p^{d-i}(p^{i-1}x + 1) \begin{bmatrix} i \\ 1 \end{bmatrix} \right),
\]
\[
c_i = \frac{p^{i-1}(x-1)(px^2 - 1)(p^{d-1}x + 1)(p^{d+i-1}x^2 - 1)}{(p^{d-1}x + 1)(p^{d}x^2 - 1)(p^{d+i}x - 1)(p^{d+i}x^2 - 1)} \begin{bmatrix} i \\ 1 \end{bmatrix},
\]
\[
c_d = \frac{p^{d-1}(x^2 - 1)(px^2 - 1)}{(p^{d}x^2 - 1)(p^{d-2}x^2 - 1)} \begin{bmatrix} d \\ 1 \end{bmatrix},
\]
\[
b_0 = \frac{(px^2 - 1)(p^{d-1}x^3 + 1)}{x(p^{d-1}x + 1)(p^{d}x^2 - 1)} \begin{bmatrix} d \\ 1 \end{bmatrix},
\]
\[
b_i = \frac{p'(x-1)(px^2 - 1)(p^{i-1}x^2 - 1)(p^{d+i-1}x + 1)}{x(p^{d-1}x + 1)(p^{d}x^2 - 1)(p^{d+i}x - 1)(p^{d+i}x^2 - 1)} \begin{bmatrix} d-i \\ 1 \end{bmatrix},
\]

where

\[
\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{cases} i \\ p' - 1 \\ p - 1 \end{cases}, \text{ if } p = 1,
\]

Moreover, all denominators are non-zero in these expressions.

The self-duality of \(A\) will be shown in Section 3. We treat the case \(p^2 = 1\) in Section 5, the case \(p^2 \neq 1\) and \(a_1 = 0\) in Section 6, and the case \(p^2 = 1\) and \(a_1 > 0\) in Section 7.

This result contains all of the aforementioned examples of spin models in distance-regular graphs as special cases. We postpone further remarks until Section 8, after the proofs are complete.

2. PRELIMINARIES

For more details concerning the material reviewed in this section, we refer the reader to \([6, 10, 5, 22]\).

Let \(X\) be a nonempty finite set of size \(n\). Let \(M_X\) denote the set of square matrices with complex entries whose rows and columns are indexed by \(X\). For \(A \in M_X\) and \(x, y \in X\), we write \(A(x, y)\) to denote the entry of \(A\) indexed by \(x\) and \(y\). We denote by \(I\) the identity matrix, by \(J\) the all ones matrix, by \(A^T\) the transpose of \(A\), and by \(A \cdot B\) the Hadamard product of \(A\) and \(B\) which is defined by \((A \cdot B)(x, y) = A(x, y)B(x, y)\).
A d-class association scheme on $X$ is a set $\{A_i\}_{i=0}^{d-1}$ of nonzero $(0,1)$-matrices in $M_X$ such that (i) $A_0=I$, (ii) $\sum_{i=0}^{d-1} A_i = J$, (iii) for all $i$ ($0 \leq i \leq d$), there exists $i'$ ($0 \leq i' \leq d$) such that $A_i A_{i'} = A_{i'} A_i$, and (iv) for all $i$, $j$ ($0 \leq i, j \leq d$) there exists a scalar $p_{ij}^0$ such that $A_i A_j = \sum_{i'=0}^{d-1} p_{ij}^0 A_{i'}$. Observe that $A_i \cdot A_j = \delta_{ij} A_j$. Let $\mathcal{A}$ be the C-linear span of $A_0, A_1, ..., A_d$ in $M_X$. Then $\mathcal{A}$ is a commutative algebra with respect to Hadamard multiplication (respectively matrix multiplication) with unity $J$ (respectively $I$), and $\mathcal{A}$ is closed under transposition. The algebra $\mathcal{A}$ is called the Bose-Mesner algebra of the association scheme $\{A_i\}_{i=0}^{d-1}$. Clearly $A_0, A_1, ..., A_d$ form a basis of the primitive idempotents of $\mathcal{A}$ with respect to Hadamard multiplication. Since $\mathcal{A}$ is commutative with respect to matrix multiplication, there exists a unique basis of primitive idempotents $E_0, E_1, ..., E_d$ with respect to matrix multiplication. Since $n^{-J}$ is an idempotent of rank 1 (and hence a primitive idempotent), $n^{-J}$ belongs to $\{E_0, E_1, ..., E_d\}$. We always choose notation so that $E_0 = n^{-J}$. Since $E_i$ is the conjugate of a diagonal $(0,1)$-matrix by a unitary matrix, $E_i$ is hermitian (see [6, Section II.3]), so that $E_i^* = E_i$.

Now we have two C-linear basis $\{A_0, A_1, ..., A_d\}$ and $\{E_0, E_1, ..., E_d\}$ of $\mathcal{A}$. Hence there are square matrices $P$ and $Q$ of size $d+1$, called the eigenmatrices of the association scheme (or of the Bose-Mesner algebra $\mathcal{A}$), such that $A_i = \sum_{j=0}^{d-1} P_{ij} E_j$ and $E_i = n^{-1} \sum_{j=0}^{d-1} Q_{ij} A_j$ ($0 \leq i \leq d$). Thus $P_{ij}$ is the eigenvalue of $A_i$ on the eigenspace $E_j V$, where $V$ denotes the linear space of column vectors whose entries are indexed by $X$. Clearly the eigenmatrices satisfy $PQ = nI$. Observe that the eigenmatrices $P$ and $Q$ depend on the ordering of the primitive idempotents $E_0, E_1, ..., E_d$.

A duality $\Psi$ of the Bose-Mesner algebra $\mathcal{A}$ is a linear isomorphism of $\mathcal{A}$ into itself satisfying $\Psi(AB) = \Psi(A) \cdot \Psi(B)$, $\Psi(A \cdot B) = n^{-1} \Psi(A) \Psi(B)$ and $\Psi(\Psi(A)) = n' A$. Clearly $\Psi$ maps $\{E_0, E_1, ..., E_d\}$ onto $\{A_0, A_1, ..., A_d\}$. Thus we can choose the ordering of the primitive idempotents $E_0, E_1, ..., E_d$ so that $\Psi(E_i) = A_i$ ($0 \leq i \leq d$). We call such an ordering the standard ordering under the duality $\Psi$. In this case, we have $\Psi(A_i) = \Psi(E_i) = n' E_i = n E_i$. In addition, $\Psi(A_i) = \Psi(\sum_{j=0}^{d-1} P_{ij} E_j) = \sum_{j=0}^{d-1} P_{ij} A_j$. Hence $E_i = n^{-1} \sum_{j=0}^{d-1} P_{ij} A_j$. Together with $E_i = n^{-1} \sum_{j=0}^{d-1} Q_{ij} A_j$, this implies $P = Q$. A Bose-Mesner algebra $\mathcal{A}$ is said to be self-dual if there exists a duality of $\mathcal{A}$.

Let $\Gamma = (X, R)$ be a finite, undirected, connected graph of diameter $d$ without loops and multiple edges, and let $\partial$ denote the shortest path distance function on $\Gamma$. For a vertex $u$ in $X$, let $\Gamma_x(u)$ denote the set of vertices $x$ at distance $i$ from $u$. $\Gamma$ is said to be distance-regular if the matrices $A_i$ ($0 \leq i \leq d$) defined by $A_i(x,y) = 1$ if $\partial(x,y) = i$ and 0 otherwise, form an association scheme. Since the graph is undirected, the adjacency matrices satisfy $A_i = A_j$ ($0 \leq i \leq d$), so the corresponding Bose-Mesner algebra $\mathcal{A}$ is symmetric. In particular, the eigenmatrix $P$ has real entries.
The eigenvalues \( \theta_j = P_{ij} \) (0 \( \leq j \leq d \)) of \( A_1 \) are also called the eigenvalues of \( \Gamma \). The dual eigenvalues of \( \Gamma \) are defined by \( \theta_j^* = Q_{ij} \) (0 \( \leq j \leq d \)).

The intersection numbers \( p'_{ij} \) of \( \mathcal{A} \) (which are also called the intersection numbers of \( \Gamma \)) satisfy \( p'_{ij} = 0 \) if \( i < [i-j] \) or \( i > i+j \). Put \( c_i = p'_{1,i-1} \), \( a_i = p'_{1,i} \), \( b_i = p'_{1,i+1} \). Then for two vertices \( u \) and \( x \) with \( \mathcal{A}(u,x) = i \), there are precisely \( c_i \) neighbors of \( x \) in \( \Gamma_{i-1}(u) \), \( a_i \) neighbors in \( \Gamma_i(u) \) and \( b_i \) neighbors in \( \Gamma_{i+1}(u) \). Clearly \( c_0 = a_0 = b_0 = 0 \), \( c_1 = 1 \), \( \Gamma \) is regular of valency \( b_0 \), and \( c_i + a_i + b_i = b_0 \). The equation \( A_i A_j = \sum_{i=0}^d p'_{ij} A_j \) at \( i = 1 \) becomes \( A_i A_j = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \). Using this recurrence relation, it can be easily shown that there exist polynomials \( v_i(x) \) of degree \( i \) such that \( A_i = v_i(A_1) \) (0 \( \leq i \leq d \)). Thus the eigenvalue \( P_{ij} \) of \( A_i \) on \( E_j V \) is given by \( P_{ij} = v_i(\theta_j) \) since \( \theta_j = P_{ij} \) is the eigenvalue of \( A_i \) on \( E_j V \). Hence the eigenvalues \( \theta_0, \theta_1, \ldots, \theta_d \) are mutually distinct. We also have \( \theta_0 = b_0 \) (the valency) since the eigenspace \( E_0 V = n^{-1} J V \) is spanned by the all ones vector.

\( \Gamma \) is said to be Q-polynomial if, for some ordering of the primitive idempotents \( E_0, E_1, \ldots, E_d \), there exist polynomials \( v_i(x) \) of degree \( i \) such that \( Q_{ij} = v_i(\theta_j^*) \) (0 \( \leq i, j \leq d \)). Such an ordering of the primitive idempotents is called a Q-polynomial ordering.

Suppose \( \mathcal{A} \) is self-dual with duality \( \mathcal{P} \) (in this case, we will simply say that \( \Gamma \) is self-dual). Then we have \( P = Q \) and \( \theta_j = \theta_j^* \) (0 \( \leq i, j \leq d \)) with respect to the standard ordering of the primitive idempotents under the duality \( \mathcal{P} \), and hence \( \Gamma \) is Q-polynomial with \( v_i(x) = v_i(x) \). The well-known three-term recurrence becomes (see for instance [10, Lemma 2.2.1 (v)]) \( \theta_i \theta_j = c_i \theta_{i-1} + a_i \theta_j + b_i \theta_{i+1} \) (1 \( \leq i \leq d \)), with an arbitrary value for \( \theta_{d+1} \). Writing \( a_i = \theta_i - b_i - c_i \), this becomes

\[
(\theta_i - \theta_0) \theta_j = c_i (\theta_{i-1} - \theta_0) + b_i (\theta_{i+1} - \theta_0) \quad (1 \leq i \leq d). \tag{1}
\]

3. THE ALGEBRA \( N(W) \)

In this section, we briefly review some material from [22] concerning the relationship between spin models and association schemes.

Let \( X \) be a finite nonempty set of size \( n \). A spin model is a matrix \( W \) in \( M_X \) with nonzero entries which satisfies the following equations for all \( a, b, c \in X \):

\[
\sum_{x \in X} W(x, b) W(x, c)^{-1} = n \delta_{b, c}, \tag{2}
\]

\[
\sum_{x \in X} W(x, a) W(x, b) W(x, c)^{-1} = \sqrt{n} W(a, b) W(a, c)^{-1} W(c, b)^{-1}. \tag{3}
\]
Setting $b = c$ in the second equation shows that diagonal entries of $W$ are equal to the same number $x$, which is called the \textit{modulus} of $W$.

In the literature, (2) is often referred to as the \textit{type II condition}, and (3) is referred to as the \textit{type III condition} or the \textit{star-triangle relation}. The entries of $W$ are called the \textit{(Boltzmann)} weights.

For all $b, c \in X$, define $Y_{bc}$ to be the column vector with $x$ entry

$$Y_{bc}(x) = W(x, b) W(x, c)^{-1} \quad (x \in X).$$

Then $N(W)$ is defined to be the set of all matrices $A$ in $M_X$ such that, for all $b, c \in X$, the vector $Y_{bc}$ is an eigenvector of $A$. For $A \in N(W)$, let $\mathcal{P}(A) \in M_X$ be defined by $A Y_{bc} = \mathcal{P}(A)(b, c) Y_{bc}$ for all $(b, c) \in X \times X$. This defines a mapping $\mathcal{P}: N(W) \to M_X$. The following result appears as Theorem 11 in [22].

\begin{theorem} [Jaeger–Matsumoto–Nomura [22]] \end{theorem}

Let $W$ be a spin model. Then

(i) $N(W)$ is the Bose–Mesner algebra of some association scheme;
(ii) $W \in N(W)$;
(iii) $N(W)$ is self-dual with the duality $\mathcal{P}$;
(iv) $\mathcal{P}(A) = x^{-1} W \cdot (W^{-1} W \cdot A)$ for all $A \in N(W)$, where $x$ denotes the modulus of $W$ and $W^{-1}$ is defined by $W^{-1}(x, y) = W(y, x)^{-1}$ ($x, y \in X$).

Let $W$ be a spin model, and let $\mathcal{A}$ be a Bose–Mesner algebra of an association scheme $\{A_i\}_{i=0}^d$. Henceforth, we will consider the case $W \in \mathcal{A} \subseteq N(W)$. We note that all examples in Section 9 satisfy this condition and that $\mathcal{A} \neq N(W)$ for Examples 6 and 7.

\begin{lemma} \end{lemma}

Suppose $W \in \mathcal{A} \subseteq N(W)$. Then $\mathcal{P}(\mathcal{A}) = \mathcal{A}$. In particular, $\mathcal{A}$ is self-dual with duality $\mathcal{P}|_{\mathcal{A}}$.

\textbf{Proof.} Since $W \in \mathcal{A}$, $W$ is a linear combination of the adjacency matrices $A_i$, i.e., $W = \sum_{i=0}^d t_i A_i$. Thus $W^{-1} = \sum_{i=0}^d t_i^{-1} A_i$. In particular, $W^{-1} \in \mathcal{A}$. For $A \in \mathcal{A}$, Theorem 3.1(iv) implies that $\mathcal{P}(A) \in \mathcal{A}$ since $\mathcal{A}$ is closed under transposition, Hadamard product, and matrix product. This shows that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{A}$, and this implies that $\mathcal{P}(\mathcal{A}) = \mathcal{A}$ since $\mathcal{P}$ is a linear bijection.

For any Bose–Mesner algebra $\mathcal{A}$ such that $W \in \mathcal{A} \subseteq N(W)$, we will always take the standard ordering of the primitive idempotents $E_0, E_1, ..., E_d$ of $\mathcal{A}$ under the duality $\mathcal{P}|_{\mathcal{A}}$. 
Lemma 3.3. Suppose \( W \in \mathcal{A} \subseteq N(W) \). Let \( P \) be the eigenmatrix of \( \mathcal{A} \). Then \( A_i Y_{bc} = P_{hi} Y_{bc} \) for all \( h, i \) (0 \( \leq h, i \leq d \)) and for all \( (b, c) \) such that \( A_i(b, c) \neq 0 \).

Proof. By the definition of \( \Psi \), \( A_i Y_{bc} = \Psi(A_i)(b, c) Y_{bc} \). We have \( \Psi(A_i) = \sum_{j=0}^{d} \frac{1}{\mu_j} A_j \). Hence \( A_i Y_{bc} = \sum_{j=0}^{d} P_{hi} A_j(b, c) Y_{bc} = P_{hi} Y_{bc} \).

4. Equations

Let \( \Gamma = (X, R) \) be a distance-regular graph of diameter \( d \geq 2 \), let \( \mathcal{A} \) be the Bose–Mesner algebra of \( \Gamma \), and let \( A_0, A_1, \ldots, A_d \) denote the adjacency matrices. Let \( W \) be a spin model such that \( W \in \mathcal{A} \subseteq N(W) \). In this section we present some equations relating the weights, the intersection numbers, and the eigenvalues.

Since \( W \in \mathcal{A} \), there are nonzero complex numbers \( t_0, t_1, \ldots, t_d \) such that \( W = \sum_{i=0}^{d} t_i A_i \). We set \( x_i = t_i t_{i-1}^{-1} \) (1 \( \leq i \leq d \)).

By Lemma 3.2, \( \mathcal{A} \) is self-dual with duality \( \Psi|_{\mathcal{A}} \). Let \( \theta_0, \theta_1, \ldots, \theta_d \) be the eigenvalues in the standard ordering under the duality \( \Psi|_{\mathcal{A}} \). To simplify arguments, we set \( x_0 = 1 \) and \( \theta_i = x_i = 1 \) for \( i < 0 \) and for \( i > d \).

Lemma 4.1. Fix \( u, v \in X \), and set \( h = \partial(u, v) \). Then
\[
A_1 Y_{uv} = \theta_h Y_{uv}.
\] (4)

Proof. Clear from Lemma 3.3.

For all vertices \( u \) and \( v \), and for all integers \( i \) and \( j \), we write
\[
D^j(u, v) = \Gamma_i(u) \cap \Gamma_j(v).
\]
For any vertex \( x \) and for any subset \( Z \subseteq X \), we write
\[
e(x, Z) = |\Gamma_i(x) \cap Z|.
\]
Observe that \( e(x, Z) \) is precisely the number of edges from \( x \) into \( Z \).

Lemma 4.2. Fix \( u, v \in X \), set \( h = \partial(u, v) \), and write \( D^j = D^j(u, v) \). Then for all \( r, s \) (0 \( \leq r, s \leq d \)) and for all \( w \in D^r \),
Proof. We compute the \textit{w}-entry of each side of (4). On one hand,

\[
(A_1 Y_w)(w) = \sum_{x \in X} \sum_{s \in F_1(w)} Y_w(x) = \sum_{x \in F_1(w)} A_1(w, x) Y_w(x) = \sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{x \in D_i \cap F_1(w)} \frac{W(x, v)}{W(x, u)} \theta_i t_j^{-1} = \sum_{i=0}^{d} \sum_{j=0}^{d} e(w, D_i) t_j^{-1} = \sum_{i=0}^{d} \sum_{j=0}^{d} e(w, D_i) t_j^{-1} = \frac{d}{i} t_j^{-1}.
\]

On the other hand, the \textit{w}-entry of the right side of (4) is \(\theta_h t_0 t_j^{-1}\). Hence (5) holds. Equation (6) is proved similarly using \(Y_vu\) in place of \(Y_w\).

Let \(c_r, a_r, b_r\) \((0 \leq r \leq d)\) denote the usual intersection numbers of \(\Gamma\).

**Lemma 4.3.** For all \(r \ (1 \leq r \leq d)\),

\[
x_i \theta_r = c_r x_i^{-1} + a_r + b_r x_{i+1},
\]

(7)

\[
x_i \theta_r = c_r x_i + a_r + b_r x_{i+1}.
\]

(8)

**Proof.** We apply Lemma 4.2 with \(w = v\). In this case (5) implies that

\[
\theta_h t_0 t_j^{-1} = e(v, D_h^{-1}) t_h^{-1} + e(v, D_h) t_h^{-1} + e(v, D_h + 1) t_h + 1 t_j^{-1} = c_h t_h^{-1} + a_h t_h + b_h t_h + 1 t_j^{-1}.
\]

Multiplying both sides by \(t_h t_j^{-1}\), this becomes \(\theta_h t_h t_j^{-1} = c_h t_h^{-1} + a_h + b_h t_h + 1 t_j^{-1}\), so (7) holds. Equation (8) is proved similarly using (6) in place of (5).

**Corollary 4.4.** For all \(r \ (1 \leq r \leq d)\),

\[
x_i \theta_r - \theta_0 = c_r (x_i^{-1} - 1) + b_r (x_{i+1} - 1),
\]

(9)

\[
x_i \theta_r - \theta_0 = c_r (x_i - 1) + b_r (x_{i+1} - 1).
\]

(10)

**Proof.** Write \(a_r = \theta_0 - c_r - b_r\) in (7) and (8).
Lemma 4.5. For all \( r \) \((1 \leq r \leq d)\),

\[
\frac{x_r - x_r^{-1}}{\theta_{r-1} - \theta_r} = \frac{x_1 - x_1^{-1}}{\theta_0 - \theta_1}. \tag{11}
\]

Proof. We show that (11) holds for all \( r \) \((1 \leq r \leq d)\) by induction. Clearly it holds for \( r = 1 \), so assume \( 1 \leq r < d \) and suppose (11) holds for all \( j \leq r \).

First observe that by (7) and (8)

\[
c_r(x_r - x_r^{-1}) - b_r(x_{r+1} - x_{r+1}^{-1}) = -\theta_r(x_1 - x_1^{-1}), \tag{12}
\]

and that by (1)

\[
c_r(\theta_{r-1} - \theta_r) - b_r(\theta_r - \theta_{r+1}) = -\theta_r(\theta_0 - \theta_1). \tag{13}
\]

Now subtracting \((\theta_{r-1} - \theta_r)\) times (12) from \((x_r - x_r^{-1})\) times (13),

\[
b_r((x_{r+1} - x_{r+1}^{-1})(\theta_{r-1} - \theta_r) - (x_r - x_r^{-1})(\theta_r - \theta_{r+1}))
= \theta_r((x_1 - x_1^{-1})(\theta_{r-1} - \theta_r) - (\theta_0 - \theta_1)(x_r - x_r^{-1})).
\]

Observe that the right side is 0 by induction. This implies

\[
\frac{x_{r+1} - x_{r+1}^{-1}}{\theta_{r-1} - \theta_{r+1}} = \frac{x_r - x_r^{-1}}{\theta_{r-1} - \theta_r},
\]

since \( b_r \neq 0 \), so we obtain (11).

Corollary 4.6. Suppose \( x_i^2 = 1 \) for some \( i \) \((1 \leq i \leq d)\). Then \( x_r^2 = 1 \) for all \( r \) \((1 \leq r \leq d)\).

Proof. Clear from Lemma 4.5.

Corollary 4.7. Suppose \( x_i^2 \neq 1 \). Then \( x_r x_{r+1} \neq 1 \) for all \( r \) \((1 \leq r < d)\).

Proof. From Lemma 4.5,

\[
\frac{x_{r+1} - x_{r+1}^{-1}}{\theta_{r-1} - \theta_{r+1}} = \frac{x_r - x_r^{-1}}{\theta_{r-1} - \theta_r}.
\]

If \( x_{r+1} = x_r^{-1} \), then this implies \((x_r - x_r^{-1})(\theta_{r-1} - \theta_{r+1}) = 0\). Thus \( x_r - x_r^{-1} = 0 \) since the eigenvalues are distinct, so that \( x_r^2 = 1 \), contradicting Corollary 4.6.
LEMMA 4.8. Fix $r$ $(1 \leq r \leq d)$, and let $x, y, z$ be vertices such that
$
\delta(x, y) = r - 1, \quad \delta(x, z) = r, \quad \delta(y, z) = 1. \quad Set \quad \gamma = |\Gamma_{r-1}(x) \cap \Gamma_{r}(y) \cap \Gamma_{r}(z)|.
$
Then
$
\gamma(x_2 - 1)(x_1 - 1) = x_1^2 \theta_{r-1} - x_2 \theta_r + x_r(x_1 x_2 - 1) + a_1 x_1(x_2 - 1),
$
(14)
$
\gamma x_1(x_2 - 1)(x_1 - 1) = x_2 x_1 \theta_{r-1} - x_1 x_2 \theta_r + 1 - x_1 x_2 + a_1 x_1(1-x_2),
$
(15)
$
\gamma x_1(x_2 - 1)(x_1 - 1) = x_2^2 \theta_r - x_1 \theta_{r-1} + x_r(1-x_1 x_2) + a_1 x_1(1-x_2),
$
(16)
$
\gamma(x_2 - 1)(x_1 - 1) = x_1^2 \theta_r - x_2 x_1 \theta_{r-1} + x_1 x_2 - 1 + a_1 x_1(x_2 - 1),
$
(17)

Proof. To prove (14) and (15), we apply Lemma 4.2 with $u = x, v = z, w = y$, and $h = r$. Set $D'_j = D'(x, z)$, and observe that $y \in D'_{r-1}$. By elementary counting arguments, we obtain $e(y, D'_{r}) = 1, e(y, D'_{r-1}) = e(y, \Gamma_{r-1}(x)) = c_{r-1}, e(y, D'_{r}) = e(y, \Gamma_{r}(x)) = e(y, \Gamma_{r-1}(x)) = e(y, D'_{r-1}) = a_1 - \gamma, e(y, D'_{r-1}) = e(y, \Gamma_{r-1}(x)) - e(y, D'_{r-1}) = a_{r-1} - \gamma$, and $e(y, D'_1) = e(y, \Gamma_{r}(x)) - e(y, D'_1) = b_{r-1} - 1 - (a_1 - \gamma)$. Therefore (5) implies that

$$
\theta_r t_{r-1} t_1^{-1} = c_{r-1} t_{r-2} t_2^{-1} + (a_{r-1} - \gamma) t_{r-1} t_2^{-1} + (b_{r-1} - 1 - a_1 + \gamma) t_r t_2^{-1} + (a_1 - \gamma) t_r t_1^{-1} + \gamma t_{r-1} t_1^{-1} + t_1 t_0^{-1}.
$$

Multiplying both sides by $t_2 t_1^{-1}$, this becomes

$$
x_2 \theta_r = -\gamma(x_2 - 1)(x_1 - 1) + (c_{r-1} x_2^{-1} + a_{r-1} + b_{r-1} x_r) + x_r(x_1 x_2 - 1) + a_1 x_1(x_2 - 1).
$$

By (7) $c_{r-1} x_2^{-1} + a_{r-1} + b_{r-1} x_r = x_1 \theta_{r-1}$. This substitution yields (14).

Equation (15) is proved similarly using (6) in place of (5).

To prove (16) and (17), we apply Lemma 4.2 with $u = x, v = y, w = z, h = r - 1$. Set $D'_j = D'(x, y)$, and observe that $z \in D'_{r-1}$. By elementary counting arguments, we obtain $e(z, D'_{r-1}) = 1, e(z, D'_{r+1}) = e(z, \Gamma_{r+1}(x)) = b_r,$
\( e(z, D_1^{r-1}) = \gamma \), \( e(z, D_1^y) = e(z, \Gamma_1(y)) - e(z, D_1^{r-1}) = a_1 - \gamma \), \( e(z, D_0^y) = e(z, \Gamma_1(x)) - e(z, D_1^y) = a_r - (a_1 - \gamma) \), and \( e(z, D_0^{r-1}) = e(z, \Gamma_{r-1}(x)) - e(z, D_0^{r-1}) = \epsilon_r - 1 - \gamma \). Therefore (5) implies that

\[
\theta_{r-1} t_i^{-1} = b_r x_{r+1} + a_r + c_r x_i^{-1} + (a_r - a_1 + \gamma) t_i x_i^{-1} + (c_r - 1 - \gamma) t_{r-1} x_i^{-1} + \gamma t_{r-1} x_i^{-1} + t_{r-1} t_0^{-1}.
\]

Multiplying both sides by \( t_2 t_r^{-1} \), this becomes

\[
x_2 \theta_{r-1} = (b_r x_{r+1} + a_r + c_r x_i^{-1} + (1 - x_i^{-1})(1 - x_2) + a_1(x_2 - 1) + x_i^{-1}(x_1 x_2 - 1).
\]

By (7) \( b_r x_{r+1} + a_r + c_r x_i^{-1} = x_1 \theta_r \). Making this substitution and multiplying through by \( x_r \), yields (17). Equation (16) is proved similarly using (6) in place of (5).

Now (18) is obtained by adding (14) and (16), and (19) is obtained by subtracting (15) from (16).

Remark 4.9. In Lemma 4.8, it is obvious that \( \gamma = 0 \) when \( r = 1 \). It is also clear that \( \gamma = 0 \) if \( a_1 = 0 \), since \( I_1(y) \cap I_1(z) = \emptyset \) in this case.

5. THE CASE \( p^2 = 1 \)

In this section we consider the case \( p^2 = 1 \). We continue with the notation of Section 4. We set \( x = x_1 \) and \( p = x_1^{1} x_2 \).

First we consider the case \( p = 1 \).

Lemma 5.1. Suppose \( p = 1 \). Then \( x_r = x \ (1 \leq r \leq d) \). Moreover, if \( x^2 = 1 \), then \( x_r = -1 \ (1 \leq r \leq d) \).

Proof. Observe that (19) becomes

\[
x(x_r + 1)(\theta_r - \theta_{r-1}) = (x^2 - 1)(x_r - 1) \quad (1 \leq r \leq d).
\]

Thus if \( x^2 = 1 \), then \( x_r = -1 \ (1 \leq r \leq d) \).

We now consider the case \( x^2 \neq 1 \). Observe that \( x_r^2 \neq 1 \) for all \( r \ (1 \leq r \leq d) \) by Corollary 4.6. Thus the above equation implies

\[
(\theta_r - \theta_{r-1}) = \frac{(x^2 - 1)(x_r - 1)}{x(x_r + 1)}.
\]
Substituting this expression into (11),
\[
\frac{x(x_r + 1)(x_r - x_r^{-1})}{(x^2 - 1)(x_r - 1)} = \frac{x(x + 1)(x - x^{-1})}{(x^2 - 1)(x - 1)},
\]
and this implies \(x + x_r^{-1} = x + x^{-1}\). Hence \(x_r = x\) or \(x_r = x^{-1}\) (\(1 \leq r \leq d\)). If \(x_r = x^{-1}\) for some \(i\), then there is some \(j\) such that \(x_{j-1} = x\) and \(x_j = x^{-1}\) since \(x_1 = x\). This contradicts Corollary 4.7. Thus \(x_r = x\) (\(1 \leq r \leq d\)).

**Lemma 5.2.** Suppose \(p = 1\) and \(x \neq -1\). Then \(c_i = i\), \(a_i = i(q - 2)\), \(b_i = (d - i)(q - 1)\), and \(\theta_i = (d - i) - d\) \((0 \leq i \leq d)\), where \(q = -x^{-1}(x - 1)^2\).

**Proof.** We have \(x_r = x\) \((1 \leq i \leq d)\) by Lemma 5.1. Subtracting (8) from (7) and dividing by the factor \(x - x^{-1}\), we find that
\[
\theta_i = b_i - c_i, \quad (0 \leq i \leq d).
\]
By (11),
\[
\theta_i - \theta_{i-1} = \theta_1 - \theta_0 \quad (1 \leq i \leq d). \tag{21}
\]
Now set \(q = b_0 - b_1 + 1\) and \(D = b_0/(b_0 - b_1)\), where we observe that \(b_0 > b_1\). Then \(b_0 = \theta_0 = D(q - 1)\) and \(b_1 = (D - 1)(q - 1)\). Using (20), \(\theta_1 = b_1 - 1 = (D - 1)(q - 1) - 1 = q(D - 1) - D\), and \(\theta_1 - \theta_0 = q(D - 1) - D\). Now (21) implies \(\theta_i = \theta_1 - \theta_0 = \theta_{i-1} - q\), so that \(\theta_i = \theta_0 - iq\) \((0 \leq i \leq d)\) by induction. Thus \(\theta_i = D(q - 1) - iq = q(D - i) - D\). From (19) at \(r = 1\), we have \((\theta_1 - \theta_0)(x + x^2) = (x^2 - 1)(x - 1)\), and this implies \(x^{-1}(x - 1)^2 = \theta_1 - \theta_0 = -q\).

We now compute the intersection numbers. Eliminating \(c_i\) from (20) and (9) at \(r = i\), \(x\theta_i - \theta_0 = b_i(x - 1) + (\theta_1 - \theta_0)(x - x^{-1})\). Thus \(b_i(x - 1)^2 x^{-1} = (x^2 - x + 1) x^{-1}\theta_i - \theta_0 = ((x - 1)^2 x^{-1} + 1) \theta_i - \theta_0\), and this implies \(b_iq = (q - 1)(\theta_i + \theta_0 = (q - 1)(q(D - i) - D) + D(q - 1) = q(q - 1)(D - i)\). Thus \(b_i = (D - i)(q - 1)\). Observe that \(b_i = 0\), so \(D = d\). Now using this value in (20), we find that \(c_i = b_i - \theta_i = (d - i)(q - 1) - q(d - i) + d = i\).

Next we consider the case \(p = -1\).

**Lemma 5.3.** Suppose \(p = -1\). Then \(x^2 = 1\), and \(x_i = (-1)^{i-1} x (1 \leq i \leq d)\).

**Proof.** Equation (11) at \(r = 2\) with \(x_2 = px = -x\) implies \((\theta_0 - \theta_2)\) \((x - x^{-1}) = 0\). Hence \(x - x^{-1} = 0\), so that \(x^2 = 1\). Then \(x_r^2 = 1\) for all \(r \leq i \leq d\) by Corollary 4.6.

First suppose that \(x = 1\). Observe that if \(x_r = x_{r+1} = 1\) for some \(i (> 0)\), then (7) implies \(\theta_i = c_i + a_i + b_i = \theta_0\), a contradiction. Also, (19) implies that \((\theta_{i-1} + \theta_r + 2)(x_r - 1) = 0\). Thus if \(x_r = x_{r+1} = -1\) for some \(i\), then
\[ \theta_{i-1} + \theta_i + 2 = 0 \] and \[ \theta_i + \theta_{i+1} + 2 = 0 \], contradicting the fact that \( \theta_{i-1} \neq \theta_{i+1} \). Thus \( x_i \neq x_{i+1} \) for all \( i \) such that \( 1 \leq i \leq d \), and hence \( x_i = (-1)^{i-1} \) for all \( i \) such that \( 1 \leq i \leq d \).

Next suppose that \( x = -1 \). Then (14) with \( x_2 = 1 \) implies \( \theta_{i-1} + \theta_i = -2x_i \). Thus, if \( x_i = x_{i+1} \), then \( \theta_{i-1} = \theta_{i+1} \), a contradiction. Hence \( x_i \neq x_{i+1} \) for all \( i \) such that \( 1 \leq i \leq d \), and so \( x_i = (-1)^i \) for all \( i \) such that \( 1 \leq i \leq d \).

**Lemma 5.4.** Suppose \( p = -1 \) and \( a_1 = 0 \). Then \( c_i = i \) for all \( i \) such that \( 0 < i \leq d \) and \( b_i = \theta_0 - i \) for all \( i \) such that \( 0 \leq i < d \). Hence \( a_i = 0 \) for all \( i \) such that \( 0 \leq i < d \).

**Proof.** We have \( x \in \{1, -1\} \) and \( x_r = (-1)^{r-1} x \) for all \( r \) such that \( 1 \leq r \leq d \) by Lemma 5.3. Observe that \( x = 0 \) in Lemma 4.8 by our assumption \( a_1 = 0 \) (see Remark 4.9). Hence (14) implies \( \theta_{r-1} + \theta_r = (-1)^{r-1} \) for all \( r \) such that \( 1 \leq r \leq d \), so that by induction

\[ \theta_r = (-1)^r (\theta_0 - 2r) \quad (1 \leq r \leq d). \]

On the other hand, from (9)

\[ x\theta_r = \theta_0 + c_r((-1)^{r-1} x - 1) + b_r((-1)^r x - 1), \]

and this implies (noting \( x \in \{1, -1\} \))

\[ \theta_r = \begin{cases} (-1)^{r-1} (\theta_0 - 2b_r) & \text{if } x = (-1)^{r-1}, \\ (-1)^{r} (\theta_0 - 2c_r) & \text{if } x = (-1)^r. \end{cases} \]

Comparing (22) and (23),

\[ \begin{align*} b_r &= \theta_0 - r & \text{if } x = (-1)^{r-1}, \\
    c_r &= r & \text{if } x = (-1)^r. \end{align*} \]

Now fix \( i \) such that \( 0 < i < d \). First suppose \( x = (-1)^{i-1} \). Using (22) and (24), we obtain \( b_i = \theta_0 - i \), \( \theta_1 = \theta_0 + 2 \), \( \theta_2 = (-1)^i (\theta_0 - 2i) \), \( \theta_{r-1} = (-1)^{r-1} (\theta_0 - 2(r-1) + 2) \), \( \theta_r = (-1)^{r+1} (\theta_0 - 2(r-1)) \). Hence (1) implies

\[ (-2\theta_0 + 2) \cdot (\theta_0 - 2i) = c_i \cdot (-1)^{i-1} (2\theta_0 - 4i + 2) \]

\[ + (\theta_0 - i) \cdot (-1)^{i+1} (2\theta_0 - 4i - 2), \]

and this becomes \( (c_i - i)(\theta_0 - 2i + 1) = 0 \). If \( \theta_0 - 2i + 1 = 0 \), then \( \theta_* = (-1)^i \)

\[ ((2i - 1) - 2i) = (-1)^{i-1} \]

and \( \theta_{r-1} = (-1)^{r-1} ((2i - 1) - 2(i-1)) = (-1)^{r-1} \), contradicting \( \theta_* \neq \theta_{r-1} \). Hence we must have \( c_i = i \).

Next suppose \( x = (-1)^i \). In the same way as above, we obtain

\[ (-2\theta_0 + 2) \cdot (\theta_0 - 2i) = i \cdot (-1)^{i-1} (2\theta_0 - 4i + 2) \]

\[ + b_i \cdot (-1)^{i+1} (2\theta_0 - 4i - 2), \]
and this becomes \((b_i - \theta_0 + i)(\theta_0 - 2i - 1) = 0\). If \(\theta_0 - 2i - 1 = 0\), then \(\theta_i = (-1)^i ((2i + 1) - 2i) = (-1)^i\) and \(\theta_{i+1} = (-1)^{i+1} ((2i + 1) - 2(i + 1)) = (-1)^i\), contradicting \(\theta_i \neq \theta_{i+1}\). Hence we must have \(b_i = \theta_0 - i\), as desired.

Proof of Theorem 1.1 (Case \(p^2 = 1\)). (i) When \(p = 1\), we have \(x_i = x\) by Lemma 5.1. When \(p = -1\), we have \(x_i = (-1)^{i-1} x\) by Lemma 5.3. (ii) Suppose \(a_1 = 0\). If \(p = -1\), then \(a_r = 0\) \((0 \leq i < d)\) by Lemma 5.4. Now suppose \(p = 1\). If \(x^2 = 1\), then \(x_i = -1\) for all \(i\) by Lemma 5.1, and so (9) at \(r = 1\) becomes \(-\theta_1 - \theta_0 = -2 + b_1(-2) = -2 - 2(\theta_0 - a_1 - 1)\), so that \(\theta_1 - \theta_0 = 2a_1 = 0\), a contradiction. Hence \(x^2 \neq 1\). Then from Lemma 5.2 we have \(a_i = i(q - 2)\) for all \(i\), so that \(a_r = 0\) by our assumption \(a_1 = 0\). (iii) Suppose \(x^2 \neq 1\). Then \(p \neq -1\) by Lemma 5.3, so that \(p = 1\). Then the eigenvalues and the intersection numbers are given by Lemma 5.2. As easily checked, these coincide with the values (at \(p = 1\)) of the formulas given in Theorem 1.1.

6. THE CASE \(p^2 \neq 1\) AND \(a_1 = 0\)

In this section we consider the case \(p^2 \neq 1\) and \(a_1 = 0\). Observe that \(x^2 \neq 1\) \((1 \leq r < d)\) and \(x_r \neq x_{r+1}\) \((1 \leq r < d)\) in the case \(p^2 \neq 1\) by Corollaries 4.6 and 4.7.

**Lemma 6.1.** Suppose \(a_1 = 0\). Then for all \(i\) \((1 \leq i \leq d)\)

\[-\theta_{i-1} + p\theta_i = x^{-1}x_i(px^2 - 1),\]  
(25)

\[p\theta_{i-1} - \theta_i = x^{-1}x_i^{-1}(px^2 - 1).\]  
(26)

**Proof.** Observe that \(\gamma = 0\) since \(a_1 = 0\) (see Remark 4.9), so (14) and (15) implies (25) and (26) respectively.

**Lemma 6.2.** Suppose \(p^2 \neq 1\) and \(a_1 = 0\). Then

\[x_i = p^{i-1}x\]  
(27)

\[\theta_i = \frac{(px^2 - 1)(p^{2i-1}x^2 + 1)}{p^{i-1}x^2}\]  
(28)
Proof. Obviously (27) holds at \( i = 0 \) and 1. From (25) and (26) at \( r = 1 \), we see that
\[
\theta_0 = \frac{(px^2 - 1)(x^2 + p)}{x^2(p^2 - 1)},
\]
\[
\theta_1 = \frac{(px^2 - 1)(px^2 + 1)}{x^2(p^2 - 1)}.
\]

Thus (28) holds at \( i = 0 \) and 1.

Now we show (27) and (28) for \( i \geq 2 \) by induction. From (25),
\[
p\theta_i = \theta_{i-1} + x^{-1}x_i(p x^2 - 1)
= \frac{(px^2 - 1)(p^{2i-1}x^2 + 1)}{p^{i-1}x^2(p^2 - 1)} + x^{-1}x_i(p x^2 - 1),
\]
and hence
\[
\theta_i = \frac{(px^2 - 1)(p^{2i-3}x^2 + 1 + x_i x_{p^{i-2}}(p^2 - 1))}{p^{i-1}x^2(p^2 - 1)},
\]
(29)

Now
\[
p\theta_{i-1} - \theta_i = \frac{p(px^2 - 1)(p^{2i-3}x^2 + 1)}{p^{i-1}x^2(p^2 - 1)} - \frac{(px^2 - 1)(p^{2i-3}x^2 + 1 + x_i x_{p^{i-2}}(p^2 - 1))}{p^{i-1}x^2(p^2 - 1)}
= \frac{(px^2 - 1)(p^{2i-3}x^2 + 1 - p^{i-2}xx_i)}{p^{i-1}x^2}.
\]
Comparing this equation with (26), and observing that \( px^2 = x_i x_{i+1} \neq 1 \), we obtain \( (p^{i-2}xx_i - 1)(x_i - p^{i-1}x) = 0 \). If \( p^{i-2}xx_i - 1 = 0 \), then \( x_{i+1} = x_i \), a contradiction. Hence \( x_i = p^{i-1}x \). Now (29) implies (28).

Lemma 6.3. Suppose \( p^2 \neq 1 \). Then
\[
c_i = \frac{(x + x^{-1}x_{i+1}) \theta_i - (x_{i+1} + 1) \theta_0}{(x_i - 1)(x_{i+1} - x_i^{-1})} \quad (1 \leq i < d),
\]
(30)
\[
c_d = \frac{x^{-1}\theta_d - \theta_0}{x_d - 1},
\]
(31)
\[
b_i = \frac{(x^{-1} + xx_i) \theta_i - (x_i + 1) \theta_0}{(x_{i+1} - 1)(x_i - x_{i+1}^{-1})} \quad (1 \leq i < d).
\]
(32)
Proof. Equation (30) is obtained by adding (9) and $x_{r+1}$ times (10), Eq. (31) is obtained from (10) at $r=d$, and Eq. (32) is obtained by adding (10) and $x_r$ times (9).

**Lemma 6.4.** Suppose $p^2 \neq 1$ and $a_1 = 0$. Then $a_i = 0$ $(1 \leq i < d)$, and

\[
c_i = \frac{(p^{2i}-1)(px^2-1)}{(p^2-1)(p^{2i-1}x^2-1)} \quad (1 \leq i < d),
\]

\[
c_d = \frac{(px^2-1)[1 + p^{2d-1}x^2 - p^{d-1}x^3 - p^d x]}{p^{d-1}x^2(p^2-1)(p^{d-1}x-1)},
\]

\[
b_i = \frac{p(px^2-1)[p^{2i-2}x^4 - 1]}{x^2(p^2-1)(p^{2i-1}x^2-1)} \quad (1 \leq i < d).
\]

Proof. Equations (33), (34), and (35) are obtained from (30), (31), and (32), respectively, using (27) and (28). Observe that for all $i$ $(0 \leq i < d)$

\[
b_i + c_i = \frac{p(px^2-1)[p^{2i-2}x^4 - 1]}{x^2(p^2-1)(p^{2i-1}x^2-1)} + \frac{(p^2-1)(px^2-1)}{(p^2-1)(p^{2i-1}x^2-1)}
\]

\[
= \frac{(px^2-1)(x^2+p)}{x^2(p^2-1)}.
\]

Thus $b_i + c_i = \theta_0$ by (28), so that $a_i = 0$. \]

**Lemma 6.5.** Suppose $p^2 \neq 1$ and $a_1 = 0$. Then either $p^{d-1}x^2 + 1 = 0$ or $p^d x - 1 = 0$.

Proof. Equations (9) and (10) at $r = d$ become, respectively, $c_d(x_d^{-1} - 1) = x_d \theta_d - \theta_0$ and $c_d(x_d - 1) = x^{-1} \theta_d - \theta_0$. Eliminating $c_d$ from these two equations,

\[(xx_d + x^{-1}) \theta_d - (x_d + 1) \theta_0 = 0.
\]

Using (27) and (28), this implies

\[(p^d x - 1)(p^{d-1}x^2 + 1)(p^{d-1}x^2 - 1) = 0.
\]

If $d$ is even, say $d = 2\ell$, then by (27), $p^{d-1}x^2 = p^{2\ell-1}x^2 = (p^\ell x)(p^\ell x) = x_{\ell+1}x_{\ell} \neq 1$, so $p^{d-1}x^2 - 1 \neq 0$. If $d$ is odd, say $d = 2\ell + 1$, then $p^{d-1}x^2 = p^{2\ell}x^2 = (p^\ell x)^2 = x_{\ell+1} \neq 1$, so $p^{d-1}x^2 - 1 \neq 0$. Thus either $p^\ell x - 1 = 0$ or $p^{\ell-1}x^2 - 1 = 0$ holds. \]
Lemma 6.6. Suppose $p^2 \neq 1$ and $a_1 = 0$.

(i) If $p^{d-1}x^2 + 1 = 0$, then

\[
\theta_i = \frac{(p^d + p^2)(p^d - p^{2i})}{p^{2i}(p^d - 1)} \quad (0 \leq i \leq d),
\]
\[
c_i = \frac{(p^d + p^2)(p^{2i} - 1)}{(p^d - 1)(p^{2i} + p^d)} \quad (1 \leq i \leq d - 1),
\]
\[
c_d = \frac{(p^d + p^2)(p^d - 1)}{p^{2d}(p^d - 1)},
\]
\[
b_0 = \frac{(p^d + p^2)(p^d - 1)}{p^d(p^d - 1)},
\]
\[
b_i = \frac{(p^d + p^2)(p^{2d} - p^{2i})}{p^{2i}(p^2 - 1)(p^d + p^2)} \quad (1 \leq i \leq d - 1).
\]

(ii) If $p^d x - 1 = 0$, then

\[
\theta_i = \frac{(1 - p^{2d-1})(p^{2d-2i} - 1 + 1)}{p^{2i}(p^d - 1)} \quad (0 \leq i \leq d),
\]
\[
c_i = \frac{(p^{2d-1} - 1)(p^{2i} - 1)}{p^{2d-2i}(p^d - 1)(p^{2d-2i} - 1)} \quad (1 \leq i \leq d - 1),
\]
\[
c_d = \frac{(p^{2d-1} - 1)(p^{2d} - 1)}{p^{2d-2i}(p^d - 1)(p - 1)},
\]
\[
b_0 = \frac{p^2(p^{2d-1} + 1)(1 - p^{2d-1})}{(p^d - 1)},
\]
\[
b_i = \frac{p^2(p^{2i-2d} - 1)(1 - p^{2d-1})}{(p^d - 1)(p^{2i-2d} - 1)} \quad (1 \leq i \leq d - 1).
\]

Proof. In the formulas given in Lemmas 6.2 and 6.4, replace $x$ with each of $x = \sqrt{-1} p^{-(d-1)/2}$, $x = -\sqrt{-1} p^{-(d-1)/2}$, and $x = p^{-d}$, and simplify.

Proof of Theorem 1.1 (Case $p^2 \neq 1$ and $a_1 = 0$). (i) Obtained by Lemma 6.2. (ii) Obtained by Lemmas 6.4 and 6.5. (iii) In the formulas given in Theorem 1.1, substitute for $x$ the values given in the proof of Lemma 6.6, and compare the results with the formulas given in Lemma 6.6.
7. THE CASE $p^2 \neq 1$ AND $a_i > 0$

In this section we consider the case $p^2 \neq 1$ and $a_i > 0$. Recall that $x_r^2 \neq 1$ (1 $\leq r \leq d$) and $x_r x_{r+1} \neq 1$ (1 $< r < d$) in the case $p^2 \neq 1$ by Corollaries 4.6 and 4.7.

We need a result by Terwilliger [32] concerning Q-polynomial distance-regular graphs. This result gives a relation between the dual eigenvalues and the kite numbers $e_i(x, y, z)$ (1 $\leq i \leq d$) defined for each triple $x, y,$ and $z$ of mutually adjacent vertices by

$$e_i(x, y, z) = |D_{i-1}(x, y)|^{-1} |D_i(x, y) \cap \Gamma_{i-1}(z)|.$$

**Theorem 7.1 (Terwilliger [32]).** Let $\Gamma$ be a Q-polynomial distance-regular graph of diameter $d$, and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ be the dual eigenvalues of $\Gamma$ with respect to a Q-polynomial ordering of the primitive idempotents. Let $x, y, z$ be mutually adjacent vertices. Then

$$e_i(x, y, z) = \alpha_i e_i(x, y, z) + \beta_i \quad (2 \leq i \leq d),$$

where

$$\alpha_i = \frac{(\theta_i^* - \theta_0^*) (\theta_i^* + \theta_i^* - \theta_{i-1}^*)}{(\theta_i^* - \theta_0^*) (\theta_i^* - \theta_{i-1}^*)},$$

$$\beta_i = \frac{(\theta_i^* - \theta_0^*) (\theta_i^* - \theta_0^* - \theta_i^*)}{(\theta_i^* - \theta_0^*) (\theta_i^* - \theta_{i-1}^*)}.$$

Since our graph $\Gamma$ is self-dual by Lemma 3.2, we have $P = Q$. In particular, $\Gamma$ is Q-polynomial, and $\theta_i = \theta_i^*(0 \leq i \leq d)$.

As shown below, the kite numbers are closely related to the number $\gamma$ in Lemma 4.8. In the case $p^2 \neq 1$, we have $x_i \neq 1$ for all $i$ (1 $\leq i \leq d$), so that the number $\gamma$ is independent of the choice of defining vertices by (14). Thus we may define the numbers $\gamma_1, \ldots, \gamma_d$ by

$$\gamma_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y) \cap \Gamma_i(z)|$$

for vertices $x, y, z$ such that $\partial(x, y) = r - 1$, $\partial(x, z) = r$, $\partial(y, z) = 1$. Observe that the Eqs. (14)–(19) hold with $\gamma = \gamma_i$.

**Lemma 7.2.** Suppose $p^2 \neq 1$. Then for all triples $x, y, z$ of mutually adjacent vertices, $\gamma_i = a_i e_i(x, y, z) \quad (2 \leq i \leq d)$.

**Proof.** Fix two adjacent vertices $x, y$, and set $D_i^j = D_i^j(x, y)$. We count the number $m$ of pairs $(z, u)$ such that $z \in D_i^1$, $u \in D_i^j_{i-1}$, and $\partial(u, z) = i - 1$. On one hand, for each $z \in D_i^1$, the number of such vertices $u$ is precisely...
\[ |D'_{i-1}| e_i(x, y, z), \text{ so that } m = |D'_{i-1}||e_i(x, y, z)\]. On the other hand, for each \( u \in D'_{i-1} \), there are precisely \( \gamma_i \) vertices \( z \) in \( \Gamma_{i-1}(u) \cap \Gamma_i(y) \cap \Gamma_i(x) \), so that \( m = |D'_{i-1}| \gamma_i \). Hence \( \gamma_i = |D'_{i-1}| e_i(x, y, z) = a_1 e_i(x, y, z) \).

**Lemma 7.3.** Suppose \( p^2 \neq 1 \). Then
\[
\gamma_i = a_i \gamma_2 + \beta_i a_1 \quad (2 \leq i \leq d). \quad (36)
\]

**Proof.** Clear from Theorem 7.1 and Lemma 7.2.

We set
\[
\rho_i = x_i - x_i^{-1} \quad (1 \leq i \leq d).
\]

**Lemma 7.4.** Suppose \( p^2 \neq 1 \). Then
\[
(p, \gamma_i - \rho_{i-1} \gamma_{i-1})(\rho_2 + \rho_2 \gamma_i) = \rho_2 (\rho_i + \rho_{i-1}) \gamma_2 + (\rho_i \rho_i - \rho_2 \rho_{i-1}) a_1 \\
(2 \leq i \leq d). \quad (37)
\]

**Proof.** By Lemma 4.5, we have \( \theta_{i-1} - \theta_i = p_1^{-1} \rho_i (\theta_0 - \theta_1) \). Hence for \( 0 \leq j \leq \ell \leq d \), \( \theta_j - \theta_\ell = \sum_{r=j+1}^{\ell} (\theta_{r-1} - \theta_r) = (\theta_0 - \theta_1) p_1^{-1} \sum_{r=j+1}^{\ell} \rho_r \). Thus for \( j \geq 2 \),
\[
\gamma_j = \frac{\rho_2}{(\rho_1 + \rho_2) \beta_1 \left( \sum_{r=1}^{j-1} \rho_r + \sum_{r=2}^{j} \rho_r \right)},
\]
\[
\beta_j = \frac{1}{(\rho_1 + \rho_2) \beta_1 \left( \rho_1 \sum_{r=1}^{j-1} \rho_r - \rho_2 \sum_{r=2}^{j-1} \rho_r \right)},
\]
where we take any empty sum to be zero. Clearly (37) holds at \( i = 2 \) since \( \gamma_1 = 0 \), so we may assume \( i \geq 3 \). Then by Lemma 7.3, we find that
\[
\rho_i \gamma_i - \rho_{i-1} \gamma_{i-1} = \rho_i (x_i \gamma_2 + \beta_i a_1) - \rho_{i-1} (x_{i-1} \gamma_2 + \beta_{i-1} a_1) = \gamma_2 \rho_2 (\rho_i \gamma_i - \rho_{i-1} \gamma_{i-1}) + a_1 (\rho_i \beta_i - \rho_{i-1} \beta_{i-1}) = \frac{\gamma_2 p_2 (\rho_i \rho_i - \rho_2 \rho_{i-1})}{\rho_1 + \rho_2} + \frac{a_1 (\rho_i \beta_i - \rho_{i-1} \beta_{i-1})}{\rho_1 + \rho_2}.
\]
Thus we obtain (37).

**Lemma 7.5.** Suppose \( p^2 \neq 1 \) and \( a_1 > 0 \). Then \( x_r = p^{r-1} x \) (1 \leq i \leq d).

**Proof.** We may assume \( i \geq 3 \). Observe that (18) implies that
\[
\gamma_r = \frac{a_1 (x_r - x_1)}{(x_r - 1)(x_1 + 1)} \quad (2 \leq r \leq d).
\]
Replacing \(\gamma_i, \gamma_{i-1}\), and \(\gamma_2\) in (37) with this expression,

\[
\frac{\rho_i a_i(x_i - x_i)}{(x_i - 1)(x_i + 1)} \cdot \frac{\rho_{i-1} a_i(x_{i-1} - x_i)}{(x_{i-1} - 1)(x_i + 1)} (\rho_1 + \rho_2)
\]

Multiplying both sides by \((x_1 + 1) a_i\), and using \(\gamma_i = (x_i - 1)(1 + x_i^{-1})\),
the above equation becomes

\[
((x_i - x_1)(1 + x_i^{-1}) - (x_{i-1} - x_1)(1 + x_{i-1}^{-1}))(\rho_1 + \rho_2)
= (1 + x_2^{-1})(\rho_i + \rho_{i-1})(x_2 - x_1) + (\rho_i \rho_j - \rho_2 \rho_{i-1})(x_1 + 1).
\]

Replacing each \(\rho_j\) with \(x_j - x_j^{-1}\), multiplying by \(x_1 x_2 x_i x_i^{-1}\), and simplifying, the above equation implies

\[x_2 x_{i-1} - x_1 x_i = 0.\]

Thus \(x_2 x_{i-1} x_1 x_i = 0\), so that \(x_i = (x_i^{-1} x_2) x_{i-1} = px_{i-1}^{-1}\).  

**Lemma 7.6.** Suppose \(p^2 \neq 1\) and \(a_1 > 0\). Then

\[
\theta_1 = \frac{1 + px^3 + (\theta_0 - 1)(px + 1) x}{x(p + x)}. \quad (38)
\]

**Proof.** From Lemma 4.3 at \(i = 1\),

\[
x \theta_1 = x^{-1} + a_1 + b_1 px, \quad (39)
\]

\[
x^{-1} \theta_1 = x + a_1 + b_1 p^{-1} x^{-1}. \quad (40)
\]

Adding (39) and \(px\) times (40),

\[
(p + x) \theta_1 = x^{-1} + px^2 + (a_1 + b_1)(px + 1)
= x^{-1} + px^2 + (\theta_0 - 1)(px + 1). \quad (41)
\]

We show that \(p + x \neq 0\). Assume \(p + x = 0\), so that \(p = -x\) and \(x_2 = px = -x^2\). Then (41) implies that \(x^{-1} - x^3 + (\theta_0 - 1)(-x^2 + 1) = 0\), so that \(x + x^{-1} = 1 - \theta_0\). On the other hand, (18) at \(r = 2\) implies that \(\gamma_2(x_1 + 1) (x_2 - 1) = a_1(x_2 - x_1)\), so that \(\gamma_2(x + x^{-1}) = a_1\). Hence \(\gamma_2(1 - \theta_0) = a_1\). This is a contradiction since \(\gamma_2 \geq 0\), \(1 - \theta_0 < 0\), and \(a_1 > 0\). Hence \(p + x \neq 0\). Now (41) implies (38).  

LEMMA 7.7. Suppose \( p^2 \neq 1 \) and \( a_1 > 0 \). Then
\[
\theta_i - \theta_0 = \frac{(p^{i-1}x^2 - 1)(px^2 - 1 + \theta_0 x(p - 1))(p^i - 1)}{p^{i-1}x(x+1)(p+x)(p-1)} \quad (0 \leq i \leq d).
\] (42)

Proof. Observe that (42) holds at \( i = 0 \). By Lemma 7.6,
\[
\theta_1 - \theta_0 = \frac{1 + px^3 + (\theta_0 - 1)(px + 1)x - \theta_0 x(p + x)}{x(p + x)} = \frac{(x^2 - 1)(px^2 - 1 + \theta_0 x(p - 1))}{x(x+1)(p+x)},
\]
so (42) holds at \( i = 1 \). We show (42) for \( i \geq 2 \) by induction. From (11),
\[
\theta_i = \theta_{i-1} + \frac{x_i - x_{i-1}}{x - x^{-1}} (\theta_1 - \theta_0).
\]
Hence by Lemma 7.5,
\[
\theta_i - \theta_0 = (\theta_{i-1} - \theta_0) + \frac{p^{i-1}x - p^{-i+1}x^{-1}}{x - x^{-1}} (\theta_1 - \theta_0)
\]
\[
= \frac{(p^{i-2}x^2 - 1)(px^2 - 1 + \theta_0 x(p - 1))(p^{i-1} - 1)}{p^{i-2}x(x+1)(p+x)(p-1)}
\]
\[
+ \frac{p^{i-2}x^2 - 1}{p^{i-1}x(x - x^{-1})} \cdot \frac{(x^2 - 1)(px^2 - 1 + \theta_0 x(p - 1))}{x(x+1)(p+x)}
\]
\[
= \frac{(p^{i-1}x^2 - 1)(px^2 - 1 + \theta_0 x(p - 1))(p^i - 1)}{p^{i-1}x(x+1)(p+x)(p-1)}.
\]
Hence (42) holds. □

LEMMA 7.8. Suppose \( p^2 \neq 1 \) and \( a_1 > 0 \). Then
\[
\theta_0 = \frac{(px^2 - 1)(p^{d-1}x^3 + 1)(p^d - 1)}{x(p^{d-1}x+1)(1 - p^d x^2)(p-1)}.
\] (43)

Proof. From (9) and (10) at \( i = d \),
\[
x \theta_d = c_d (x_d^{-1} - 1) + \theta_0, \quad (44)
\]
\[
x^{-1} \theta_d = c_d (x_d-1) + \theta_0. \quad (45)
\]
Adding \((45)\) and \(x_d\) times \((44)\), we obtain \((xx_d + x^{-1}) \theta_d = \theta_d(x_d + 1)\), and this becomes \((p^{d-1}x^3 + 1) \theta_d = \theta_d x(p^{d-1}x + 1)\). Observe that \(p^{d-1}x^3 + 1 \neq 0\) since the right-hand side is nonzero by \(\theta_d > 0\) and \(x_d + 1 \neq 0\). Hence
\[
\theta_d = \frac{\theta_d x(p^{d-1}x + 1)}{p^{d-1}x^3 + 1},
\]
so that
\[
\theta_d - \theta_0 = \frac{\theta_d x(p^{d-1}x + 1) - (p^{d-1}x^3 + 1))}{p^{d-1}x^3 + 1} = \frac{\theta_d (1 - x)(p^{d-1}x^2 - 1)}{p^{d-1}x^3 + 1}.
\]
On the other hand, from \((42)\) we have
\[
\theta_d - \theta_0 = \frac{(p^{d-1}x^3 - 1)(px^2 - 1 + \theta_0 x(p - 1))(p^{d-1})}{p^{d-1}x(x + 1)(p + x)(p - 1)}.
\]
Since \(\theta_d \neq \theta_0\), we have \(p^{d-1}x^3 - 1 \neq 0\) and \(p^{d-1} - 1 \neq 0\). Then the above two equations imply
\[
\frac{(px^2 - 1 + \theta_0 x(p - 1))(p^{d-1})}{p^{d-1}x(x + 1)(p + x)(p - 1)} = \frac{\theta_d (1 - x)}{p^{d-1}x^3 + 1}.
\]
This implies
\[
\frac{(-1)(px^2 - 1)(p^{d-1})}{p^{d-1}x(x + 1)(p + x)(p - 1)} = \frac{\theta_d (x - 1)}{p^{d-1}x^3 + 1} + \frac{\theta_d (p^{d-1})}{p^{d-1}x^3 + 1} = \frac{\theta_d (p^{d-1}x^2 - 1)(p^{d-1}x + 1)}{p^{d-1}x^3 + 1(x + 1)(p + x)}.
\]
so that
\[
\theta_d (p^{d-1}x^2 - 1)(p^{d-1}x + 1) = \frac{(px^2 - 1)(p^{d-1})}{x(p - 1)}.
\]
Observe that the right-hand side is nonzero since \(px^2 = x_1x_2 \neq 1\). Hence the above equation implies \((43)\).

**Lemma 7.9.** Suppose \(p^2 \neq 1\) and \(a_i > 0\). Then for all \(i\) \((0 \leq i \leq d)\)
\[
\theta_i = \frac{(px^2 - 1)((p^{d-i}+1)x^3 + 1)(p^{d-1} - 1) - p^{d-i} x(p^{d-1} x + 1)(p^{d-1} - 1))}{x(p^{d-i}x + 1)(1 - p^{d-i}x^2)(p - 1)}.
\]

\((46)\)
Proof. Observe that (46) holds for $i=0$ by Lemma 7.8, so we may assume that $i \geq 1$. From Lemma 7.7,

$$\theta_i = \theta_0 + \frac{(p^{i-1}x^2 - 1)(px^2 - 1 + \theta_0x(p-1))(p'-1)}{p^{i-1}x(x+1)(p+x)(p-1)}$$

$$= \frac{(p^{i-1}x^2 - 1)(px^2 - 1)(p'-1)}{p^{i-1}x(x+1)(p+x)(p-1)} + \frac{\theta_0(p'x+1)(p^{i-1}x+1)}{p^{i-1}(x+1)(p+x)}.$$

Using (43) and simplifying, we obtain (46).

**Lemma 7.10.** Suppose $p^2 \not= 1$ and $\alpha_1 > 0$. Then

$$b_i = \frac{p(1-x)(px^2 - 1)(p^{d-i}+1)(p^{d-i}x + 1)}{x(p^{d-i}+1)(p^{d-i}x^2 - 1)(p'-1)} \quad (1 \leq i < d).$$

**Proof.** From (32) and Lemma 7.5,

$$b_i = \frac{(p^{i+1}x^2 + 1)}{(p^{i+1} - p^{-i}x^{-1})(p'x-1)}.$$

Using (43) and (46), this becomes

$$b_i = \frac{(px^2 - 1)}{(p^{i+1} - p^{-i}x^{-1})(p'x-1)(1-x-p^2x^2(p-1))},$$

where

$$M = (p^{i+1}x^2 + 1)(p^{d-i}+1)(p^{d-i}x + 1) + p^{d-i}x(p^{i+1}x + 1)(p'-1)$$

$$= x^{-1}(p^{d-i} - 1)(p^{d-i}+1)(p^{d-i}x + 1) - (p^{i+1} - 1)(p^{d-i}x^2 + 1) - (p^{i+1}x^2 - 1)(p'-1).$$

This implies the result.

**Lemma 7.11.** Suppose $p^2 \not= 1$ and $\alpha_1 > 0$. Then

$$c_i = \frac{p^{i+1}(x-1)(px^2 - 1)(p^{d-i}x + 1)(p^{d-i}+1)x^2 - 1)(p'-1)}{(p^{d-i}+1)(p^{d-i}x^2 - 1)(p^{d-i}x - 1)(p^{d-i}+1)(p'-1)} \quad (1 \leq i < d),$$

$$c_d = \frac{p^{d-1}(x^2 - 1)(px^2 - 1)(p'-1)}{(p^{d-1}x^2 - 1)(p'-1)(p^{d-1}x^2 - 1)(p'-1)}.$$
Proof. From (30) and Lemma 7.5,

\[ c_i = \frac{(p^i + x) \theta_i - (p^i x + 1) \theta_0}{(p^i x - p^{-(r-1)} x^{-1})(p^{r-1} x - 1)}. \]

Using (43) and (46), this becomes

\[ c_i = \frac{(p x^2 - 1) M}{(p^i x - p^{-i+1} x^{-1})(p^{r-1} x - 1) x(p^{d-1} x + 1)(1 - p^d x^2)(p - 1)}, \]

where

\[ M = \frac{(p^i + x)((p^{d+i-1} x^3 + 1)(p^{d-i} - 1) + p^{d-i} x(p^{d-1} x + 1)(p^i - 1))}{(p^i x + 1)(p^{d-1} x + 1)(p^d - 1)} \]

\[ = (p^i - 1)(1 - x)(p^{d-i} x + 1)(p^{d+i-1} x^2 - 1). \]

This gives \( c_i \) for \( 1 \leq i < d \). From (31), Lemma 7.5, (43) and (46), we obtain \( c_d \).

Proof of Theorem 1.1 (Case \( p^2 \neq 1 \) and \( a_1 > 0 \)). (i) Obtained by Lemma 7.5. (iii) Obtained by Lemmas 7.9, 7.10, and 7.11.

8. CLOSING REMARKS

We now comment on Theorem 1.1 in the context of some open problems.

Problem 8.1. Classify spin models which have only two weights.

This is the only case which falls outside the scope of Theorem 1.1.

We claim that if \( W \) is a spin model with just two weights and there exists a distance-regular graph \( I \) such that \( I \) and \( W \) satisfy the assumptions of Theorem 1.1 (so \( d \geq 2 \)), then the weights of \( W \) are \( \pm t_0 \). Indeed, \( t_1 = t_0 \) implies that \( x = 1 \), and \( t_1 \neq t_0 \) implies that either \( x_1 = 1 \) or \( x_1 x_2 = 1 \) since \( t_2 \in \{ t_0, t_1 \} \). Now Corollaries 4.6 and 4.7 imply that in any of these cases \( x_1^2 = 1 (1 \leq i \leq d) \). Thus the weights of \( W \) are just \( \pm t_0 \). Conversely, \( x_1^2 = 1 \) implies that \( W \) only has weights \( \pm t_0 \) by Corollary 4.6.

It is also possible to apply our method to obtain some strong restrictions on the eigenvalues and on the intersection numbers in this case. We conjecture that no distance-regular graph other than the 4-cycle (Example 2 of Section 9) satisfying the assumptions of Theorem 1.1 for a spin model with just two weights. However, we leave this case to the slightly more general Problem 8.1 (where no assumption about a distance-regular graph is made).
(See also [17] where this problem is studied in the more general context of four-weight spin models).

Problem 8.2. Characterize the distance-regular graphs which have the parameters of Theorem 1.1(iii) (with $x^2 \neq 1$).

The parameterization of Theorem 1.1(iii) can be viewed as a special case of Leonard’s theorem—a characterization of the Q-polynomial distance regular graphs as those distance-regular graphs whose intersection numbers can be described by certain formulas involving 5 free parameters (see [6, Theorem III.5.1]). In the notation of Bannai and Ito [6], the parameters of Leonard’s theorem here are $(q, s) \in \{(p, x^2/p^3), (1/p, p^2/x^2)\}$, $s^* = s$, and $(r_1, r_2) = \{-x/p, -p^d - 2x^3\}$, where we adopt the convention that $r_d = p^{-d-1}$.

Problem 8.2 can be viewed as part of the characterization project for the Q-polynomial distance-regular graphs. For the study of spin models it would be enough to assume some additional properties, such as the existence of the combinatorial structure constants $\gamma$, or the thin condition of Terwilliger (see [13]).

Note that all graphs listed in Section 9 (with sufficiently large diameter) have either $p^2 = 1$ or $a_1 = 0$. Thus make the following conjecture.

Conjecture 8.3. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 4$ and intersection numbers given by Theorem 1.1(iii) (with $x^2 \neq 1$). Then either $p^2 = 1$ or $a_1 = 0$.

The distance-regular graphs with intersection numbers given by Theorem 1.1(iii) (with $x^2 \neq 1$) and either $p^2 = 1$ or $a_1 = 0$ are (essentially) known.

First suppose $p^2 = 1$ and $x^2 \neq 1$. By Lemma 5.3, we need not treat the case $p = -1$, so suppose $p = 1$. Then the intersection numbers given in Lemma 5.2 are exactly those of the Hamming graph $H(d, q)$, so that the graph is isomorphic to $H(d, q)$ or the Doob graph by [31, 15].

Now suppose that $a_1 = 0$. We claim that $\Gamma$ is either a 2-homogeneous bipartite or 2-homogeneous almost bipartite distance-regular graph. First let us recall some definitions. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 2$. $\Gamma$ is said to be almost bipartite whenever $a_i = 0$ (1 ≤ $i < d$) and $a_d \neq 0$, and $\Gamma$ is said to be bipartite whenever $a_i = 0$ (1 ≤ $i < d$). Suppose $\Gamma$ is bipartite or almost-bipartite. Then $\Gamma$ is said to be 2-homogeneous whenever for all integers $i$ (1 ≤ $i ≤ D$) the number $|\Gamma_i(x) \cap \Gamma_i(y) \cap \Gamma_i(z)|$ is independent of the choice of $x, y, z \in X$ with $d(x, y) = 2$, $d(x, y) = d(x, z) = i$.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 4$ and intersection numbers given by Theorem 1.1(iii) (with $x^2 \neq 1$) and $a_1 = 0$. Then $\Gamma$ is bipartite or almost bipartite by Theorem 1.1(ii). Moreover, since
\( \Gamma \) is self-dual, it has a nontrivial eigenvalue with multiplicity precisely \( b_0 \).
But it was shown in [29] that all bipartite and all almost bipartite distance-regular graphs with a nontrivial eigenvalue of multiplicity at most \( b_0 \) are 2-homogeneous.

The 2-homogeneous bipartite and 2-homogeneous almost bipartite distance-regular graphs were characterized in [29]. There are three families with arbitrary diameter—the cycles, the Hamming cubes, and the folded Hamming cubes. In addition there are six families with diameter at most five—the complete graphs, the complete bipartite graphs, the complement of the \( 2 \times (k+1) \)-grid, the Hadamard graphs, and two families with intersection arrays \( \{k, k - 1; 1, c\} \) and \( \{k, k - 1, k - c, c, 1; 1, c, k - c, k - 1, k\} \), where \( k = \gamma(\gamma^2 + 3\gamma + 1) \) and \( c = \gamma(\gamma + 1) \) for \( \gamma \geq 2 \) a positive integer. The only known members of the last two families are the Higman–Sims graph and its antipodal double cover (for \( \gamma = 2 \)). This is a major challenge in giving a strong solution to Problem 8.2.

The paper [12] studied the 2-homogeneous bipartite distance-regular graphs in connection with the Q-polynomial property. The parameterization given there coincides with the specialization of Theorem 1.1(iii) to the bipartite case.

Problem 8.4. Characterize the distance-regular graphs and spin models which together satisfy the assumptions of Theorem 1.1.

Given a strong solution to Problem 8.2, this is likely to be straightforward. However, this problem is still nontrivial, given only a parameterization of the intersection numbers and restrictions on the parameters as a solution to Problem 8.2.

Although Conjecture 8.3 predicts that most examples of spin models on distance-regular graphs are known, we suspect that there are more distance-regular graphs with diameter 3 which contain a spin model in their Bose–Mesner algebra.

We note that the distance-regular graphs having intersection numbers given by the parameterization of Theorem 1.1(iii) do not necessarily contain a spin model in their Bose–Mesner algebra. As noted after Conjecture 8.3, the Doob graphs satisfy the parameterization of Theorem 1.1. We may, however, show that no Doob graph contains a spin model in its Bose–Mesner algebra as follows. Suppose \( x_1 = x_2 \neq -1 \). Pick any path \((x, y, z)\) of length two connecting two vertices \( x, z \) at distance \( \delta(x, z) = 2 \), and apply Lemma 4.8 at \( r = 2 \). Then (18) implies \( \gamma = 0 \), so that there is no vertex \( u \) which is adjacent to each of \( x, y, z \). This is not the case for the Doob graphs.

Problem 8.5. Characterize the self-dual distance-regular graphs which have the modular invariance property.
A self-dual association scheme \( \{ A_i \}_{i=0}^{d} \) is said to have the **modular invariance property** if there exists a diagonal matrix \( T \) of size \( d \) which satisfies \((PT)^3 = I \), where \( P \) denotes the eigenmatrix of the scheme. The self-dual P-polynomial association schemes with the modular invariance property were studied in [11].

The identity of Theorem 3.1(iv) is equivalent to the equation \((PT)^3 = I\) (where \( x \) denotes the modulus) by [22, Proposition 12].

**Conjecture 8.6.** Let \( G \) be a self-dual distance-regular graph of diameter \( d > 1 \), and let \( P \) denote the eigenmatrix. Suppose there exists a diagonal matrix \( T = \text{diag}[t_0, t_1, \ldots, t_d] \) such that \((PT)^3 = I\). Let \( x = t_1^{-1} \) (1 \( \leq i \leq d \)), \( x = x_1 \) and \( p = x_1^{-1} x_2 \). Then parts (i)-(iii) of Theorem 1.1 hold.

Observe that Conjecture 8.6 implies Theorem 1.1 by [22, Proposition 12]. The authors have confirmed this conjecture for the case \( d = 3 \). Conversely, we make the following conjecture.

**Conjecture 8.7.** Any distance-regular graph which has the parameters of Theorem 1.1(iii) (with \( x^2 \neq 1 \)) has the modular invariance property.

Recall that \( P_{ij} = v_i(\theta_j) \), where \( v_0(y) = 1 \), \( v_i(y) = y \), and \( v_i(y) (2 \leq i \leq d) \) is the polynomial of degree \( i \) defined by the recurrence

\[
yv_i(y) = b_{i-1}v_{i-1}(y) + av_i(y) + cv_{i+1}(y).
\]

(See [6, Proposition III.1.1].)

We conjecture that this construction for \( P \) admits a solution \( T \) to the equation \((PT)^3 = I\) whenever \( c, a, b, \) and \( \theta \) are given by the parameterization of Theorem 1.1(iii) with any values of \( p \) and \( x \) such that \( x^2 \neq 1 \) and the denominators are non-zero. This does not require the existence of such a distance-regular graph. Such a solution \( T \) would be of the form \( T = \text{diag}[t_0, t_1, \ldots, t_d] \), where \( t_0/t_1 = p^{-1}x \) (the number \( t_0 \) must be chosen so as to give the proper normalization). We expect that this can be verified with a clever computation.

9. KNOWN EXAMPLES

In this section we present a list of all examples of spin models, known to the authors, which are associated with a distance-regular graph.

For each of the following distance-regular graphs \( G = (X, R) \) with diameter \( d \), let \( A_i \) denote the \( i \)th adjacency matrix (0 \( \leq i \leq d \)). Then the matrix \( W = \sum_{i=0}^{d} t_i A_i \) is a spin model on \( X \) such that the Bose–Mesner
algebra $\mathcal{A}$ of $\Gamma$ satisfies $W \in \mathcal{A} \subseteq N(W)$. The presentation of the data is explained in “Example 0.”

0) Name of spin model $W$ [Reference to introduction].
   — Name of graph $\Gamma$, diameter $d$, intersection array $\{b_0, b_1, ..., b_d\}$.
   — Boltzmann weights $t_0, t_1, ..., t_d$ of $W$ (with any auxiliary parameters used in their definition).
   — The parameters $p$ and $x$ of Theorem 1.1 (when defined).

1) Potts model [23].
   — Complete graph, $d = 1$, $n > 1$, $\{n - 1; 1\}$.
   — $t_1$ is any solution of $t_1^2 + t_1^{-2} + \sqrt{n} = 0$, $t_0 = -t_1^{-3}$.

2) Square model [23].
   — 4-cycle, $d = 2$, $\{2, 1; 1, 2\}$
   — $t_0$ is any nonzero complex number, $t_1 = t_0^{-1}$, $t_2 = -t_0$.
   — $p = t_0^4$, $x = t_0^2$.

3) Jaeger’s Higman–Sims model [18].
   — Higman–Sims graph, $d = 2$, $\{22, 21; 6, 16\}$.
   — Set $\tau = (1 + \sqrt{5})/2$:
     $t_0 = (5\tau + 3)\sqrt{-1}$, $t_1 = \tau \sqrt{-1}$, $t_2 = (-\tau + 1)\sqrt{-1}$.
   — $p = (5\tau + 3)(-\tau + 1)/\tau^2$, $x = \tau/(5\tau + 3)$.

4) Hadamard model [26].
   — Hadamard graph, $d = 4$, $\{4m, 4m - 1, 2m, 1; 1, 2m, 4m - 1, 4m\}$ with $m > 0$ an integer.
   — Let $s$ be a solution of $s^2 + 2(2m - 1)s + 1 = 0$:
     $t_0$ is any number such that $t_0^s = 2\sqrt{m}/((4m - 1)s + 1)$,
     $t_1$ is any number such that $t_1^s = 1$, $t_2 = st_0$, $t_3 = -t_1$, $t_4 = t_0$.
   — $p = st_0^2/t_1^2$, $x = t_1/t_0$.

5) Doubled Higman–Sims model [25].
   — Double cover of the Higman–Sims graph, $d = 5$, $\{22, 21, 16, 6; 1, 6, 16, 21, 22\}$.
— Set \( \omega = \exp(\pi \sqrt{-1}/8) \), \( \tau = (1 + \sqrt{5})/2 \).

\[
\begin{align*}
t_0 &= (5\tau + 3) \omega \sqrt{-1}, \quad t_1 = \tau \omega, \quad t_2 = (\tau - 1) \omega \sqrt{-1}, \\
t_3 &= (-\tau + 1) \omega, \quad t_4 = \tau \omega \sqrt{-1}, \quad t_5 = (5\tau + 3) \omega.
\end{align*}
\]

— \( p = -(5\tau + 3)(-\tau + 1)/\tau^2 \), \( x = -\tau \sqrt{-1}(-(5\tau + 3)) \).

(6) Odd cyclic model [23].
— Odd cycle, \( d \geq 1 \), \( \{2, 1, \ldots, 1; 1, 1, \ldots, 1\} \) with \( n = 2d + 1 \) vertices.
— Set \( \omega = \exp(2\pi \sqrt{-1}/n) \) and \( L = \sum_{l=0}^{n-1} \omega^l \).

\( t_0 \) is any number such that \( t_0^2 = \sqrt{n}/L, \quad t_i = t_0 \omega^i \quad (i = 0, 1, \ldots, d) \).

— \( p = \omega^2, \quad x = \omega \).

(7) Even cyclic model [2].
— Even cycle, \( d \geq 2 \), \( \{2, 1, \ldots, 1; 1, 1, \ldots, 1, 2\} \) with \( n = 2d \) vertices.
— Set \( \omega = \exp(\pi \sqrt{-1}/n) \) and \( L = \sum_{l=0}^{n-1} \omega^l \).

\( t_0 \) is any number such that \( t_0^2 = \sqrt{n}/L, \quad t_i = t_0 \omega^i \quad (i = 0, 1, \ldots, d) \).

— \( p = \omega^2, \quad x = \omega \).

(8) Direct product of Potts models [4].
— Hamming graph \( H(d, q), \quad d \geq 1, \quad q \geq 2 \), \( c_i = i \), \( b_i = (n-i)(q-1) \) \((i = 0, 1, \ldots, d) \).
— Let \( \lambda \) be a solution of \( \lambda + \lambda^{-1} + (q-2) = 0 \):

\( t_0 \) is any number such that \( t_0^2 = (\sqrt{\lambda}/(1 + (q-1) \lambda))^d \)

(where \( \sqrt{\lambda} \) denotes one of the complex square roots of \( \lambda \)).

\( t_i = t_0 \lambda^i \quad (i = 0, 1, \ldots, d) \).

— \( p = 1, \quad x = \lambda \).

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