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On certain Riesz families in vector-valued de Branges–Rovnyak spaces

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ABSTRACT

We obtain criteria for the Riesz basis property for families of reproducing kernels in vector-valued de Branges–Rovnyak spaces $\mathcal{H}(b)$. In particular, it is shown that in several situations the property implies a special form for the function b. We also study the completeness of a related family.

1. Introduction

Starting with the works of Paley–Wiener [23], a whole direction of research has investigated families of exponentials in $L^2(\mathbb{R})$, looking for properties as completeness, minimality, or being an unconditional basis. A classical result is Ingham's Theorem [14], which says roughly that a small perturbation of the standard exponentials e^{int} remains a basis in $L^2(-\pi,\pi)$. This line of approach, the consideration of a given family of exponentials as a small perturbation of one that is known to be complete or a Riesz basis has subsequently yielded many stability results. One should also mention that the theory of geometric properties of scalar or vector valued exponential families has found applications in various areas such as convolution equations, string scattering theory or controllability of dynamical systems (see [1,13,19] for a survey on exponentials systems and their applications).

In this context, functional models have been used in [13], allowing the use of tools from operator theory on a Hilbert space. The model spaces are subspaces of the Hardy space H^2, invariant under the adjoints of multiplications; their theory is connected to dilation theory for contractions on Hilbert spaces (see [25]). The approach has proved fruitful, leading to the recapture of all the classical results as well as to several generalizations.

This investigation has been pursued with respect to families of reproducing kernels in vector-valued and scalar model spaces in [9,10] and [5], and in scalar de Branges–Rovnyak spaces in [11]. Again the main goal is to obtain criteria for a family of reproducing kernels to be complete, minimal or Riesz basis. We also mention an interesting paper of A. Baranov [2] whose criteria is based on recent work of Ortega-Cerdà and Seip [22].

In this paper we investigate similar problems in the context of vector-valued de Branges–Rovnyak spaces. It appears that the functional methods used in [9,11] are no more appropriate in this situation, and we have to find a new approach. This is essentially done by using in more detail the structure of the model theory of contractions [25], and especially its relation to vector valued de Branges–Rovnyak spaces as emphasized in [20]. This new approach throws some light also on the scalar-valued case; we show, for instance, that if the scalar-valued de Branges–Rovnyak space $\mathcal{H}(b)$ admits a Riesz basis of reproducing kernels, then necessarily b is inner. Similar methods are used to investigate properties of a different family

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of functions, called difference quotients; in particular, in the scalar case, we show that the difference quotients are complete in $\mathcal{H}(b)$ if and only if either b is an extreme point of the unit ball of H^∞ or b is not pseudocontinuable.

The plan of the paper is the following. The next section contains some preliminary material. The connection of the de Branges–Rovnyak spaces to the functional model for contractions is described in detail in Section 3, where a main notion is that of *abstract functional embedding*, introduced in [21]. Section 4 shows how the problems concerning bases of reproducing kernels can be reduced, under suitable hypotheses, to the study of the invertibility of a certain operator (the *distortion* operator). Criteria for this invertibility are given in Section 5, which contains the main results of the paper. A different type of criterium appears in Section 6, while Section 7 contains some interesting examples. Finally, Section 8 studies completeness properties of the difference quotients.

2. Preliminaries

2.1. Hardy spaces and de Branges–Rovnyak spaces

If E is a separable complex Hilbert space, $L^2(E)$ is the usual L^2-space of E-valued functions f on the unit circle $\mathbb{T}$ with respect to the normalized measure m endowed with the norm

$$\|f\|_2^2 = \int_{\mathbb{T}} \|f(z)\|_E^2 \, dm(z).$$

The corresponding Hardy space $H^2(E)$ is defined as E-valued analytic functions on $\mathbb{D}$, $f(z) = \sum_{n\geqslant 0} a_n z^n$, $a_n \in E$, with $\|f\|_2 < +\infty$, where

$$\|f\|_2^2 = \sum_{n\geqslant 0} \|a_n\|_E^2.$$

Alternately, it is well known that $H^2(E)$ can be regarded as the closed subspace of $L^2(E)$ consisting of functions whose negative Fourier coefficients vanish. The symbol P_+ (respectively P_-) stands for the Riesz orthogonal projection from $L^2(E)$ onto $H^2(E)$ (respectively onto $H^2_-(E) := L^2(E) \ominus H^2(E)$).

If E, E_* are two separable Hilbert spaces, we denote by $\mathcal{L}(E, E_*)$ the space of all bounded linear operators from E to E_*. Then $L^\infty(E \to E_*)$ is the Banach space of weakly measurable essentially bounded functions defined on $\mathbb{T}$ with values in $\mathcal{L}(E, E_*)$, endowed with the essential norm. The Banach space $H^\infty(E \to E_*)$ is formed by bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(E, E_*)$; taking (strong) radial limits identifies $H^\infty(E \to E_*)$ with a subspace of $L^\infty(E \to E_*)$.

If $\varphi \in L^\infty(E \to E_*)$, we will make a standard abuse of notation and denote by the same symbol φ the multiplication operator

$$\begin{aligned} \varphi : L^2(E) &\longrightarrow L^2(E_*) \\ f &\longmapsto \varphi f \end{aligned}$$

defined by $(\varphi f)(\zeta) := \varphi(\zeta) f(\zeta)$, $\zeta \in \mathbb{T}$. The inclusion $\varphi H^2(E) \subset H^2(E_*)$ is equivalent to $\varphi \in H^\infty(E \to E_*)$, while $\|\varphi\| \leqslant 1$ (or $\|\varphi | H^2(E)\| \leqslant 1$) is equivalent to $\|\varphi(\zeta)\| \leqslant 1$ a.e. on $\mathbb{T}$. The symbol T_φ denotes the *Toeplitz* operator from $H^2(E)$ to $H^2(E_*)$ defined by

$$T_\varphi f := P_+(\varphi f).$$

Then $T_\varphi \in \mathcal{L}(H^2(E), H^2(E_*))$, $\|T_\varphi\| = \|\varphi\|_\infty$, and $T_\varphi^* = T_{\varphi^*}$, where $\varphi^* \in L^\infty(E_* \to E)$ is defined by $\varphi^*(\zeta) := (\varphi(\zeta))^*$, $\zeta \in \mathbb{T}$.

We will also use occasionally the *Hankel* operator $\mathrm{H}_\varphi : H^2(E) \to H^2_-(E_*)$ defined by

$$\mathrm{H}_\varphi(f) = P_-(\varphi f).$$

We have then

$$\varphi f = T_\varphi f + \mathrm{H}_\varphi f, \qquad \|\varphi f\|^2 = \|T_\varphi f\|^2 + \|\mathrm{H}_\varphi f\|^2. \tag{2.1}$$

The vector valued Nehari Theorem says that $\|\mathrm{H}_\varphi\| = \mathrm{dist}(\varphi, H^\infty(E \to E_*))$.

Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$. The *de Branges–Rovnyak space* $\mathcal{H}(b)$, associated to b, is the vector space of those $H^2(E_*)$ functions which are in the range of the operator $(Id - T_b T_b^*)^{1/2}$; it becomes a Hilbert space when equipped with the inner product

$$\left\langle \left(Id - T_b T_b^*\right)^{1/2} f, \left(Id - T_b T_b^*\right)^{1/2} g \right\rangle_b := \langle f, g \rangle_2,$$

where $f, g \in H^2(E_*) \ominus \ker(Id - T_b T_b^*)^{1/2}$ (see [4,6]; [24] contains an extensive presentation of the scalar case). Note that $\mathcal{H}(b)$ is contained contractively in $H^2(E_*)$ and the inner product is defined in order to make $(Id - T_b T_b^*)^{1/2}$ a coisometry from $H^2(E_*)$ to $\mathcal{H}(b)$. The norm of $\mathcal{H}(b)$ will be denoted by $\|\cdot\|_b$; it coincides with the induced norm from $H^2(E_*)$ if and only if T_b is a partial isometry on $H^2(E)$.

An important particular case is obtained for b an *inner* function, that is, a function in $H^\infty(E \to E_*)$ such that $b(\zeta)$ is an isometry for almost all $\zeta \in \mathbb{T}$. Then $\mathcal{H}(b)$ is a *closed* subspace of $H^2(E_*)$, and $\|\cdot\|_b$ coincides with the induced norm; more precisely, we have $\mathcal{H}(b) = H^2(E_*) \ominus bH^2(E)$. By the Lax–Halmos Theorem, these are the nontrivial subspaces of $H^2(E_*)$ which are invariant for the backward shift $S^*|H^2(E_*)$. They are traditionally denoted by K_b; thus, in this case, we have $\mathcal{H}(b) = K_b$.

For further use, remember that b is called $*$*-inner* if $b(\zeta)$ is a coisometry for almost all $\zeta \in \mathbb{T}$. This is equivalent to $\tilde{b}(\zeta) := b(\bar{\zeta})^*$ being inner.

2.2. Reproducing kernels

If $\lambda \in \mathbb{D}$ and $e \in E_*$, the function $k_{\lambda,e}(z) = \frac{1}{1-\bar{\lambda}z}e$ belongs to $H^2(E_*)$ and is a reproducing kernel for this space; that is, for any $f \in H^2(E_*)$ we have $\langle f(\lambda), e\rangle_{E_*} = \langle f, k_{\lambda,e}\rangle_{H^2(E_*)}$. Since $\mathcal{H}(b)$ is contained contractively in $H^2(E_*)$, this formula defines also a bounded linear functional on $\mathcal{H}(b)$, which, according to Riesz's Theorem, is given by the inner product in $\mathcal{H}(b)$ with a vector $k^b_{\lambda,e} \in \mathcal{H}(b)$; thus, for all $f \in \mathcal{H}(b)$, $\langle f, k^b_{\lambda,e}\rangle_b = \langle f(\lambda), e\rangle_{E_*}$. A computation similar to the case of scalar de Branges–Rovnyak spaces (see [24, Chapter 2]) yields the formula

$$k^b_{\lambda,e}(z) = \big(Id - T_bT_b^*\big)k_{\lambda,e} = \frac{1}{1-\bar{\lambda}z}\big(Id - b(z)b(\lambda)^*\big)e \tag{2.2}$$

for the reproducing kernels in $\mathcal{H}(b)$. Also, it follows easily that

$$\|k_{\lambda,e}\|_2^2 = \frac{\|e\|^2}{1-|\lambda|^2}, \qquad \big\|k^b_{\lambda,e}\big\|_b^2 = \frac{\|e\|^2 - \|b(\lambda)^*e\|^2}{1-|\lambda|^2}. \tag{2.3}$$

We denote by $\kappa_{\lambda,e}$ and $\kappa^b_{\lambda,e}$ the normalized reproducing kernels of $H^2(E_*)$ and $\mathcal{H}(b)$ respectively; that is

$$\kappa_{\lambda,e}(z) = \frac{\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}z)\|e\|}e$$

and

$$\kappa^b_{\lambda,e}(z) = \frac{\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}z)\sqrt{\|e\|^2 - \|b(\lambda)^*e\|^2}}\big(Id - b(z)b(\lambda)^*\big)e.$$

We will also discuss properties of another interesting family of elements of the de Branges–Rovnyak space $\mathcal{H}(b)$: the so-called *difference quotients*, defined by

$$\hat{k}^b_{\lambda,e} = \frac{1}{z-\lambda}\big(b(z) - b(\lambda)\big)e, \quad \lambda \in \mathbb{D},\ e \in E. \tag{2.4}$$

3. A geometric approach to the de Branges–Rovnyak space

The function-theoretical approach, as developed for the scalar case in [24], is no more adequate when dealing with vector-valued de Branges–Rovnyak spaces. We will use a more geometric description, connected to the model theory for contractions. The main source for this point of view is [21] (see also [20,28], as well as the exposition of [19]).

We start with an *abstract functional embedding* (*AFE*). This is a linear mapping

$$\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K,$$

satisfying the following properties:

1. the restrictions π and π_* are isometries;
2. $\pi H^2(E) \perp \pi_* H^2_-(E_*)$;
3. the range of Π is dense in K;
4. $\pi_*^*\pi$ commutes with the shift operator and maps $H^2(E)$ into $H^2(E_*)$; hence we know (see [19, Lemma 1.2.3]) that $\pi_*^*\pi = b$, with b being a contractive $H^\infty(E \to E_*)$ function.

Note that, in contrast to [21], we do not include the purity of $\pi_*^*\pi$ in the definition (since we are not interested in the correspondence with the model contraction). It follows then easily that the operator U_Π defined by the relation $U_\Pi \Pi = \Pi z$ is unitary on K.

Set $\Delta = (Id - b^*b)^{1/2}$ and $\Delta_* = (Id - bb^*)^{1/2}$. Since

$$\|(\pi - \pi_* b)f\|_K = \|\Delta f\|_2, \qquad \|(\pi_* - \pi b^*)g\|_K = \|\Delta_* g\|_2,$$

for every $f \in L^2(E)$ and $g \in L^2(E_*)$, the equalities

$$\tau\Delta = \pi - \pi_* b, \qquad \tau_*\Delta_* = \pi_* - \pi b^*$$

determine the partial isometries

$$\tau : L^2(E) \longrightarrow K, \qquad \tau_* : L^2(E_*) \longrightarrow K$$

with initial spaces $\mathrm{clos}(\Delta L^2(E))$ and $\mathrm{clos}(\Delta_* L^2(E_*))$, respectively. It is easy to see that

$$\tau^*\pi = \Delta, \qquad \tau^*\pi_* = 0, \qquad \tau_*^*\pi = 0, \qquad \tau_*^*\pi_* = \Delta_*, \qquad \tau_*^*\tau = -b, \tag{3.1}$$

and

$$Id = \pi\pi^* + \tau_*\tau_*^* = \pi_*\pi_*^* + \tau\tau^*. \tag{3.2}$$

In particular, we get from (3.1) and (3.2) the following decompositions:

$$K = \pi(L^2(E)) \oplus \tau_*(L^2(E_*)) = \pi_*(L^2(E_*)) \oplus \tau(L^2(E)). \tag{3.3}$$

Conversely, if we start with b, a contractive $H^\infty(E \to E_*)$ function, then one can construct an AFE $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ such that $\pi_*^*\pi = b$ (see for instance the Sz.-Nagy–Foias or de Branges–Rovnyak transcriptions related to the construction of the model for contractions on Hilbert spaces [21]).

Now for a given AFE Π, we define $\mathbb{H} = K \ominus (\pi(H^2(E)) \oplus \pi_*(H^2_-(E_*))$; thus

$$K = \mathbb{H} \oplus \pi(H^2(E)) \oplus \pi_*(H^2_-(E_*)), \tag{3.4}$$

and

$$P_{\mathbb{H}} = Id - \pi P_+ \pi^* - \pi_* P_- \pi_*^*. \tag{3.5}$$

The space $\mathbb{H}$ is further decomposed as

$$\mathbb{H} = \mathbb{H}' \oplus \mathbb{H}'' = \mathbb{H}'_* \oplus \mathbb{H}''_*,$$

where $\mathbb{H}'' = \mathbb{H} \cap \tau(L^2(E)) = \mathbb{H} \cap \tau(\mathrm{clos}(\Delta L^2(E)))$, $\mathbb{H}' = \mathbb{H} \ominus \mathbb{H}''$, and $\mathbb{H}''_* = \mathbb{H} \cap \tau_*(L^2(E_*)) = \mathbb{H} \cap \tau_*(\mathrm{clos}(\Delta L^2(E_*)))$, $\mathbb{H}'_* = \mathbb{H} \ominus \mathbb{H}''_*$. Note also that (3.3) and (3.4) imply that actually $\mathbb{H}'' = \mathbb{H} \cap \pi_*(H^2(E_*))^\perp$ and $\mathbb{H}''_* = \mathbb{H} \cap \pi(H^2_-(E))^\perp$.

We will also denote, for further use,

$$\mathcal{R} = \mathrm{clos}(\Delta L^2(E)) \ominus \mathrm{clos}(\Delta H^2(E)), \qquad \mathcal{R}_* = \mathrm{clos}(\Delta_* L^2(E_*)) \ominus \mathrm{clos}(\Delta_* H^2_-(E_*)). \tag{3.6}$$

The following simple lemma will be used several times in the sequel.

Lemma 3.1. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^*\pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. Then we have*

$$\mathbb{H}'' = \tau(\mathcal{R}), \qquad \mathbb{H}''_* = \tau_*(\mathcal{R}_*).$$

Consequently, $\mathbb{H} = \mathbb{H}'$ if and only if $\mathrm{clos}(\Delta H^2(E)) = \mathrm{clos}(\Delta L^2(E))$, and $\mathbb{H} = \mathbb{H}'_*$ if and only if $\mathrm{clos}(\Delta_* H^2_-(E_*)) = \mathrm{clos}(\Delta_* L^2(E_*))$.

Proof. Suppose $\chi = \tau g$ with $g \in \mathrm{clos}(\Delta L^2(E))$. By (3.1), it follows that $\chi \perp \pi_* L^2(E_*)$; in particular, $\chi \perp \pi_* H^2_-(E_*)$. Then

$$\chi \in \mathbb{H}'' \iff \chi \in \mathbb{H} \iff \chi \perp \pi(H^2(E)).$$

Using again (3.1), one obtains that $\chi \perp \pi(H^2(E))$ is equivalent to

$$0 = \langle \tau g, \pi h\rangle = \langle g, \tau^*\pi h\rangle = \langle g, \Delta h\rangle$$

for all $h \in H^2(E)$, which proves the first assertion of the lemma. The second assertion follows from similar arguments, while for the last part we have only to remember that τ and τ_* are isometries on $\mathrm{clos}(\Delta L^2(E))$ and $\mathrm{clos}(\Delta_* L^2(E_*))$, respectively. □

In the end of this section we connect the abstract functional embeddings with the de Branges–Rovnyak spaces.

Lemma 3.2. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^* \pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. Then*

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}} \pi_* \big| H^2(E_*) = \pi_*^* P_{\mathbb{H}'} \pi_* \big| H^2(E_*).$$

Proof. Using (3.5) as well as the relations $\pi_*^* \pi_* = Id$ and $\pi_*^* \pi = b$, we obtain

$$\pi_*^* P_{\mathbb{H}} \pi_* = \pi_*^* \pi_* - \pi_*^* \pi P_+ \pi^* \pi_* - \pi_*^* \pi_* P_- \pi_*^* \pi_* = P_+ - b P_+ b^*,$$

whence

$$\pi_*^* P_{\mathbb{H}} \pi_* \big| H^2(E_*) = \big(P_+ - b P_+ b^*\big)\big| H^2(E_*) = Id - T_b T_b^*.$$

The second equality follows since $\mathbb{H} \ominus \mathbb{H}' = \mathbb{H}'' = \tau(\mathcal{R})$ is contained in the kernel of π_*^* according to (3.1). □

Proposition 3.3. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^* \pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. The operator π_*^* is a coisometry from $\mathbb{H}$ onto $\mathcal{H}(b)$, with $\ker \pi_*^* | \mathbb{H} = \mathbb{H}''$. In particular, if $\operatorname{clos}(\Delta L^2(E)) = \operatorname{clos}(\Delta H^2(E))$, then $\pi_*^* : \mathbb{H} \to \mathcal{H}(b)$ is unitary.*

Proof. First, using (3.4), we see that $\pi_*^* \mathbb{H} \subset H^2(E_*)$. Then, if we put $A = \pi_*^* | \mathbb{H} : \mathbb{H} \to H^2(E_*)$ and $B = (Id - T_b T_b^*)^{1/2}$, Lemma 3.2 shows that $AA^* = B^2$. There exists therefore a partial isometry $U : \mathbb{H} \to H^2(E_*)$, with initial space $(\ker A)^\perp$ and final space the closure of the range of B, such that $A = BU$. The definition of the norm of $\mathcal{H}(b)$ implies then that $A = \pi_*^* | \mathbb{H}$ is a partial isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$. Its kernel is

$$\ker \pi_*^* \big| \mathbb{H} = \mathbb{H} \cap \ker \pi_*^* = \mathbb{H} \cap (\operatorname{Im} \pi_*)^\perp = \mathbb{H}'',$$

whence the proof is ended using Lemma 3.1. □

Remark 3.4. In [20] (see also [18, pp. 84–86]), some conditions equivalent to $\operatorname{clos}(\Delta H^2(E)) = \operatorname{clos}(\Delta L^2(E))$ are given; in particular, one of them is the density of the polynomials in $L^2(E, \Delta)$. In [26], S. Treil shows that b is an extreme point in the unit ball of $H^\infty(E \to E_*)$ if and only if either $\operatorname{clos}(\Delta H^2(E)) = \operatorname{clos}(\Delta L^2(E))$ or $\operatorname{clos}(\Delta_* H^2_-(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$, which is equivalent (according to Lemma 3.1) to $\mathbb{H} = \mathbb{H}'$ or $\mathbb{H} = \mathbb{H}'_*$. In the scalar case $\dim E = \dim E_* = 1$, b is extreme if and only if $\log(1 - |b|)$ is not integrable on $\mathbb{T}$ (see [7]).

4. Riesz bases of reproducing kernels

The main problems that we intend to study are the following: given $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$, given a sequence $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and a sequence $(e_n)_{n\geqslant 1} \subset E_*$, $\|e_n\| = 1$, $n \geqslant 1$, find criteria for the sequence $(\kappa^b_{\lambda_n, e_n})_{n\geqslant 1}$ to form

(P1) a Riesz basis of its closed linear hull;
(P2) a Riesz basis of $\mathcal{H}(b)$.

We will not, however, study these problems in the most general form. First, note that if $\dim E_* = +\infty$ and $(e_n)_{n\geqslant 1}$ is an orthonormal sequence in E_*, then $(\kappa_{\lambda_n, e_n})_{n\geqslant 1}$ is an orthonormal sequence in $H^2(E_*)$, for any choice of sequence $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{D}$. In some sense, if E_* is an infinite dimensional Hilbert space, there is too much freedom for the vectors e_n to hope to get a satisfactory criterion for Riesz basis. That is why interesting results have usually been obtained under the condition $\dim E_* < +\infty$ (see [1,27]). This condition will be assumed in this section.

Secondly, it is easy to see that if $(\kappa^b_{\lambda_n, e_n})_{n\geqslant 1}$ is a Riesz basis, then $(\kappa_{\lambda_n, e_n})_{n\geqslant 1}$ is minimal, which implies that $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence [1, pp. 65–67]. Therefore we will also suppose, in the sequel, that the sequence $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$. We have then $\operatorname{span}(\kappa_{\lambda_n, e_n}\colon\ n \geqslant 1) = H^2(E_*) \ominus B H^2(E_*) = K_B$, where the inner function $B \in H^\infty(E_* \to E_*)$ is a Blaschke–Potapov product (see, for instance, [15]).

At this point, the technique originating in [16] (and which is used in [9,11]) regards the family $(\kappa^b_{\lambda_n, e_n})_{n\geqslant 1}$ as a "distortion" of $(\kappa_{\lambda_n, e_n})_{n\geqslant 1}$. It is then assumed that $Id - T_b T_b^*$ does not distort very much the norms of the reproducing kernels, in the sense that

$$\sup_{n\geqslant 1} \frac{\|k_{\lambda_n, e_n}\|_2}{\|(Id - T_b T_b^*) k_{\lambda_n, e_n}\|_b} < +\infty.$$

Using (2.3), we see that this condition is equivalent to

$$\sup_{n\geqslant 1} \big\| b(\lambda_n)^* e_n \big\| < 1. \tag{4.1}$$

Under this further condition, we can state the following result.

Theorem 4.1. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$, let $(\lambda_n)_{n\geqslant 1}$ be a Blaschke sequence in $\mathbb{D}$ and let $(e_n)_{n\geqslant 1} \subset E_*$, $\|e_n\| = 1$. Assume that $\dim E_* < +\infty$ and that condition (4.1) is satisfied. Then the following are equivalent*:

(a) *the sequence $(\kappa^b_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$);*
(b) *the sequence $(\kappa_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of K_B and the operator*

$$\left(Id - T_b T_b^*\right)\big| K_B : K_B \longrightarrow \mathcal{H}(b)$$

is an isomorphism onto its range (resp. onto $\mathcal{H}(b)$).

Proof. (a) $\Rightarrow$ (b) By formula (2.2) we have $(Id - T_b T_b^*)(k_{\lambda_n,e_n}) = k^b_{\lambda_n,e_n}$ and condition (4.1) implies that $\|k^b_{\lambda_n,e_n}\|_b \asymp \|k_{\lambda_n,e_n}\|_2$. It follows then (see, for instance, [13, p. 228]) that the uniform minimality of $(k^b_{\lambda_n,e_n})_{n\geqslant 1}$ implies the uniform minimality of $(k_{\lambda_n,e_n})_{n\geqslant 1}$. But, according to a result of S. Treil [27], since $\dim E_* < \infty$, the latter is equivalent to the fact that the sequence $(\kappa_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of K_B. Since the operator $(Id - T_b T_b^*)|K_B$ maps one Riesz basis onto another, it is an isomorphism of K_B onto $\operatorname{span}(\kappa^b_{\lambda_n,e_n}\colon n \geqslant 1)$.

(b) $\Rightarrow$ (a) Conversely, if $(Id - T_b T_b^*)|K_B$ is an isomorphism onto its range and the sequence $(\kappa_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of K_B, then $((Id - T_b T_b^*)\kappa_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of its closed linear hull. But

$$\left(Id - T_b T_b^*\right)\kappa_{\lambda_n,e_n} = \frac{\|k^b_{\lambda_n,e_n}\|_b}{\|k_{\lambda_n,e_n}\|_2}\kappa^b_{\lambda_n,e_n},$$

and since $\frac{\|k^b_{\lambda_n,e_n}\|_b}{\|k_{\lambda_n,e_n}\|_2}$ is bounded from below and above, we obtain that the sequence $(\kappa^b_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of its closed linear hull. Moreover, if $(Id - T_b T_b^*)|K_B$ is an isomorphism onto $\mathcal{H}(b)$, we have

$$\operatorname{span}\left(\kappa^b_{\lambda_n,e_n}\colon n \geqslant 1\right) = \operatorname{span}\left(\left(Id - T_b T_b^*\right)k_{\lambda_n,e_n}\colon n \geqslant 1\right) = \mathcal{H}(b). \qquad \square$$

Remark 4.2. Until now the elaborated theory [9,11,16] works under condition (4.1) only. This is not surprising in view of the method used, which is based on projecting a basis from K_B; therefore the first thing to require is that the size of the individual elements of the base should not be changed too drastically. In the particular case of exponential families, this condition means that the imaginary parts of the frequencies of the exponentials are bounded below, which is the case for families arising from control theory. It should be mentioned however that [2] gives certain criteria for a family of reproducing kernels to be a Riesz basis in a model subspace associated to a meromorphic inner function, without using the assumption (4.1).

Remark 4.3. Theorem 4.1 reduces the problem of finding Riesz bases in $\mathcal{H}(b)$ to the case of K_B. To apply it, we should be able first to decide when a reproducing sequence of kernels in $H^2(E_*)$ forms a Riesz sequence. Such a criterion has been given by S. Ivanov (see [1, p. 73]); we need some further notations in order to state it.

We define, for $\lambda \in \mathbb{D}$ and $r > 0$, the pseudo-hyperbolic disc

$$\omega(\lambda, r) := \left\{z \in \mathbb{D}\colon \left|b_\lambda(z)\right| < r\right\}, \quad \text{where } b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Then, for a sequence $\Lambda = (\lambda_n)_{n\geqslant 1}$ in $\mathbb{D}$, we set

$$G(\Lambda, r) = \bigcup_{n\geqslant 1} \omega(\lambda_n, r).$$

For $m \geqslant 1$, we denote by $G_m(\Lambda, r)$ the connected components of the set $G(\Lambda, r)$ and we write

$$E_m(r) := \left\{n \geqslant 1\colon \lambda_n \in G_m(\Lambda, r)\right\}.$$

Then the sequence $(\kappa_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of its closed linear hull if and only if the two following conditions are satisfied:

(a) the sequence $(\lambda_n)_{n\geqslant 1}$ is the union of at most $\dim E_*$ Carleson sets;
(b) there exists $r > 0$ such that

$$\inf_{m\geqslant 1} \min_{n\in E_m(r)} \alpha\left(e_n, \operatorname{span}\left(e_p\colon p \in E_m(r),\ p \neq n\right)\right) > 0,$$

where $\alpha(e_n, Y)$ denotes the angle between the vector e_n and the subspace Y.

We will call $(Id - T_b T_b^*)|K_B : K_B \to \mathcal{H}(b)$ the *distortion operator*. By Theorem 4.1 and Remark 4.3, problems (P1) and (P2) are reduced, in case condition (4.1) is satisfied, to the following: find criteria for the distortion operator to be

(P1′) an isomorphism onto its range;
(P2′) an isomorphism onto $\mathcal{H}(b)$.

These problems will be addressed in the next section.

We end this section by stating a stability result. The proof is similar to the analogous result for model subspaces (i.e. the inner case) obtained in [11, Theorem 3.4], and will therefore be omitted.

Theorem 4.4. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$, $\Lambda = (\lambda_n)_{n\geqslant 1}$ be a Blaschke sequence in $\mathbb{D}$ and let $(e_n)_{n\geqslant 1} \subset E_*$, $\|e_n\| = 1$ such that the sequence $(\kappa^b_{\lambda_n,e_n})_{n\geqslant 1}$ is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$). Assume that $\dim E_* < +\infty$ and that condition (4.1) is satisfied. Then there exists $\varepsilon > 0$ such that any sequence $(\kappa^b_{\mu_n,a_n})_{n\geqslant 1}$ satisfying*

$$\left|b_{\lambda_n}(\mu_n)\right| \leqslant \varepsilon \quad \text{and} \quad \|a_n - e_n\| \leqslant \varepsilon, \quad n \geqslant 1,$$

is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$).

Let us also mention that in the scalar de Branges–Rovnyak spaces, using a different approach based on Bernstein type inequalities, a stability result was found in [3] without the assumption (4.1). However, the techniques used therein do not seem adaptable to the vector case.

5. The distortion operator

We will discuss in this section the invertibility of the distortion operator $(Id - T_b T_b^*)|K_\Theta : K_\Theta \to \mathcal{H}(b)$, for a general inner function $\Theta \in H^\infty(F \to E_*)$ and $b \in H^\infty(E \to E_*)$ contractive. As noted above, the methods used in the scalar case in [11,13] and in the inner vector case in [9], are no more appropriate, and we have to use a different approach, based on the AFE introduced in Section 3.

We start by reminding a simple lemma, whose proof we omit.

Lemma 5.1. *Suppose we have two orthogonal decompositions of a Hilbert space $\mathfrak{H}$:*

$$\mathfrak{H} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 = \mathfrak{Y}_1 \oplus \mathfrak{Y}_2.$$

Then the following statements are all equivalent:

(1) *$P_{\mathfrak{Y}_1}|\mathfrak{X}_1$ is surjective.*
(2) *$P_{\mathfrak{X}_1}|\mathfrak{Y}_1$ is bounded below.*
(3) *$\|P_{\mathfrak{X}_2}|\mathfrak{Y}_1\| < 1$.*
(4) *$\|P_{\mathfrak{Y}_1}|\mathfrak{X}_2\| < 1$.*
(5) *$P_{\mathfrak{Y}_2}|\mathfrak{X}_2$ is bounded below.*
(6) *$P_{\mathfrak{X}_2}|\mathfrak{Y}_2$ is surjective.*

Here for a (closed) subspace E of $\mathfrak{H}$, the notation P_E denotes the orthogonal projection of $\mathfrak{H}$ onto E.

The first result gives the answer to problem (P1′).

Theorem 5.2. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$ and let $\Theta \in H^\infty(F \to E_*)$ be an inner function. The distortion operator is an isomorphism onto its range if and only if $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$.*

Proof. Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE such that $\pi_*^* \pi = b$. Recall that, according to Lemma 3.2, we have

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}} \pi_* \big| H^2(E_*).$$

Moreover, by Proposition 3.3, π_*^* is a partial isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$ with kernel equal to $\mathbb{H}''$. Since $P_{\mathbb{H}} \pi_* L^2(E_*) \subset (\ker \pi_*^*|\mathbb{H})^\perp$, we have that $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is an isomorphism onto its range if and only if $P_{\mathbb{H}}|\pi_* K_\Theta$ is bounded below. Applying Lemma 5.1, this last assertion is equivalent to

$$\|P_{K\ominus\mathbb{H}}|\pi_* K_\Theta\| < 1.$$

Now, $K \ominus \mathbb{H} = \pi(H^2(E)) \oplus \pi_*(H^2_-(E_*))$, and the second term in the orthogonal sum is orthogonal to $\pi_* K_\Theta$. Thus the condition is equivalent to $\|P_{\pi(H^2(E))}|\pi_* K_\Theta\| < 1$, or, passing to the adjoint, $\|P_{\pi_* K_\Theta}|\pi(H^2(E))\| < 1$.

But we obviously have

$$\left\| P_{\pi_* K_\Theta} \middle| \pi\left(H^2(E)\right) \right\| = \left\| \pi_* P_{K_\Theta} \pi_*^* \pi \middle| H^2(E) \right\| = \left\| P_{K_\Theta} b \middle| H^2(E) \right\|,$$

while, using the vector valued Nehari Theorem,

$$\left\| P_{K_\Theta} b \middle| H^2(E) \right\| = \left\| \Theta P_- \Theta^* b \middle| H^2(E) \right\| = \| \mathrm{H}_{\Theta^* b} \| = \operatorname{dist}\left(\Theta^* b, H^\infty(E \to F)\right).$$

This string of equalities proves the theorem. $\square$

The next theorem is an answer to problem (P2′); it is not, however, as explicit as the answer to problem (P1′).

Theorem 5.3. *The distortion operator is an isomorphism onto* $\mathcal{H}(b)$ *if and only if* $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$ *and the operator*

$$\Gamma_b := \begin{pmatrix} P_+ b^* \Theta & P_+ \Delta \end{pmatrix} : \begin{matrix} H^2(F) \\ \oplus \\ \operatorname{clos}\left(\Delta H^2(E)\right) \end{matrix} \longrightarrow H^2(E)$$

is bounded below.

Proof. Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE such that $\pi_*^* \pi = b$. It follows from Proposition 3.3 that the operator π_*^* is an isometry from $\mathbb{H}'$ onto $\mathcal{H}(b)$; therefore, using Lemma 3.2, we get that

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}'} \pi_* \middle| H^2(E_*),$$

and $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is an isomorphism onto $\mathcal{H}(b)$ if and only if $P_{\mathbb{H}'} | \pi_* K_\Theta$ is bounded below and surjective. According to Theorem 5.2, $P_{\mathbb{H}'} | \pi_* K_\Theta$ is bounded below if and only if $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$; thus it remains to show that $P_{\mathbb{H}'} | \pi_* K_\Theta$ is surjective if and only if Γ_b is bounded below.

Now, since

$$\mathbb{H}' \oplus \pi_*\left(H^2_-(E_*)\right) = K \ominus \left[\pi\left(H^2(E)\right) \oplus \mathbb{H}''\right],$$
$$\pi_*(K_\Theta) \oplus \pi_*\left(H^2_-(E_*)\right) = \pi_*\left(L^2(E_*)\right) \ominus \pi_*\left(\Theta H^2(F)\right),$$

it follows that $P_{\mathbb{H}'} | \pi_* K_\Theta$ is surjective if and only if $P_{K \ominus [\pi(H^2(E)) \oplus \mathbb{H}'']} | \pi_*(L^2(E_*)) \ominus \pi_*(\Theta H^2(F))$ is surjective. Applying then Lemma 5.1 to the case $\mathfrak{X}_1 = \pi_*(L^2(E_*)) \ominus \pi_*(\Theta H^2(F))$, $\mathfrak{X}_2 = \pi_*(\Theta H^2(F)) \oplus \pi_*(L^2(E_*))^\perp$, $\mathfrak{Y}_1 = K \ominus [\pi(H^2(E)) \oplus \mathbb{H}'']$, $\mathfrak{Y}_2 = [\pi(H^2(E)) \oplus \mathbb{H}'']$, the surjectivity of $P_{\mathfrak{Y}_1} | \mathfrak{X}_1$ is equivalent to $P_{\mathfrak{Y}_2} | \mathfrak{X}_2$ bounded below.

Since $\mathbb{H}'' \subset \pi_*(L^2(E_*))^\perp$, this last condition is equivalent to

$$P_{\pi(H^2(E))} | \pi_*\left(\Theta H^2(F)\right) \oplus \left[\pi_*\left(L^2(E_*)\right)^\perp \ominus \mathbb{H}''\right]$$

bounded below. Now we note that $P_{\pi H^2(E)} = \pi P_+ \pi^*$ and, according to (3.3) and Lemma 3.1, we have

$$\pi_*\left(L^2(E_*)\right)^\perp \ominus \mathbb{H}'' = \tau\left(L^2(E)\right) \ominus \mathbb{H}'' = \tau\left(\operatorname{clos}\left(\Delta L^2(E)\right)\right) \ominus \mathbb{H}'' = \tau\left(\operatorname{clos}\left(\Delta H^2(E)\right)\right).$$

Therefore, $P_{\mathbb{H}'} | \pi_* K_\Theta$ is surjective if and only if

$$\pi P_+ \pi^* (\pi_* \Theta \quad \tau) : H^2(F) \oplus \operatorname{clos}\left(\Delta H^2(E)\right) \longrightarrow K$$

is bounded below. But it follows from (3.1) that

$$\pi P_+ \pi^* (\pi_* \Theta \quad \tau) = \pi \left(P_+ \pi^* \pi_* \Theta \quad P_+ \pi^* \tau\right) = \pi \left(P_+ b^* \Theta \quad P_+ \Delta\right) = \pi \Gamma_b.$$

Since π is an isometry, we obtain the desired conclusion. $\square$

Theorem 5.3 may be compared to a basic result in the scalar case, namely the *Theorem on Close Subspaces* in [17, p. 201] (and its complement on p. 204), where conditions are given for the projection from K_θ to $K_{\theta'}$ to be an isomorphism (θ, θ' scalar inner functions). For instance, an equivalent condition therein is the invertibility of the scalar Toeplitz operator $T_{\theta \bar{\theta}'}$. Although the necessary and sufficient condition obtained above may be hard to check in practice (see, however, Example 7.3), it leads to several useful corollaries. They show that the invertibility of the distortion operator often implies strong conditions on the function b.

Corollary 5.4. *If* $\operatorname{clos}(\Delta H^2(E)) = \operatorname{clos}(\Delta L^2(E))$ *and the distortion operator is invertible, then b is inner.*

Proof. If Γ_b is bounded below, then, for any $f \in \operatorname{clos}(\Delta L^2(E)) = \operatorname{clos}(\Delta H^2(E))$, we have

$$c\|f\|_2 = c\|\bar{z}^n f\|_2 \leqslant \|\Gamma_b(0 \oplus \bar{z}^n f)\|_2 = \|P_+ \Delta \bar{z}^n f\|_2 = \|P_+ \bar{z}^n \Delta f\|_2.$$

Since the right side of the last inequality tends to 0 as $n \to +\infty$, we obtain $\operatorname{clos}(\Delta L^2(E)) = \{0\}$, which is equivalent to b inner. □

An interesting result can be obtained in the case the inner function Θ has full range.

Corollary 5.5. *Suppose* $\dim F = \dim E_* < +\infty$. *If the distortion operator is invertible, then b is $*$-inner.*
If also $\dim E = \dim E_*$, *then b is inner.*

Proof. If Γ_b is bounded below, then $(b^*\Theta \quad \Delta)$ is bounded below as an operator from $H^2(F) \oplus \operatorname{clos}(\Delta H^2(E))$ to $L^2(E)$. Since for any $f \in L^2$ and $\epsilon > 0$ one can find $g \in H^2$ and $N \in \mathbb{N}$ such that $\|z^N f - g\|_2 < \epsilon$, a standard argument shows that $(b^*\Theta \quad \Delta)$ is bounded below from $L^2(F) \oplus \operatorname{clos}(\Delta L^2(E))$ to $L^2(E)$. If $\dim F = \dim E_* < +\infty$, then Θ inner implies that $\Theta L^2(F) = L^2(E_*)$, and thus

$$\begin{pmatrix} b^* & \Delta \end{pmatrix} : \begin{array}{c} L^2(E_*) \\ \oplus \\ \operatorname{clos}(\Delta L^2(E)) \end{array} \longrightarrow L^2(E)$$

is bounded below. But the adjoint of this last operator is an isometry. Since a coisometry that is bounded below is necessarily unitary, it is easily seen that multiplication with b must be a coisometry from $L^2(E)$ to $L^2(E_*)$, whence b is $*$-inner. The last assertion is then obvious. □

In particular, Corollary 5.5 can be applied to our original problem, namely the Riesz property of a family of reproducing kernels in $\mathcal{H}(b)$. Indeed, in that case the inner function Θ is actually a Blaschke–Potapov product corresponding to a Blaschke sequence (λ_n), which verifies the condition $\dim F = \dim E_*$, and we have assumed that $\dim E_* < +\infty$.

Corollary 5.6. *Suppose* $\dim E = \dim E_* = 1$. *Then the distortion operator is invertible exactly in the two following cases*:

(i) *b is inner,* $\operatorname{dist}(\bar{\Theta} b, H^\infty) < 1$ *and* $\operatorname{dist}(\bar{b}\Theta, H^\infty) < 1$.
(ii) $F = \{0\}$ *and* $\|b\|_\infty < 1$.

Proof. From Corollary 5.5 it follows that the invertibility of the distortion operator implies either b inner or $F = \{0\}$. If b is inner, then the conditions in (i) are known to be equivalent to the invertibility of the distortion operator (see, for instance, [17]). If $\{F\} = \{0\}$, then $K_\Theta = H^2$, and the invertibility of the distortion operator is equivalent to $\|b\|_\infty < 1$. □

Remark 5.7. Provided condition (4.1) is satisfied, Corollary 5.6 generalizes Proposition 5.1 in [11], where it is shown that in the scalar nonextreme case we cannot have bases of reproducing kernels for $\mathcal{H}(b)$.

The next corollary discusses the connection between different conditions that are related to our original problem.

Corollary 5.8. *Consider the assertions*:

(i) *the operator* $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ *is invertible*;
(ii) $\operatorname{dist}(\Theta^* b, H^\infty(E, F)) < 1$ *and* $T_{b^*\Theta}$ *is left invertible*;
(iii) $\operatorname{dist}(\Theta^* b, H^\infty(E, F)) < 1$ *and* $\operatorname{dist}(b^*\Theta, H^\infty(E, F)) < 1$.

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
If b is inner ($b^* b = Id$), *then* (ii) $\Rightarrow$ (i), *while, if b is $*$-inner, then* (iii) $\Rightarrow$ (ii).

Proof. For $f \in H^2(F)$ we have

$$\|\Gamma_b(f \oplus 0)\|_2 = \|T_{b^*\Theta} f\|_2 \tag{5.1}$$

and (i) $\Rightarrow$ (ii) follows immediately from Theorem 5.3. If b is inner, then $\Gamma_b = T_{b^*\Theta}$ and we use again Theorem 5.3 to conclude that (ii) $\Rightarrow$ (i).

Since we have, by (2.1),

$$\|f\|_2 \geqslant \|b^*\Theta f\|_2^2 = \|T_{b^*\Theta} f\|_2^2 + \|\mathrm{H}_{b^*\Theta} f\|_2^2,$$

the vector valued Nehari Theorem yields (ii) $\Rightarrow$ (iii). If b is $*$-inner, the first inequality becomes an equality, giving the converse. $\square$

In general, none of the implications in Corollary 5.8 can be reversed; examples will be given in Section 7.

6. A different characterization

In the scalar case, another equivalent condition for the orthogonal projection from K_θ to $K_{\theta'}$ to be an isomorphism is given by the two relations $\operatorname{dist}(\theta'\bar{\theta}, H^\infty) < 1$, $\operatorname{dist}(\bar{z}\theta'\bar{\theta}, H^\infty) = 1$. A similar condition for scalar de Branges spaces appears in [11, Theorem 4.1]. We will obtain below an alternate answer along that line to problem (P2′); however, the formulation in the case of vector-valued de Branges–Rovnyak spaces is less elegant.

Some notations are needed: for any $x \in F$, let P_x be the orthogonal projection onto the subspace generated by x and define $\theta_x \in H^\infty(F \to F)$ by $\theta_x(z) := zP_x + (Id - P_x)$. It is immediate that θ_x is an inner function in $H^\infty(F \to F)$ and we have $K_{\theta_x} = \mathbb{C}x$ (the subspace of constant functions equal to a multiple of x).

Proposition 6.1. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$ and let $\Theta \in H^\infty(F \to E_*)$ be an inner function. Assume that the operator $Id - T_bT_b^* : K_\Theta \to \mathcal{H}(b)$ is left invertible. Then the following assertions are equivalent*:

(i) *$Id - T_bT_b^*$: $K_\Theta \to \mathcal{H}(b)$ is an isomorphism*;
(ii) *for all $x \in F$, we have $\operatorname{dist}(\theta_x^*\Theta^*b, H^\infty(E \to F)) = 1$.*

Proof. Once more, we will use an AFE $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ such that $\pi_*^*\pi = b$. By Lemma 3.2 and Proposition 3.3, the assertion (i) is equivalent to the invertibility of $P_{\mathbb{H}'}\pi_*|K_\Theta : K_\Theta \to \mathbb{H}'$, while from Theorem 5.2 it follows that

$$\text{(ii)} \quad \Longleftrightarrow \quad \text{for all } x \in F, \quad P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x} : K_{\Theta\theta_x} \to \mathbb{H}' \text{ is not left invertible.}$$

We will also use repeatedly the equality

$$K_{\Theta\theta_x} = K_\Theta \oplus \Theta K_{\theta_x} = K_\Theta \oplus \mathbb{C}\Theta x. \tag{6.1}$$

(i) $\Rightarrow$ (ii) Since (6.1) implies $K_\Theta \subsetneq K_{\Theta\theta_x}$, it follows that if $P_{\mathbb{H}'}\pi_*|K_\Theta$ is invertible, then $P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x}$ is not one-to-one. Therefore it cannot be left invertible.

(ii) $\Rightarrow$ (i) We argue by contradiction, assuming that $P_{\mathbb{H}'}\pi_*|K_\Theta$ is not invertible. Since this operator is left invertible, that means that $P_{\mathbb{H}'}\pi_*K_\Theta$ is not dense in $\mathbb{H}'$; there exists thus $\chi \in \mathbb{H}'$, $\chi \neq 0$ such that $\chi \perp \pi_*K_\Theta$.

Since $P_{\mathbb{H}'}\pi_*|K_\Theta$ is left invertible and $P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x}$ is not, this last operator is not one-to-one, and we may choose $g_x \in K_{\Theta\theta_x} \setminus K_\Theta$ such that $P_{\mathbb{H}'}\pi_*g_x = 0$. Since $g_x \in K_{\Theta\theta_x} \subset H^2(E_*)$, it follows that $\pi_*g_x \in \pi_*H^2(E_*) \subset (\pi_*H^2_-(E_*))^\perp = \mathbb{H} \oplus \pi H^2(E)$. However, by definition we have $\pi_*g_x \in \mathbb{H}'^\perp$, while, using (3.3) and the definition of $\mathbb{H}''$, we also have $\pi_*g_x \in \mathbb{H}''^\perp$. Therefore $\pi_*g_x \in \mathbb{H}^\perp$, whence $\pi_*g_x \in \pi(H^2(E))$. But the space $\pi H^2(E)$ is U_Π-invariant, which implies that $U_\Pi^k\pi_*g_x \in \pi H^2(E)$. In particular, we have $\pi_*(z^kg_x) = U_\Pi^k(\pi_*g_x) \perp \chi$ (since $\pi(H^2(E)) \perp \mathbb{H}'$).

We claim now that

$$\operatorname{span}\big(K_\Theta, z^kg_x\colon\ k \geqslant 0,\ x \in F\big) = H^2(E_*). \tag{6.2}$$

To prove it, let $f \in H^2(E_*)$ and assume that $f \perp \operatorname{span}(K_\Theta, z^kg_x\colon\ k \geqslant 0,\ x \in F)$. Since $f \perp K_\Theta$, there exists $f_1 \in H^2(F)$ such that $f = \Theta f_1$. We will show by induction that for all $k \geqslant 0$, $f_1^{(k)}(0) = 0$, which of course will imply that $f_1 \equiv 0$, and thus the truth of (6.2).

First, by (6.1) there exist $g_x^\Theta \in K_\Theta$ and $\lambda_x \in \mathbb{C}^*$ such that

$$g_x = g_x^\Theta + \lambda_x\Theta x.$$

We have then

$$0 = \langle f, g_x\rangle = \big\langle \Theta f_1, g_x^\Theta + \lambda_x\Theta x\big\rangle_2 = \bar{\lambda}_x\langle f_1, x\rangle_2 = \overline{\lambda_x}\big\langle f_1(0), x\big\rangle_F.$$

Since $\lambda_x \neq 0$, this implies that $\langle f_1(0), x\rangle_F = 0$ for all $x \in F$, whence $f_1(0) = 0$.

Assume now that $f_1^{(k)}(0) = 0$. This means that there exists $f_{k+2} \in H^2(F)$ such that $f_1 = z^{k+1}f_{k+2}$. Therefore

$$0 = \big\langle f, z^{k+1}g_x\big\rangle = \big\langle \Theta z^{k+1}f_2, z^{k+1}g_x\big\rangle_2 = \langle \Theta f_{k+2}, g_x\rangle_2.$$

As before we deduce that $f_{k+2}(0) = 0$, which implies that $f_1^{(k+1)}(0) = 0$. The property for f_1 follows now by induction, concluding the proof of (6.2).

Since π_* is an isometry, (6.2) implies that

$$\operatorname{span}\big(\pi_* K_\Theta, \pi_*\big(z^k g_x\big):\ k \geqslant 0,\ x \in F\big) = \pi_* H^2(E_*).$$

Recall that by construction $\chi \perp \pi_* K_\Theta$, while we have shown that $\chi \perp \pi_*(z^k g_x)$, for all $k \geqslant 0$ and for all $x \in F$. Consequently $\chi \perp \pi_* H^2(E_*)$. On the other hand, since $\chi \in \mathbb{H}'$, we also have $\chi \perp \pi_* H^2_-(E_*)$, whence $\chi \perp \pi_* L^2(E_*)$. Finally, we obtain that $\chi \in \mathbb{H} \cap (\pi_* L^2(E_*))^\perp = \mathbb{H}''$. Therefore $\chi \in \mathbb{H}' \cap \mathbb{H}'' = \{0\}$ which is absurd and ends the proof of the proposition. □

The next result is then a consequence of Theorem 5.2 and Proposition 6.1.

Theorem 6.2. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$ and let $\Theta \in H^\infty(F \to E_*)$ be an inner function. Then the operator $(Id - T_b T_b^*)|K_\Theta$ is an isomorphism from K_Θ onto $\mathcal{H}(b)$ if and only if*

$$\begin{cases} \operatorname{dist}\big(\Theta^* b, H^\infty(E \to F)\big) < 1, \\ \operatorname{dist}\big(\theta_x^* \Theta^* b, H^\infty(E \to F)\big) = 1, \quad \forall x \in F. \end{cases}$$

In the scalar case $\dim E = \dim E_* = \dim F = 1$, we have $\theta_x(z) = z$; thus Theorem 6.2 generalizes part of [11, Theorem 4.1] and of the theorem on Close Subspaces in [17].

7. Some examples and remarks

The first two examples show that the two implications in Corollary 5.8 cannot be reversed even in the scalar case $\dim E = \dim E_* = \dim F = 1$.

Example 7.1. Define

$$f\big(e^{i\vartheta}\big) := \begin{cases} 1 & \text{if } \vartheta \in [0, \pi], \\ 1/2 & \text{if } \vartheta \in]\pi, 2\pi[, \end{cases}$$

and consider the outer function g, positive at the origin and with modulus equal to $|f|$ a.e. on $\mathbb{T}$. Set $b = \Theta g$, where Θ is any inner function. Since $\log(1 - |b|)$ is not integrable, b is an extreme point of the unit ball of H^∞. Since $f, f^{-1} \in L^\infty$, it is immediate that g is invertible in H^∞, whence $T_{\bar{b}\Theta} = T_{\bar{g}} = T_g^*$ is invertible. Also, $\bar{\Theta} b = g \in H^\infty$, so $\operatorname{dist}(\bar{\Theta} b, H^\infty) = 0 < 1$.

On the other hand, Corollary 5.6 shows that the distortion operator $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ cannot be invertible. Consequently, the implication (i) ⇒ (ii) in Corollary 5.8 cannot be reversed.

Example 7.2. Let $h : \mathbb{D} \to \mathbb{D}$ be the conformal transform of the disk $\mathbb{D}$ onto the simply connected domain

$$\Omega = \left\{ z \in \mathbb{C}:\ |z| < 1,\ -\frac{1}{4} < \Re\mathrm{e}\, z < 0 \right\}.$$

If we regard h as an element of H^∞, then $|h(e^{it})| = 1$ on an arc of positive measure, while 0 is in the essential range of h; also, $|\Re\mathrm{e}\, h| \leqslant \frac{1}{4}$ everywhere.

Let Θ be an arbitrary inner function, and define $b = \Theta h$; b is then an extreme function. Since $\bar{\Theta} b = h \in H^\infty$, $\operatorname{dist}(\bar{\Theta} b, H^\infty) = 0 < 1$. Also, $\bar{b}\Theta = \bar{h}$; since $|\bar{h} + h| = 2|\Re\mathrm{e}\, h| \leqslant 1/2$ everywhere, it follows that $\operatorname{dist}(\bar{b}\Theta, H^\infty) \leqslant 1/2 < 1$. Thus condition (iii) of Corollary 5.8 is satisfied.

On the other hand, 0 is in the essential range of $\bar{b}\Theta = h$; it is then known (see, for instance, [18, B.4.2]) that $T_{\bar{b}\Theta}$ cannot be left invertible. Thus the implication (ii) ⇒ (iii) in Corollary 5.8 cannot be reversed.

As shown in Corollaries 5.4–5.6, the invertibility of the distortion operator often implies the already studied case of b inner. It is therefore interesting to see some concrete examples when this does not happen.

Example 7.3. This is a simple case when b is neither inner nor $*$-inner, with both $\mathcal{H}(b)$ and K_Θ infinite dimensional.

Let $\alpha, \beta \in \mathbb{R}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$, $\alpha > \beta$ and $\alpha\beta > -1/2$ and let θ be a scalar inner function. Then take $b(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{z}{\sqrt{2}} & \frac{z}{2\sqrt{2}} \end{pmatrix}$ and $\Theta(z) = \begin{pmatrix} \alpha\theta \\ z\beta\theta \end{pmatrix}$. It is easy to check that b is in the unit ball of $H^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$ and Θ is an inner function in $H^\infty(\mathbb{C} \to \mathbb{C}^2)$. Moreover straightforward computations show that

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad b^* \Theta = \begin{pmatrix} \frac{(\alpha-\beta)\theta}{\sqrt{2}} \\ \frac{(\alpha+\beta)\theta}{2\sqrt{2}} \end{pmatrix},$$

whence $\mathrm{clos}(\Delta H^2(\mathbb{C}^2)) = H^2$ and

$$\Gamma_b = \begin{pmatrix} T_{\frac{(\alpha-\beta)\theta}{\sqrt{2}}} & 0 \\ T_{\frac{(\alpha+\beta)\theta}{2\sqrt{2}}} & \frac{\sqrt{3}}{2} \end{pmatrix} : \begin{matrix} H^2 \\ \oplus \\ H^2 \end{matrix} \to H^2(\mathbb{C}^2).$$

Now it is well known that if $T := \left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$ then T is bounded below if A and C are bounded below. But since $\alpha > \beta$, it is clear that $T_{\frac{(\alpha-\beta)\theta}{\sqrt{2}}}$ is bounded below; thus Γ_b is bounded below. On the other hand, we have

$$\Theta^* b = \begin{pmatrix} \frac{\alpha-\beta}{\sqrt{2}}\bar{\theta} \\ \frac{\alpha+\beta}{2\sqrt{2}}\bar{\theta} \end{pmatrix} = \begin{pmatrix} \frac{\alpha-\beta}{\sqrt{2}} \\ \frac{\alpha+\beta}{2\sqrt{2}} \end{pmatrix} \bar{\theta},$$

and thus since θ is inner and $|\alpha|^2 + |\beta|^2 = 1$, we obtain

$$\mathrm{dist}\big(\Theta^* b, H^\infty\big(\mathbb{C}^2 \to \mathbb{C}\big)\big) \leqslant \|\Theta^* b\|_\infty = \left\| \begin{pmatrix} \frac{\alpha-\beta}{\sqrt{2}} \\ \frac{\alpha+\beta}{2\sqrt{2}} \end{pmatrix} \right\|_2 = \frac{\sqrt{5-6\alpha\beta}}{2\sqrt{2}}.$$

Now the condition $\alpha\beta > -1/2$ implies that $\mathrm{dist}(\Theta^* b, H^\infty(\mathbb{C}^2 \to \mathbb{C})) < 1$. We may then apply Theorem 5.3 to conclude that the distortion operator is invertible.

In the next example b is $*$-inner, but not inner.

Example 7.4. Consider a $*$-inner function $b = (\alpha \quad \beta) \in H^\infty(\mathbb{C}^2 \to \mathbb{C})$; that is, $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\big(I - T_b T_b^*\big) f = \alpha P_-(\bar{\alpha} f) + \beta P_-(\bar{\beta} f).$$

Therefore the image of $(I - T_b T_b^*)$ as well as the image of $(I - T_b T_b^*)^{1/2}$ are contained in the invariant subspace for T_z^* generated by $T_z^*\alpha$ and $T_z^*\beta$. In particular, if α and β are rational, then $\mathcal{H}(b)$ is finite dimensional, and thus equal as a set to K_Θ for some Blaschke product Θ. But in general the norm on $\mathcal{H}(b)$ is different from the usual H^2 norm on K_Θ, and the distortion operator corresponding to Θ is invertible, but not equal to the identity.

On the other hand, if we take $b = (1/\sqrt{2} \quad 1/\sqrt{2}B) \in H^\infty(\mathbb{C}^2 \to \mathbb{C})$ with B an infinite Blaschke product, then $(I - T_b T_b^*)^{1/2} = 1/\sqrt{2} P_{K_B}$. Thus $\mathcal{H}(b)$ is just K_B with the norm divided by $\sqrt{2}$, and the corresponding distortion operator is again invertible.

One might expect that if b and Θ are sufficiently close, in the sense that

$$\|b - \Theta\|_\infty < 1, \tag{7.1}$$

then the distortion operator should be invertible. In the scalar case, if b is inner, then it was pointed in [17, p. 202] that this condition is indeed sufficient. If b is a vector-valued inner function, then condition (7.1) remains sufficient to ensure the invertibility of the distortion operator. Indeed, it follows from (7.1) that $\|1 - \Theta^* b\|_\infty < 1$, whence $T_{\Theta^* b} = Id + T_{1-\Theta^* b}$ is invertible. In particular, $T_{b^*\Theta} = (T_{\Theta^* b})^*$ is left invertible; therefore, condition (ii) of Corollary 5.8 is satisfied, which implies that the distortion operator is invertible.

However, even in the general case of an arbitrary extreme point b in the unit ball of H^∞, condition (7.1) is no longer sufficient to ensure the invertibility of the distortion operator. Actually, it seems improbable that a condition expressed only in terms of functions might be found. The next example shows that indeed no condition similar to (7.1) is sufficient.

Example 7.5. For $\varepsilon > 0$, let h be the conformal transform of the disk $\mathbb{D}$ onto the simply connected domain

$$\Omega = \left\{ z \in \mathbb{C}\colon\ |z| < 1,\ \Re\mathrm{e}\, z > 1 - \frac{\varepsilon}{2} \right\}.$$

If we regard h as an element of H^∞, then $|h| = 1$ on an arc of positive measure but h is not inner. Moreover, on $\mathbb{T}$, we have

$$|1 - h|^2 = 1 + |h|^2 - 2\Re\mathrm{e}\, h \leqslant 2(1 - \Re\mathrm{e}\, h) < \varepsilon.$$

Now take $b = \Theta h$ with Θ is an arbitrary inner function. We have $\|\Theta - b\|_\infty = \|1 - h\|_\infty \leqslant \varepsilon$, while Corollary 5.6 implies that the distortion operator $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is not invertible. Consequently, no condition of closeness in the H^∞ norm can ensure the invertibility of the distortion operator.

8. Completeness of the difference quotients

Recall that the difference quotients are the elements $\hat{k}^b_{\lambda,e}$ of $\mathcal{H}(b)$ defined by (2.4). For $\dim E = \dim E_* = 1$ and b an extreme point in the unit ball of H^∞, the set $\{\hat{k}^b_\lambda\colon \lambda \in \mathbb{D}\}$ (which does not depend on e in this case) has been shown to be complete in $\mathcal{H}(b)$ in [11]; this completeness is further used therein to obtain results about properties of reproducing kernels. If, moreover, b is inner, then the completeness of the difference quotients can easily be obtained by noting that the mapping $f \mapsto \bar{z}b\bar{f}$ is an antilinear surjective isometry which maps k^b_λ onto $\hat{k}^b_\lambda$.

The study of the completeness of the family of the difference quotients in the general case may present independent interest. Together with the kernels, the difference quotients represent the main examples of "concrete" elements of the de Branges space $\mathcal{H}(b)$; they also appear in the study of model spaces and related questions (see, for instance, [4,12]). We devote this section to the investigation of their completeness.

We start with an equivalent condition.

Lemma 8.1. *Let $b \in H^\infty(E \to E_*)$. Then the following two conditions are equivalent*:

(1) $\operatorname{span}\{\hat{k}^b_{\lambda,e}\colon \lambda \in \mathbb{D},\ e \in E\} = \mathcal{H}(b)$.
(2) $\operatorname{span}\{S^{*n+1}be\colon n \geqslant 0,\ e \in E\} = \mathcal{H}(b)$.

Proof. As in the scalar case [24, II-8] it is easily seen that, for $\lambda \in \mathbb{D}$ and $f \in H^2(E)$, we have

$$\frac{f(z) - f(\lambda)}{z-\lambda} = \left(Id - \lambda S^*\right)^{-1} S^* f.$$

In particular, applying this formula to $f(z) := b(z)e$, we obtain

$$\frac{b(z) - b(\lambda)}{z-\lambda} e = \left(Id - \lambda S^*\right)^{-1} S^* be = \sum_{n=0}^{\infty} \lambda^n S^{*n+1} be. \tag{8.1}$$

Now according to (8.1), we have $f \in \mathcal{H}(b) \ominus \operatorname{span}\{\hat{k}^b_{\lambda,e}\colon \lambda \in \mathbb{D},\ e \in E\}$ if and only if

$$\sum_{n=0}^{\infty} \lambda^n \left\langle S^{*n+1} be, f\right\rangle_b = 0 \quad (\lambda \in \mathbb{D},\ e \in E),$$

and, since the function $\lambda \mapsto \sum_{n=0}^\infty \lambda^n \langle S^{*n+1}be, f\rangle_b$ is analytic in a neighbourhood of 0, this is equivalent to

$$\left\langle S^{*n+1} be, f\right\rangle_b = 0 \quad (n \geqslant 0,\ e \in E),$$

which gives the result. □

The scalar case has been discussed in [11]; we give a different proof of Lemma 4.2 therein, which seems to us of independent interest. Note that in Corollary 8.4 below we will obtain a more general result.

Theorem 8.2. *(See [11, Lemma 4.2].) Let b be an extreme point of the unit ball of H^∞. Then*

$$\operatorname{span}\left\{\hat{k}^b_\lambda\colon \lambda \in \mathbb{D}\right\} = \mathcal{H}(b).$$

Proof. As in the case of b inner, we will construct an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$ which maps k^b_λ onto $\hat{k}^b_\lambda$. Consider $\Pi = (\pi, \pi_*) : L^2 \oplus L^2 \to K$ be an AFE such that $\pi_*^*\pi = b$ and let $W : K \to K$ be the operator defined (on a dense set) by

$$W(\pi f + \pi_* g) = \pi J g + \pi_* J f,$$

with $J : L^2 \to L^2$ the antilinear map defined by $Jf = \bar{z}\bar{f}$ (note that since b is a scalar function, then the maps π and π_* act on scalar L^2-space). Then standard arguments show that W is an antilinear surjective isometry, keeping the model space $\mathbb{H}$ and interchanging the subspaces πH^2 and $\pi_* H^2_-$. In other words, we have

$$W P_{\mathbb{H}} = P_{\mathbb{H}} W. \tag{8.2}$$

Now, for $\chi \in \mathbb{H}$, set

$$\Omega\left(\pi_*^* \chi\right) = \pi_*^* (W\chi).$$

Since b is an extreme point of the unit ball of H^∞, it follows from Proposition 3.3 that π_*^* is an isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$, and then we can easily verify that Ω is an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$. For $\lambda \in \mathbb{D}$, recall that k_λ denotes the reproducing kernel of H^2 and consider the function $\Xi_\lambda \in \mathbb{H}$ defined by $\Xi_\lambda = P_{\mathbb{H}}\pi_* k_\lambda$. According to Lemma 3.2 and (2.2), we have

$$\pi_*^*\Xi_\lambda = \pi_*^* P_{\mathbb{H}}\pi_* k_\lambda = \left(Id - T_b T_b^*\right)k_\lambda = k_\lambda^b,$$

and then we get using (8.2),

$$\begin{aligned}\Omega\left(k_\lambda^b\right) &= \Omega\left(\pi_*^*\Xi_\lambda\right) = \pi_*^* W\,\Xi_\lambda = \pi_*^* W\,P_{\mathbb{H}}\pi_* k_\lambda = \pi_*^* P_{\mathbb{H}} W\pi_* k_\lambda \\ &= \pi_*^* P_{\mathbb{H}}\pi\, J\left(\frac{1}{1-\bar{\lambda}z}\right) = \pi_*^* P_{\mathbb{H}}\pi\left(\frac{1}{z-\lambda}\right).\end{aligned}$$

Now it is easy to see that (3.5) implies

$$\pi_*^* P_{\mathbb{H}}\pi = P_+ b - bP_+, \tag{8.3}$$

and therefore since $(z-\lambda)^{-1} \in H_-^2$, we obtain

$$\Omega\left(k_\lambda^b\right) = P_+\left(\frac{b(z)}{z-\lambda}\right) = P_+\left(\frac{b(z)-b(\lambda)}{z-\lambda} + \frac{b(\lambda)}{z-\lambda}\right) = \frac{b(z)-b(\lambda)}{z-\lambda} = \hat{k}_\lambda^b,$$

and the proof is complete. □

The nonextreme scalar case will be discussed below, as a consequence of Theorem 8.3.

To go now beyond the scalar case, we will use the abstract functional embedding introduced in Section 3. We have then the following general result.

Theorem 8.3. *Suppose $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$ and consider $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE such that $\pi_*^*\pi = b$. The following assertions are equivalent*:

(1) $\operatorname{span}\{\hat{k}_{\lambda,e}^b\colon \lambda \in \mathbb{D},\ e \in E\} = \mathcal{H}(b)$;
(2) $\mathbb{H}' \cap \mathbb{H}_*'' = \{0\}$;
(3) $\mathbb{H}'' \vee \mathbb{H}_*' = \mathbb{H}$;
(4) $\ker(P_{\mathcal{R}} b^*|\mathcal{R}_*) = \{0\}$ *($\mathcal{R}, \mathcal{R}_*$ defined by (3.6)).*

Proof. (1) ⇔ (2) Denote $\xi_n = P_{\mathbb{H}}\pi(\bar{z}^{n+1}e)$, $n \geqslant 0$. One can easily check that

$$\operatorname{span}(\xi_n) = \operatorname{clos}\left(P_{\mathbb{H}}\pi H_-^2(E)\right). \tag{8.4}$$

We know from Proposition 3.3 that π_*^* is a coisometry from $\mathbb{H}$ onto $\mathcal{H}(b)$ with kernel $\mathbb{H}'' = \mathbb{H} \ominus \mathbb{H}'$. Denoting $\eta_n = P_{\mathbb{H}'}\xi_n$, we have

$$\pi_*^*\eta_n = \pi_*^*(P_{\mathbb{H}'} + P_{\mathbb{H}\ominus\mathbb{H}'})\xi_n = \pi_*^* P_{\mathbb{H}}\xi_n = \pi_*^* P_{\mathbb{H}}\pi\left(\bar{z}^{n+1}e\right),$$

and using (8.3), we get

$$\pi_*^*\eta_n = P_+ b\bar{z}^{n+1}e = S^{*n+1}be.$$

It follows that π_*^* is a unitary from $\operatorname{span}(\eta_n)$ onto $\operatorname{span}(S^{*n+1}be\colon n \geqslant 0,\ e \in E)$. Then, according to Lemma 8.1, the difference quotients are not complete iff there exists a non-null vector χ in $\mathbb{H}'$ that is orthogonal to all η_n; or equivalently, that is orthogonal to all ξ_n. By (8.4), this is equivalent to being orthogonal to $\pi(H_-^2(E))$, which is the same as saying that $\chi \in \mathbb{H}_*''$. The equivalence is thus proved.

(2) ⇔ (3) follows easily from the definition.

(2) ⇔ (4) Let $\chi \in K$. Using Lemma 3.1, we have $\chi \in \mathbb{H}' \cap \mathbb{H}_*''$ if and only if there is $g \in \mathcal{R}_*$ such that $\chi = \tau_* g$ and $\tau_* g \perp \tau f$, for every $f \in \mathcal{R}$. Now it follows from (3.1) that this is equivalent to the existence of $g \in \mathcal{R}_*$ such that $\chi = \tau_* g$ and $P_{\mathcal{R}} b^* g = 0$. Since τ_* is an isometry on $\mathcal{R}_*$, we get the conclusion. □

Corollary 8.4. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leqslant 1$. If $\operatorname{clos}(\Delta_* H_-^2(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$, then*

$$\operatorname{span}\left\{\hat{k}_{\lambda,e}^b\colon \lambda \in \mathbb{D},\ e \in E\right\} = \mathcal{H}(b).$$

In particular, if b is $*$-inner, then the difference quotients are complete.

Proof. The hypothesis implies that $\mathcal{R}_* = \{0\}$ and the conclusion follows from Theorem 8.3. □

Corollary 8.5. *Let b be an extreme point of the unit ball of $H^\infty(E \to E_*)$. Then the two following conditions are equivalent*:

(i) $\operatorname{span}\{\hat{k}^b_{\lambda,e}\colon \lambda \in \mathbb{D},\ e \in E\} = \mathcal{H}(b)$.
(ii) $\operatorname{clos}(\Delta_* H^2_-(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$.

Proof. (ii) ⇒ (i) follows from Corollary 8.4. As for (i) ⇒ (ii), we know from [26] (see Remark 3.4) that b is an extreme point of the unit ball of $H^\infty(E \to E)$ if and only if $\mathcal{R} = \{0\}$ or $\mathcal{R}_* = \{0\}$. Assume that (ii) is not satisfied, which means $\mathcal{R}_* \neq \{0\}$. Then we necessarily have $\mathcal{R} = \{0\}$ and thus $\ker(P_{\mathcal{R}} b^* | \mathcal{R}_*) \neq \{0\}$. But if the difference quotients are complete, then, by Theorem 8.3, we obtain a contradiction. □

For the nonextreme scalar case, we have to recall that a function f in the Nevanlinna class of the unit disc $\mathbb{D}$ is said to be *pseudocontinuable* (across $\mathbb{T}$) if there exists $g, h \in \bigcup_{p>0} H^p$ such that

$$f = \bar{h}/\bar{g}$$

a.e. on $\mathbb{T}$. The function $\widetilde{f} := \bar{h}/\bar{g}$ is the (nontangential) boundary function of the meromorphic function $\widetilde{f}(z) := \bar{h}(\frac{1}{\bar{z}})/\bar{g}(\frac{1}{\bar{z}})$ defined for $|z| > 1$, which is called a pseudocontinuation of f. R. Douglas, H. Shapiro and A. Shields have obtained [8] the following characterization: a function $f \in H^2$ is pseudocontinuable if and only if it is not S^*-cyclic, that is $\operatorname{span}(S^{*n} f\colon n \geqslant 0) \neq H^2$. It follows then from the structure of invariant subspaces of S^* that there exists an inner function θ such that $f \perp \theta H^2$; in particular, $\bar{f}\theta \in H^2$.

Theorem 8.6. *Suppose b is not an extreme point in the unit ball of H^∞. Then*

$$\operatorname{span}\{\hat{k}^b_\lambda\colon \lambda \in \mathbb{D}\} = \mathcal{H}(b) \quad \Longleftrightarrow \quad b \text{ is not pseudocontinuable.}$$

Proof. According to Theorem 8.3, it is sufficient to prove that b is not pseudocontinuable if and only if $\ker(P_{\mathcal{R}} \bar{b} | \mathcal{R}_*) \neq \{0\}$. Note first that nonextremality of b implies that $\log \Delta \in L^1$ and in particular $\Delta \neq 0$ almost everywhere on $\mathbb{T}$; thus $\operatorname{clos}(\Delta L^2) = L^2$. On the other hand, it follows easily from the Beurling–Helson Theorem on shift-invariant subspaces that there exists an outer function $\phi \in H^2$ with $|\phi| = \Delta$ such that $\operatorname{clos} \Delta H^2 = \frac{\Delta}{\phi} H^2$.

Now assume that $\ker(P_{\mathcal{R}} \bar{b} | \mathcal{R}_*) \neq \{0\}$. Since

$$\mathcal{R} = L^2 \ominus \frac{\Delta}{\phi} H^2 \quad \text{and} \quad \mathcal{R}_* = L^2 \ominus \operatorname{clos}(\Delta H^2_-),$$

there exists $h \in L^2$, $h \neq 0$ such that $g = \Delta h \in H^2$ and $\bar{b} h = \frac{\Delta}{\phi} h_1$, with $h_1 \in H^2$. Multiplying the first equality by $\bar{b}$, we obtain

$$\bar{b} g = \bar{b} \Delta h = \frac{1 - |b|^2}{\phi} h_1,$$

whence $\bar{b}(\phi g + b h_1) = h_1$, or

$$b = \frac{\bar{h}_1}{\overline{\phi g + b h_1}}$$

(note that $h_1 \neq 0$ and thus $\phi g + b h_1 \neq 0$). Consequently, b is pseudocontinuable.

Conversely, if b is pseudocontinuable, there exists an inner function θ such that $h_1 = \bar{b}\theta \in H^2$. Then $g = \frac{\theta - b h_1}{\phi} = \frac{\Delta^2 \theta}{\phi}$ is in the Nevanlinna class of $\mathbb{D}$, while on $\mathbb{T}$ we have $|g| = |\Delta|$. It follows that g belongs to H^2 (even to H^∞); moreover, $h = g/\Delta$ is a unimodular function in L^2 with $\bar{b} h = \frac{\Delta}{\phi} h_1$. This means that $h \in \ker(P_{\mathcal{E}} \bar{b} | \mathcal{E}_*)$, which is therefore different from $\{0\}$. □

Example 8.7. As a consequence of Theorem 8.6, it is simple to give two examples of de Branges–Rovnyak spaces (both corresponding to nonextreme functions b), with the completeness of the difference quotients false for the first and true for the second. Note first that, if $\sup_{z \in \mathbb{T}} |b(z)| < 1$, then $\log(1 - |b|)$ is integrable, and thus b is not extreme. This condition is satisfied by both functions $b_1(z) := 1/(z-3)$ and $b_2(z) := \exp((z-2)^{-1})$. The first is pseudocontinuable, and thus the difference quotients are not complete in $\mathcal{H}(b_1)$, while the second is not, whence the difference quotients are complete in $\mathcal{H}(b_2)$. We see then that extremality is not a necessary condition for the completeness of the difference quotients.

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References

[1] S.A. Avdonin, S.A. Ivanov, Families of Exponentials, Cambridge Univ. Press, Cambridge, 1995.
[2] A. Baranov, Completeness and Riesz basis of reproducing kernels in model subspaces, Int. Math. Res. Not. (2006), Art. ID 81530, 34 pp.
[3] A. Baranov, E. Fricain, J. Mashreghi, Weighted norm inequalities for de Branges–Rovnyak spaces and their applications, Amer. J. Math., in press.
[4] J.A. Ball, T.L. Kriete, Operator-valued Nevanlinna–Pick kernels and the functional models for contraction operators, Integral Equations Operator Theory 10 (1987) 17–61.
[5] I. Chalendar, E. Fricain, D. Timotin, Functional models and asymptotically orthonormal sequences, Ann. Inst. Fourier (Grenoble) 53 (5) (2003) 1527–1549.
[6] L. de Branges, J. Rovnyak, Canonical models in quantum scattering theory, in: C.H. Wilcox (Ed.), Perturbation Theory and Its Applications in Quantum Mechanics, Wiley, New York, 1966, pp. 295–392.
[7] K. de Leueuw, W. Rudin, Extreme points and extremum problems in H_1, Pacific J. Math. 8 (3) (1958) 467–485.
[8] R.G. Douglas, H.S. Shapiro, A. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, Ann. Inst. Fourier (Grenoble) 20 (1) (1970) 37–76.
[9] E. Fricain, Bases of reproducing kernels in model spaces, J. Operator Theory 46 (2001) 517–543.
[10] E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, Ann. Inst. Fourier (Grenoble) 52 (2) (2002) 661–686.
[11] E. Fricain, Bases of reproducing kernels in de Branges spaces, J. Funct. Anal. 226 (2) (2005) 373–405.
[12] S.R. Garcia, Conjugation and Clark operators, Contemp. Math. 393 (2006) 67–111.
[13] S.V. Hruschev, N.K. Nikolski, B.S. Pavlov, Unconditional bases of exponentials and reproducing kernels, in: Lecture Notes in Math., vol. 864, 1981, pp. 214–335.
[14] A.E. Ingham, A note on Fourier transforms, J. London Math. Soc. 9 (1934) 29–32.
[15] N.K. Nikolskiĭ, Invariant subspaces in operator theory and function theory, in: Mathematical Analysis, vol. 12, Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1974, pp. 199–412, 468, loose errata (in Russian).
[16] N.K. Nikolski, Bases of exponentials and values of reproducing kernels, Dokl. Akad. Nauk SSSR 252 (6) (1980) 1316–1320 (in Russian); English transl.: Soviet Math. Dokl. 21 (3) (1980) 937–941.
[17] N.K. Nikolski, Treatise on the Shift Operator, vol. 273, Springer-Verlag, Berlin, 1986.
[18] N.K. Nikolski, Operators, Functions, and Systems: An Easy Reading, vol. 1: Hardy, Hankel and Toeplitz, Math. Surveys Monogr., vol. 92, Amer. Math. Soc., 2002.
[19] N.K. Nikolski, Operators, Functions, and Systems: An Easy Reading, vol. 2: Model Operators and Systems, Math. Surveys Monogr., vol. 93, Amer. Math. Soc., 2002.
[20] N. Nikolski, V. Vasyunin, Notes on two function models, in: The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof, in: Math. Surveys Monogr., vol. 21, 1985, pp. 113–141.
[21] N.K. Nikolski, V. Vasyunin, Elements of spectral theory in terms of the free function model. I. Basic constructions, in: Holomorphic Spaces, Berkeley, CA, 1995, in: Math. Sci. Res. Inst. Publ., vol. 33, Cambridge Univ. Press, Cambridge, 1998, pp. 211–302.
[22] J. Ortega-Cerdà, K. Seip, Fourier frames, Ann. of Math. (2) 155 (3) (2002) 789–806.
[23] R.E. Paley, N. Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, RI, 1934.
[24] D. Sarason, Sub-Hardy Hilbert Spaces in the Unit Disk, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, Willey–Interscience Publication, 1995.
[25] B. Szökefalvi-Nagy, C. Foias, Analyse harmonique des opérateurs de l'espace de Hilbert, Akadémiai Kiado, Budapest, 1967; English transl.: Harmonic Analysis of Operators on Hilbert Space, North-Holland, New York, 1970.
[26] S. Treil, Extreme points of the unit ball of the operator Hardy space $H^\infty(E \to E)$, Zap. Nauchn. Sem. LOMI 149 (2) (1986) 160–164 (in Russian); English transl.: J. Soviet Math. 42 (2) (1988) 1653–1656.
[27] S.R. Treil, Geometric methods in spectral theory of vector-valued functions: Some recent results, in: Oper. Theory Adv. Appl., vol. 42, Birkhäuser, Basel, 1989, pp. 209–280.
[28] V.I. Vasjunin, The construction of the B. Szőkefalvi–Nagy and C. Foiaş functional model, in: Investigations on Linear Operators and the Theory of Functions, VIII, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 73 (1978) 16–23, 229.