The Exterior Dirichlet Problem for a Class of Fourth Order Elliptic Equations

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In some exterior domain \( G \) of the Euclidian \( p \)-space \( \mathbb{R}^p \) the Dirichlet boundary value problem is considered for the equation \((L + K^2)u = f\), where \( L \) is a uniformly elliptic operator and \( K \) is a real number different from 0. It can be shown that each solution \( u \) of this equation splits into \( u = u_1 + u_2 \), where \( u_1 \) and \( u_2 \) satisfy Helmholtz equations. Asymptotic conditions for \( u \) are formulated by imposing Sommerfeld radiation conditions on \( u_1 \) and \( u_2 \). If \( u_1 \) and \( u_2 \) are assumed to satisfy the same radiation condition, we prove a “Fredholm alternative theorem.” If \( u_1 \) and \( u_2 \) satisfy different radiation conditions, existence and uniqueness of the solution can be shown, provided the space dimension \( p \) is greater than 2.

1. Notations and Introduction

In some exterior domain \( G \subset \mathbb{R}^p \) let

\[
L(x, \partial) u(x) := \partial_i (a_{ij}(x) \partial_j u(x)) + a(x) u(x) \quad \text{(sum convention!)}
\]

be a uniformly elliptic symmetric operator. The bounded real valued functions \( a_{ij} = a_{ji} \) and \( a \) resp. belong to \( C^3(G) \) and \( C^0(G) \) resp., and satisfy the following asymptotic conditions for \( |x| \to \infty \):

\[
a_{ij}(x) - \delta_{ij} = O(|x|^{-3/2-\epsilon})
\]

\[
\partial_i a_{ij}(x), \partial_i \partial_m a_{ij}(x) = O(|x|^{-5/2-\epsilon}),
\]

\[
a(x), Aa(x) = O(|x|^{-3/2-\epsilon}),
\]

where

\[
\Lambda(x, \partial) := x_i \partial_i
\]

and \( \epsilon \in ]0, 1[ \). For the Sobolev spaces Agmon’s notation [1] is used. Finally let:

\[
\tilde{f}_m(G) := \{ \psi \in H^m(G) \text{ for all } \psi \in C_0^\infty(\mathbb{R}^p) \} \quad \text{(resp.: } f_m(G)),
\]

\[
F := \{ f : (1 + |x|) f \in L^0(G) \};
\]

\[
\tilde{F} := \{ f \in F : Af \in F \text{ in the distribution sense}. \}
\]
If $\kappa \in \mathbb{R} \setminus \{0\}$ and $f, A f \in L^{10e}(G)$ we prove in Section 2, that each weak solution $u$ [1, Definition 6.2] of

$$(L + \kappa^2) u = f,$$  

(8)

can be split in a unique way into

$$u = \Lambda u_1 + u_2,$$  

(9)

where $u_\sigma$ ($\sigma = 1, 2$) satisfy the equations

$$(L + \kappa^2) u_\sigma = g_\sigma,$$  

(10)

weakly. Here $g_1$ is determined by $f$ and $g_2$ by the pair $(f, u_1)$ in such a way that $f \in \mathcal{F}$ implies $g_\sigma \in \mathcal{F}$. Under the assumption $g_\sigma \in \mathcal{F}$ Jäger [3] proves an existence and uniqueness theorem for the problem, given by Eq. (10), an inhomogeneous Dirichlet boundary condition and the Sommerfeld radiation condition.

Therefore one may formulate asymptotic conditions for solutions of (8) by imposing Sommerfeld radiation conditions on $u_\sigma$.

This leads to the following definition:

1.1. DEFINITION. Let $f \in \mathcal{F}$ and $\kappa \in \mathbb{R} \setminus \{0\}$. A solution of the Dirichlet problems $\mathcal{D}^+(f, \kappa)$ and $\mathcal{D}^-(f, \kappa)$ resp. is a function $u \in \tilde{f}_2(G)$, which solves (8) weakly and satisfies the radiation conditions

(i) $D u_1 - i\kappa u_1 \in L_2(G')$,

(ii) $D u_2 + i\kappa u_2 \in L_2(G')$ resp. $D u_2 - i\kappa u_2 \in L_2(G')$,

for each $G' \subset G$. Here $D(x, \partial) := \{ x^{-1} x a_{ij} \partial_j \}$, and $G' \subset G$ means that the domain $G'$ is contained in $G$ and has positive distance from the boundary $\partial G$ of $G$.

The splitting (9) was first given by Subeika [6] in the case $L = \Delta$. Existence and uniqueness theorems for the problem $\mathcal{D}^+(f, \kappa)$ are proved by Teschke [7] under the assumptions that $p > 2$, and $f(x) = 0$ and $L(x, \partial) = \Delta$ for large $|x|$. As far as we know there exists only a very special result for the problem $\mathcal{D}^-(f, \kappa)$, i.e., an existence and uniqueness proof by Teschke [7] if $G$ is the complement of a ball in $\mathbb{R}^3$ and $L(x, \partial) = \Delta$.

In this paper we give a proof of a "Fredholm alternative theorem" for $\mathcal{D}^-$ (Section 4) and of an existence and uniqueness theorem for $\mathcal{D}^+$ (Section 5). In this last case one must assume $p > 2$, as an example by Teschke [7] shows. The existence theory is based on a Fredholm-type theorem, which is proved in Section 3 by an alternating technique similar to that used in [8] to treat the exterior oblique derivative problem for the Helmholtz equation. This
technique is a modification of the method, used by Leis [4] in order to combine Hilbert space and integral equation methods. In Section 6 we briefly mention some results for higher order equations.

2. The Splitting of the Solution

The following commutation rules are proved by simple computation:

\[ LA - AL = 2L - M, \]
\[ DA - AD = D - D_M, \]
where
\[ Mv = \partial_i((L\alpha_{ij}) \partial_j v) + (L\alpha + 2\kappa) v \]
and
\[ D_Mv = |x|^{-1} x_i(L\alpha_{ij}) \partial_j v. \]

From these rules we get the splitting theorem:

**Theorem 2.1.** For some domain \( G' \subset G \) let \( f \) and \( Af \) belong to \( L^2_{100}(G') \). Then the following two assertions hold:

(i) If \( u \in L^2_{100}(G') \) solves (8) weakly, then

\[ u_1 := -\left(\kappa^{-2}/2\right)(L + \kappa^2) u, \quad u_2 := u - Au, \]

are weak solutions of

\[ (L + \kappa^2) u_1 = -\left(\kappa^{-2}/2\right) f \]

and

\[ (L + \kappa^2) u_2 = \kappa^{-2}f + (\kappa^{-2}/2) Af + Mu, \text{ resp.} \]

(ii) If, on the other hand, \( u_1 \) and \( u_2 \) are weak solutions of (14) and (15) resp., then \( u := Au_1 + u_2 \) is a weak solution of (8) and

\[ u_1 := -\left(\kappa^{-2}/2\right)(L + \kappa^2) u. \]

**Proof.** [1, Theorem 6.3] implies that each weak solution \( u_1 \) of (14) is in \( H^2_{100}(G') \) and so \( Au_1 \) is in \( H^1_{100}(G') \). Considering (11) and the fact that

\[ A^* = -\hat{p} - A \]

is the formal adjoint of \( A \), \( Au_1 \) is easily seen to be weak solution of

\[ (L + \kappa^2) v = -((\kappa^{-2}/2) Af + \kappa^{-2}f + 2\kappa^2 u_2 + Mu_2). \]
To prove (i), [1, Theorem 6.3] is used to show \( u \in H^3_{lo}(G') \). Then \( u_1 \) belongs to \( H^{10c}(G') \) and is obviously a weak solution of (14). The assertion then follows from (17). To prove (ii), we get from [1, Theorem 6.3] \( u_1, u_2, \Lambda u_1 \in H^3_{lo}(G') \). But then with \( \phi \in C_0^\infty(G') \):
\[
(u, (L + \kappa^2)^2 \phi)_{0,G'} = ((L + \kappa^2)u, (L + \kappa^2)\phi)_0 = (f, \phi)_0,
\]
i.e., \( u \) is a weak solution of (8). A straightforward computation yields
\[
(L + \kappa^2)u = -2\kappa^2u_1.
\]

As a corollary we get

**Corollary 2.2.** Let \( u \in L^2_{lo}(G') \) be a weak solution of \( (L + \kappa^2)^2 u = 0 \), vanishing in some open set \( U \subset G' \). Then \( u \) vanishes identically in \( G' \).

For \( u_1 \) is a classical solution of \( (L + \kappa^2)u_1 = 0 \) by Weyl's Lemma [2, p. 189ff.], vanishing in \( U \) and therefore in \( G' \) because of [5]. A similar argument then holds for \( u_2 \).

The following lemma may be considered as a corollary of [1, Theorem 6.3]:

**Lemma 2.3.** Let \( v \in L^2_{lo}(G) \) be a weak solution of \( (L + \kappa^2)v = f \), and \( \mu \in \mathbb{R} \) be a number such that
\[
(1 + |x|^2)^{\mu}v, (1 + |x|^2)^{\mu}v \in L^2(G).
\]
Then \( v \in H^2_{lo}(G) \) and \( (1 + |x|^2)^{\mu} \partial_\gamma v, (1 + |x|^2)^{\mu} \partial_\gamma v \in L^2(G') \) for each \( G' \subset G \) and \( i,j = 1,\ldots,p \).

**Sketch of the Proof.** It is sufficient to suppose that \( G' \) is the complement of some cube in \( \mathbb{R}^p \). Consider a net of congruent open cubes \( \{C_\sigma \}_{\sigma \in \Sigma} \) so that
\[
\overline{G'} = \bigcup_{\sigma \in \Sigma} \overline{C_\sigma} \quad \text{and} \quad C_\sigma \cap C_\rho = \emptyset \quad \text{for} \ \sigma \neq \rho.
\]

Now [1, Theorem 6.3] implies the estimate
\[
\|(1 + |x|^2)^{\mu}v\|_{0,C_\sigma} \leq K\{(1 + |x|^2)^{\mu}v\|_{0,C_\sigma} + \|(1 + |x|^2)^{\mu}f\|_{0,C_\sigma},
\]
with \( K > 0 \) independent of \( \sigma \). Here \( \partial \) stands for a first or second derivative of \( v \), and \( C_\sigma \) is that cube, which is built up by \( C_\sigma \) and the \( 3^p - 1 \) cubes surrounding \( C_\sigma \). From this estimate we get with \( K' > 0 \) depending on \( K \) and \( p \):
\[
\|(1 + |x|^2)^{\mu}v\|_{0,G'} \leq K'\{(1 + |x|^2)^{\mu}v\|_{0,G} + \|(1 + |x|^2)^{\mu}f\|_{0,G},
\]
which proves the lemma.

Now let \( G' \) be any exterior domain contained in \( G \). Jäger [3] proves, that
for each \( g \in F \) and \( h \in H_1(G') \) there exists a uniquely determined solution \( v \) of the Dirichlet problem \( \mathcal{D}(g, h, \kappa) \), given by (18)-(20):

\[
v \text{ solves } (L + \kappa^2) v = g \text{ weakly in } G', \quad (18)
\]

\[
v - h \in \mathcal{J}_1(G'), \quad (19)
\]

\[
Dv - i\kappa v \in L_2(G'). \quad (20)
\]

Jäger's proof implies \((1 + |x|)^{-1/2} v \in L_2(G')\). Therefore we get from Lemma 2.3:

**Theorem 2.4.** With \( f \in \hat{F} \) let \( u \) be a solution of \( \mathcal{D}^\perp(f, \kappa) \). Then for each \( G' < G \), \( i, j = 1, ..., p \) and \( \sigma = 1, 2 \) we have \((1 + |x|)^{-1/2} u_\sigma \), \((1 + |x|)^{-(1+\sigma)/2} \partial_{\sigma} u_\sigma \in L_2(G')\) and \((1 + |x|)^{-(1+\sigma)/2} \partial_{\sigma} \partial_{\tau} u_\sigma \in L_2(G')\).

The proof must first be done for \( \sigma = 1 \) and then for \( \sigma = 2 \) in order to know the asymptotic behavior of \( Mu_1 \).

In the following, if \( u \) solves (8) weakly, \( u_1 \) and \( u_2 \) will always denote the "components" defined by (13).

### 3. A Fredholm — Type Theorem

Let \( R \) and \( \bar{R} \) be two real numbers with \( 0 < \bar{R} < R \) and

\[
\mathbb{R}^p - G \subset K(\bar{R}) := \{ x \in \mathbb{R}^p : |x| < \bar{R} \}.
\]

In \( G_R := G \cap K(R) \) the homogenous Dirichlet problem for \((L + \kappa^2) u = 0\) has no nontrivial solution. For if \( u \in \hat{H}_2(G_R) \) solves \((L + \kappa^2) u = 0 \) weakly, then \( u \) satisfies \(((L + \kappa^2) u, (L + \kappa^2) u)_{0, G_R} = 0 \) and therefore solves \((L + \kappa^2) u = 0\). But then \( u \) can be continued by 0 to some \( \hat{u} \in H_2(G) \). This function \( \hat{u} \) solves \((L + \kappa^2) \hat{u} = 0 \) in \( G \) even in the classical sense by Weyl's Lemma. The unique continuation principle [5] then implies \( u \equiv 0 \).

Define:

\[
\bar{G} := \mathbb{R}^p - \overline{K(\bar{R})}, \quad Z := K(R) - \overline{K(\bar{R})}.
\]

Let \( f \in \hat{F} \) be fixed and

\[
y \in H_2(G_R). \quad (21)
\]

Let \( u \) be the unique function, that solves the inhomogeneous Dirichlet problem

\[
(L + \kappa^2) u = f, \quad u - y \in \hat{H}_2(G_R) \quad \text{in } G_R. \quad (22)
\]
We form \( u_1 \) and \( u_2 \) and set

\[ h_1 = \tilde{\psi} u_1 |_Z, \quad h_2 = \tilde{\psi} u_2 |_Z, \]  

(23)

where \( \tilde{\psi} \in C_0^\infty(G_R) \) is identically 1 in a neighborhood of \( \partial K(R) \). \( h_1 \) and \( h_2 \) are considered as elements of \( H_2(G) \) by continuation as 0 in \( G \). Because of Jäger's [3] results we can form the unique solutions

\[ v_1 \in J_1(\tilde{G}) \] of the problem \( \mathcal{D}(-\kappa^{-2}/2)f, h_1, \kappa) \)  

(24)

and

\[ v_2 \in J_1(G) \] of the problem \( \mathcal{D}(\kappa^{-2}/2)\Lambda f + \kappa^{-2}f + M v_1, h_2, \mp \kappa) \)  

(25)

in \( \tilde{G} \), where the choice of the sign of \( \kappa \) in (25) depends on whether \( \mathcal{D}^+(f, \kappa) \) or \( \mathcal{D}^-(f, \kappa) \) is to be solved. Note that [1, Theorem 9.8], Jäger's existence proof and Lemma 2.3 imply \((1 + \|x\|) M v_1 \in L_2(\tilde{G})\), so that (25) makes sense.

With some \( \psi \in C_0^\infty(\tilde{G}) \) which is identically 1 in some neighborhood of \( \partial K(R) \) we finally form

\[ y^* = \psi(A v_1 + v_2) \]  

(26)

and consider \( y^* \) as an element of \( H_2(G_R) \), which is possible because \( v_1, v_2, A v_1 \in H_2^{100}(\tilde{G}) \) (note (17)!). Now one can write

\[ y^* = \Theta f + \Gamma y, \]  

(27)

\( \Theta \) and \( \Gamma \) resp. being linear operators from \( \tilde{G} \) and \( H_2(G_R) \) resp. into \( H_2(G_R) \).

**Lemma 3.1.** \( \Gamma: H_2(G_R) \rightarrow H_2(G_R) \) is compact.

**Proof.** One gets \( \Gamma y \) by going through the previous construction in the case \( f = 0 \). Since the operator \( (L + \kappa^2)^2 \) is 3-smooth [1, Definition 6.1], by [1, Theorem 6.3] the change from \( y (21) \) via (22) into \( (h_1, h_2) \) (23) is a continuous operation from \( H_2(G_R) \) into \( H_2(\tilde{G}) \times H_2(\tilde{G}) \), whence it is compact as an operation into \( H_2(\tilde{G}) \times H_2(\tilde{G}) \) by Rellich's compactness theorem [1, Theorem 3.8]. By [1, Theorem 9.8], Jäger's existence proof, and Lemma 2.3 the change from (23) to (24), (25) can be considered as a continuous operation from \( H_2(\tilde{G}) \times H_2(\tilde{G}) \) into \( J_2(\tilde{G}) \times J_2(\tilde{G}) \). (17) and [1, Theorem 6.3] imply that the change from (24), (25) into (26) is a continuous operation into \( H_2(G_R) \). Hence \( \Gamma \) is compact.

**Lemma 3.2.** Let \( f \in \tilde{F} \) be given. If \( y \in H_2(G_R) \) solves the equation

\[ y = \Theta f + \Gamma y \quad \text{(or } \Theta f - (I d - \Gamma)y), \]  

(28)
then the functions $u$ and $v: = \Lambda v_1 + v_2$ coincide in $\Omega$, where $u$ is given by (22) and $v_1, v_2$ are given by (24), (25). The function

$$q: \begin{cases} u & \text{in } G_R, \\ v & \text{in } G, \end{cases}$$

solves $\mathcal{D} (f, \kappa)$. If, on the other hand, $q$ solves $\mathcal{D}^+ (f, \kappa)$, then $y: = \psi q |_{G_R}$ solves Eq. (28).

Proof. The last assertion easily follows from the uniqueness theorems for the Dirichlet problems involved. If $u$ and $v$ coincide in $\Omega$ it is easily seen that $q$ is a solution of $\mathcal{D} (f, \kappa)$. So it remains to show $u |_{\Omega} = v |_{\Omega}$. Lemma 2.1 implies that $w: = v |_{\Omega} - u |_{\Omega}$ is a weak solution of $(L + \kappa^2) w = 0$ in $\Omega$ with the components $w_1 = v_1 |_{\Omega} - u_1 |_{\Omega}$ and $w_2 = v_2 |_{\Omega} - u_2 |_{\Omega}$, where $w_1$ and $w_2$ solve

$$(L + \kappa^2) w_1 = 0 \quad \text{and} \quad (L + \kappa^2) w_2 = M w_1 .$$

Furthermore the construction (21)-(27) gives

$$\tilde{\psi} w_1, \tilde{\psi} w_2 \in \tilde{\mathcal{H}}_1(\Omega), \quad (30)$$

$$\psi w \in \tilde{\mathcal{H}}_2(\Omega). \quad (31)$$

From (29) and (30) it follows by [1, Theorem 9.8], that $\tilde{\psi} w_1, \tilde{\psi} w_2 \in \mathcal{H}_2(\Omega)$, i.e., $\tilde{\psi} \Lambda w_1 \in \mathcal{H}_2(\Omega)$ and therefore because of (31):

$$w \in \mathcal{H}_2(\Omega).$$

The same theorem yields from (31) and the differential equation for $w$, that $\psi w \in \mathcal{H}_2(\Omega)$, and therefore

$$w_1 \in \mathcal{H}_2(\Omega). \quad (33)$$

Since the boundary $\partial \Omega$ is smooth, functions in $\mathcal{H}_2(\Omega)$ and their first derivatives resp. possess traces in $\mathcal{H}_1(\partial \Omega)$ and $L_2(\partial \Omega)$ resp., and the following partial integrations are possible (here $n_i$ denotes the $i$th component of the outward unit normal):

$$\int_{\Omega} |w_1|^2 \, dx = -\kappa^{-2}/2 \int_{\Omega} w_1 \cdot (L + \kappa^2) \bar{w} \, dx$$

$$= -\kappa^{-2}/2 \int_{\partial \Omega} (w_1 \cdot n_i a_i \bar{\psi} \bar{w} - n_i a_i \bar{\psi} \bar{w}_1 \cdot \bar{w}) \, d\sigma$$

$$= -\kappa^{-2}/2 \int_{|z|=R} (Dw_1 \cdot \Lambda \bar{w}_1) \, d\Omega .$$
the last equality following from (30) and (31). But on \( \partial K(\bar{R}) \) we have

\[
Dw_1 = \bar{R} a_i(x; x_j) \Lambda w_1 + (D - \bar{R} a_i x; x_j \Lambda) w_1.
\]

The second term is a tangential derivative of \( w_1 \) and vanishes because of (30). Since \( a_{ij}x; x_j > 0 \), (34) implies \( w_1 \in 0 \) in \( Z \).

Therefore \( w = w_2 \), and because of (31) \( w_2 \) can be continued by 0 to some \( \tilde{w}_2 \in H_2(\bar{G}) \), which solves \( (L + \kappa^2) \tilde{w}_2 = 0 \) in \( \bar{G} \) in the classical sense by Weyl's Lemma.

Hence \( 0 = w_2 = w \) in \( Z \) because of [5], which proves the lemma.

As in [8] the Riesz–Schauder theory for compact operators yields:

**THEOREM 3.3.** The number \( \nu \) of linear independent solutions of \( \mathcal{D}^\pm(0, \kappa) \) is finite and

\[
\nu = \dim N(Id - \Gamma) = \dim N(Id - \Gamma^*),
\]

holds, where \( N(T) \) denotes the kernel of the operator \( T \). The condition, that \( \Theta f \) is orthogonal to \( N(Id - \Gamma^*) \) in \( H_2(G_R) \), is necessary and sufficient for the solvability of \( \mathcal{D}^\pm(f, \kappa) \).

4. **THE CASE OF EQUAL RADIATION CONDITIONS**

In this section we prove a "Fredholm alternative theorem" for \( \mathcal{D}^-(f, \kappa) \).

The part of the adjoint problem to \( \mathcal{D}^-(\cdot, \kappa) \) is played by \( \mathcal{D}^-(\cdot, -\kappa) \), which is the problem with the opposite radiation conditions. Denote by \( \mathcal{N} \) and \( \mathcal{N}^* \) resp. the space of solutions for \( \mathcal{D}^-(0, \kappa) \) and \( \mathcal{D}^-(0, -\kappa) \) resp. Let \( \Theta, \Gamma \) be the operators, constructed in Section 2 with respect to \( \mathcal{D}^-(\cdot, \kappa) \). We shall prove the following:

**THEOREM 4.1.** \( \dim \mathcal{N} = \dim \mathcal{N}^* < \infty \). Suppose \( f \in \hat{F} \). Then \( \mathcal{D}^-(f, \kappa) \) is solvable if and only if \( f \perp \mathcal{N}^* \).

The condition \( f \perp \mathcal{N}^* \) must be clarified, since for \( f \in \hat{F} \) and \( v \in \mathcal{N}^* \) the integral \( (f, v)_{0,0} \) need not exist. But it exists as a Cauchy mean value. For let \( \phi \in C_x([0, \infty[ \times [0, 1], 0 \leq \phi \leq 1 \text{ in } [1, 2[ \text{ and } \phi \equiv 0 \text{ in } [2, \infty[, \) and denote for \( R > 0 \)

\[
\phi_R(x) := \phi(|x|/R); \quad \phi_R'(x) := \phi'(|x|/R) \in C_0^\infty(\mathbb{R}^p).
\]

(35)

Then for each \( f \in \hat{F} \) and \( v \in \mathcal{N}^* \)

\[
\langle f, v \rangle := \lim_{R \to \infty} (\phi_R f, v)_{0,0},
\]

exists. Thus \( f \perp \mathcal{N}^* \) means \( \langle f, v \rangle = 0 \) for all \( v \in \mathcal{N}^* \).
The existence of the limit in (36) is not difficult to show. If $\chi \in C_0^{\infty}(\mathbb{R}^p)$ is identically 1 in a neighborhood of $\partial G$, a simple computation yields

$$\langle f, v \rangle = (\chi f, v)_{0,G} + ((1 - \chi) f, v_2)_{0,G} - ((\rho + A) (f - \chi f), v_1)_{0,G}.$$  

We first show Lemma 4.2.

**Lemma 4.2.** The condition $f \perp N^{*}$ is necessary for the solvability of $\mathcal{D}(f, \kappa)$.

**Proof.** Let $v$ solve $\mathcal{D}^*(0, -\kappa)$ and $u \mathcal{D}^*(f, \kappa)$ with $f \in \hat{F}$. Then

$$B(u, \phi_R v) = ((L + \kappa^2) u, (L + \kappa^2)(\phi_R v))_{0,G} = (f, \phi_R v)_{0,G} \xrightarrow{R \to \infty} \langle f, v \rangle$$

and

$$B(\phi_R u, v) = 0.$$  

So it is sufficient to show, that for $R \to \infty$ we have

$$I_R = -\kappa^{-2}/2 \left[ B(\phi_R u, v) - B(u, \phi_R v) \right] \to 0.$$  

Letting

$$Z(R) = K(2R) - \overline{K(R)}$$

we get

$$I_R = \int_{Z(R)} [u_1(\Lambda \bar{u}_1 + \bar{v}_2) - \bar{v}_1(\Lambda u_1 + u_2)] \partial_i(a_{ij} \partial_j \phi_R) \, dx$$

$$+ \int_{Z(R)} [u_1 \partial_i(\Lambda \bar{u}_1 + \bar{v}_2) - \bar{v}_1 \partial_i(\Lambda u_1 + u_2)] (2a_{ij} \partial_j \phi_R) \, dx.$$  

Partial integration in the first integral yields

$$I_R = \int_{Z(R)} (a_{ij} \partial_j \phi_R) (u_1 \partial_i \bar{v}_2 - \partial_i u_1) \, dx$$

$$- \int_{Z(R)} (a_{ij} \partial_j \phi_R) (\bar{v}_1 \partial_i u_2 - \partial_i \bar{v}_1) \, dx$$

$$+ \int_{Z(R)} (a_{ij} \partial_j \phi_R) (u_1 \partial_i \Lambda \bar{v}_1 - \bar{v}_1 \partial_i \Lambda u_1 + \partial_i \bar{v}_1 \cdot \Lambda u_1 - \partial_i u_1 \cdot \Lambda \bar{v}_1) \, dx$$

$$=: I_R^{(1)} + I_R^{(2)} + I_R^{(3)}.$$  

Now

$$\partial_j \phi_R(x) = R^{-1} \left| x \right|^{-1} x_j \phi_{\kappa'}(x)$$

and

$$\partial_i (a_{ij} \partial_j \phi_R) = R^{-1} \phi_{\kappa'} \cdot D.$$  

(37)
Therefore $I_R^{(1)}$ can be estimated by

$$K \cdot \int_{Z(R)} |x|^{-1} (|u_1| |Dv_2| + |v_2| |Du_1 - i\kappa u_1|) \, dx,$$

(38)

$K > 0$ being independent of $R$. Hence $I_R^{(1)}$ tends to 0 when $R$ tends to $\infty$ by Theorem 2.4. In the same way one proves $I_R^{(2)} \to 0$.

Commuting $D$ and $A$ by (12), $I_R^{(3)}$ takes the form

$$I_R^{(3)} = \int_{Z(R)} R^{-1} \phi_R' (u_1 A Dv_1 - \bar{v}_1 A Du_1 + Dv_1 \cdot Au_1 - Du_1 \cdot A\bar{v}_1) \, dx$$

$$+ \int_{Z(R)} R^{-1} \phi_R' (v_1 Dv_1 - \bar{v}_1 Du_1) \, dx$$

$$- \int_{Z(R)} R^{-1} \phi_R' (u_1 D_M \bar{v}_1 - \bar{v}_1 D_M u_1) \, dx.$$

The second integral can be estimated as in (38) and therefore tends to 0. With some positive constant $K$, independent of $R$, the third integral can be estimated by

$$K \cdot \int_{Z(R)} |x|^{-5/2-\varepsilon} (|u_1| |\nabla v_1| + |v_1| |\nabla u_1|) \, dx,$$

(39)

and therefore tends to 0 as $R$ tends to $\infty$ by Theorem 2.4. By partial integration of the first two terms of the sum, the first integral takes the form:

$$\int_{Z(R)} R^{-1} \phi_R' (-Au_1 \cdot D\bar{v}_1 + A\bar{v}_1 \cdot Du_1 - A\bar{v}_1 \cdot Du_1 + Au_1 \cdot D\bar{v}_1) \, dx$$

$$- \int_{Z(R)} R^{-1}(\phi + A) \phi_R'(u_1 D\bar{v}_1 - \bar{v}_1 Du_1) \, dx.$$

Here the first integral vanishes identically and the second one tends to 0 as $R$ tends to $\infty$, since it can be estimated as in (38). Hence the lemma is proved.

We now prove Theorem 4.1. Certainly we have dim $N^* = \dim N$, because $v$ lies in $N$, if and only if $\bar{v}$ lies in $N^*$.

It remains to show that $f_\perp N^*$ is sufficient for the solvability of our problem. With some nonnegative test function $\psi \in C_0^\infty(G)$, $\psi \neq 0$, we define the linear operator

$$\Psi: N^* \to N(Id - \Gamma^*), \quad v \mapsto \Pi \Theta(\psi v),$$

where $\Pi$ denotes the orthogonal projection from $H_2(G_R)$ onto $N(Id - \Gamma^*)$. As in [8], we prove that $\Psi$ is bijective: If $\Psi v = 0$ for $v \in N^*$, then by Theorem
3.3 $\mathcal{D}^{-}(\psi v, \kappa)$ is solvable. By Lemma 4.2 $v \equiv 0$ in the support of $\psi$ and also in $G$ by Corollary 2.3. Thus $\Psi$ is injective. Because
\[
\dim \mathcal{N}^* = \dim \mathcal{N} = \dim (\text{Id} - \Gamma^*)
\]
(Theorem 3.3) it is also bijective.

If for $f \in \bar{F}$ $\mathcal{D}^{-}(f, \kappa)$ is not solvable, then we get from Theorem 3.3 and the fact, that $\Psi$ is bijective:
\[
0 \neq \Pi \Theta f = \Psi v \quad \text{for some } v \in \mathcal{N}^*.
\]
Then $\Pi \Theta (f - \psi v) = 0$, and Theorem 3.3 implies the solvability of $\mathcal{D}^{-}(f - \psi v, \kappa)$. Then from Lemma 4.2 we know
\[
\langle f, v \rangle = \langle \psi v, v \rangle \neq 0
\]
for $v \neq 0$ in supp $\psi$ because of (40). But this proves the theorem.

5. THE CASE OF DIFFERENT RADIATION CONDITIONS

In this section we shall prove a uniqueness theorem for the problem $\mathcal{D}^{+}(f, \kappa)$, provided the space dimension $p$ is larger than 2. Combining this and Theorem 3.3 we get

**THEOREM 5.1.** $\mathcal{D}^{+}(f, \kappa)$ is uniquely solvable for all $f \in \bar{F}$, if $p > 2$.

Let $u$ be a solution of $\mathcal{D}^{+}(0, \kappa)$. We shall prove that we have
\[
\text{Im} \int_{|z|=R} D\bar{u}_1 \cdot u_t \, do_R = o(1),
\]
for $R \to \infty$. Once having proved (41), the radiation condition implies
\[
\lim \inf_{R \to \infty} \int_{|z|=R} (\kappa^2 |u_1|^2 + |Du_1|^2) \, do_R
\]
\[
= \lim \inf_{R \to \infty} \int_{|z|=R} |Du_1 - i\kappa u_1|^2 \, do_R = 0.
\]
Then Rellich's estimate [3, Theorem 2] and the unique continuation principle [5] yield $u_1 \equiv 0$ in $G$. But then $u = u_2 \in \mathcal{L}(G)$ and vanishes by Jäger's uniqueness theorem [3, Theorem 4].
To prove (41) we define \( \phi_R \) as in (35) and get:

\[
0 = B(u, \phi_R u) = \int_G \phi_R ((L + \kappa^2) u) \cdot ((L + \kappa^2) \bar{u}) \, dx
- 2\kappa^2 \int_{Z(R)} \left[ (\partial_i (a_{ij} \partial_j \phi_R)) (u_i \bar{u}) + 2u_i (a_{ij} \partial_i \phi_R) \cdot \partial_j \bar{u} \right] \, dx.
\]

Since the first integral is real, the imaginary part of the second integral vanishes. We integrate the first term of the sum in the second integral by parts. Considering (37) and commuting \( D \) and \( \Lambda \) by (12), we get:

\[
0 = \text{Im} \int_{Z(R)} R^{-1} \phi_R' (u_i D\bar{u}_2 - D_{u_1} \cdot \bar{u}_2) \, dx
+ \text{Im} \int_{Z(R)} R^{-1} \phi_R' (u_1 \Lambda D\bar{u}_1 - D_{u_1} \cdot \Lambda \bar{u}_1) \, dx
+ \text{Im} \int_{Z(R)} R^{-1} \phi_R' u_1 D\bar{u}_1 \, dx
- \text{Im} \int_{Z(R)} R^{-1} \phi_R' u_1 D_{u_1} \bar{u}_1 \, dx.
\]

The first and the last integral tend to 0 when \( R \) tends to \( \infty \), for they can be estimated as in (38) and (39) resp. By partial integration of the first term of the second integral, this integral can be transformed into

\[
- \text{Im} \int_{Z(R)} (R^{-1} (\rho + \Lambda) \phi_R') \cdot u_1 D\bar{u}_1 \, dx
- \text{Im} \int_{Z(R)} R^{-1} \phi_R' (\Lambda u_1 \cdot D\bar{u}_1 + D_{u_1} \cdot \Lambda \bar{u}_1) \, dx.
\]

Here the second integral is real. Therefore we get finally for \( R \to \infty \):

\[
o(1) = \text{Im} \int_{Z(R)} R^{-1} (\rho - 1 + \Lambda) \phi_R' \cdot u_1 D\bar{u}_1 \, dx
= \text{Im} \int_{Z(R)} (R^{-1} \vert x \vert^{-1} x_i (\rho - 1 + x_k \partial_k) \phi_R') \cdot a_{ij} u_1 \partial_j \bar{u}_1 \, dx.
\]

Because of (37) and the identity

\[
R^{-1} \vert x \vert^{-1} x_i x_k \partial_k \phi_R (x) = x_k \partial_k \partial_i \phi_R (x)
\]
we get for $R \to \infty$:

$$o(1) = \text{Im} \int_{Z(R)} (p - 1) (\partial_1 \phi_R) (u_1 a_{ij} \partial_j \bar{u}_1) \, dx$$

$$+ \text{Im} \int_{Z(R)} (\partial_1 \partial_2 \phi_R) (x a_{ij} \partial_i \partial_j \bar{u}_1) \, dx$$

$$- \text{Im} \int_{Z(R)} (p - 1) \psi_R \partial_1 (u_1 a_{ij} \partial_j \bar{u}_1) \, dx$$

$$- \text{Im} \int_{Z(R)} (x a_1 \partial_1 \phi_R) \partial_1 (u_1 a_{ij} \partial_j \bar{u}_1) \, dx$$

$$- \text{Im} \int_{Z(R)} (\partial_1 \phi_R) (u_1 a_{ij} \partial_j \bar{u}_1) \, dx - \text{Im} \int_{|z|=R} (p - 1) u_1 D \bar{u}_1 \, d\sigma_R,$$

where partial integration was used to obtain the second equality. The first two integrals are real. Integrating the third integral by parts we get finally for $R \to \infty$:

$$(2 - p) \cdot \text{Im} \int_{|z|=R} u_1 D \bar{u}_1 \, d\sigma_R + \text{Im} \int_{Z(R)} \phi_R \partial_1 (u_1 a_{ij} \partial_j \bar{u}_1) \, dx = o(1).$$

This proves (41), since the second integral is real.

6. Higher Order Equations

Assuming that the coefficients of $L$ and $f$ are sufficiently smooth and possess a suitable asymptotic behavior for $|x| \to \infty$, one can split solutions $u$ of

$$P(L) u = f,$$

into

$$u = \sum_{j=1}^{o} \sum_{\nu=0}^{\mu_j - 1} A^\nu u_{j\nu},$$

in a similar way as in Section 2. Here $P$ is a polynomial with complex coefficients, possessing $\rho$ zeros $-\kappa_j^2 \neq 0$ $(\text{Im } \kappa_j \geq 0)$ of order $\mu_j$, i.e.,

$$P(\xi) = \prod_{j=1}^{o} (\xi + \kappa_j^2)^{\mu_j}.$$

The functions $u_{j\nu}$ satisfy equations of the kind

$$(L \mid \kappa_j^2) u_{j\nu} = g_{j\nu} \quad (j = 1, \ldots, \rho; \nu = 0, \ldots, \mu_j - 1),$$

(44)
where \( g_{jv} \) are sufficiently smooth functions with a suitable asymptotic behavior. \( g_{jv} \) is given by \( f \) and those \( u_{iv} \), for which \( (i, \sigma) \) is lexically greater than \( (j, \nu) \). Thus (44) can be considered as a "triangular" system of differential equations.

If on the other hand functions \( u_{jv} \) solve the "triangular" system (44) they may be used to construct a solution of (42) by (43).

Therefore one may formulate asymptotic conditions for \( u \) by imposing Sommerfeld radiation conditions on \( u_{jv} \). The \( u_{jv} \), by the way, can be obtained by applying some differential operators \( N_{jv} \) to \( u \). In the case \( G = \mathbb{R}^p \) Eq. (42) therefore has exactly one solution satisfying the previous asymptotic conditions. For in this case (42) and (44) are equivalent.

But there is the difficulty that the operators \( N_{jv} \) are not uniquely determined so that many splittings of the form (43), (44) are possible. However it can be shown that all those splittings yield the same solutions.

**References**