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Expansion of a Special Operator*

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1. INTRODUCTION

We consider the expansion of the operator

$$(xD_y + yD_x)^n = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y)\, D_x{}^j D_y{}^k \qquad (n = 0, 1, 2,\ldots). \tag{1.1}$$

where

$$D_x = \frac{\partial}{\partial x}, \qquad D_y = \frac{\partial}{\partial y}.$$

It is evident from (1.1) that $C_{j,k}^{(n)}(x, y)$ is a polynomial in x, y with nonnegative integral coefficients. Moreover, replacing x, y by λx, λy, where λ is an arbitrary constant, we infer that $C_{j,k}^{(n)}(x, y)$ is a homogeneous polynomial of degree $j + k$. Also it is evident that

$$C_{0,0}^{(0)}(x, y) = 1, \qquad C_{0,0}^{(n)}(x, y) = 0 \qquad (n > 0) \tag{1.2}$$

and

$$C_{j,k}^{(n)}(x, y) = C_{k,j}^{(n)}(y, x). \tag{1.3}$$

Generalizing (1.1), we take

$$(axD_y + byD_x)^n = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y \mid a, b)\, D_x{}^j D_y{}^k \tag{1.4}$$

where a, b are constants. Replacing x, y by λx, μy, where λ, μ are constants, (1.4) becomes

$$(a\lambda\mu^{-1}xD_y + b\lambda^{-1}\mu yD_x)^n = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(\lambda x, \mu y \mid a, b)\, \lambda^{-j}\mu^{-k} D_x{}^j D_y{}^k. \tag{1.5}$$

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Since the left-hand side of (1.5) is equal to

$$\sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y \mid a\lambda\mu^{-1}, b\lambda^{-1}\mu)\, D_x^{\,j} D_y^{\,k},$$

it follows that

$$C_{j,k}^{(n)}(\lambda x, \mu y \mid a, b) = \lambda^j \mu^k C_{j,k}^{(n)}(x, y \mid a\lambda\mu^{-1}, b\lambda^{-1}\mu). \tag{1.6}$$

In particular, if we take $\lambda^2 = b$, $\mu^2 = a$, (1.6) becomes

$$\begin{aligned} C_{j,k}^{(n)}(\lambda x, \mu y \mid a, b) &= a^{1/2k} b^{1/2j} C_{j,k}^{(n)}(x, y \mid (ab)^{1/2}, (ab)^{1/2}) \\ &= a^{1/2(k+n)} b^{1/2(j+n)} C_{j,k}^{(n)}(x, y \mid 1, 1) \\ &= a^{1/2(k+n)} b^{1/2(j+n)} C_{j,k}^{(n)}(x, y). \end{aligned} \tag{1.7}$$

Thus there is no real loss in generality in restricting the discussion to the case $a = b = 1$.

In the next place put

$$(xD_y + yD_x)^n\, x^r y^s = \sum_{j=0}^{n} f_j^{(n)}(r, s)\, x^{r+n-2j} y^{s-n+2j}. \tag{1.8}$$

As we shall see below, the coefficients $f_j^{(n)}(r, s)$ are polynomials in r, s; indeed if r and s are nonnegative integers then the $f_j^{(n)}(r, s)$ are also non-negative integers. We define the polynomial $F_{r,s}^{(n)}(x, y)$ by means of

$$F_{r,s}^{(n)}(x, y) = \sum_{j=0}^{n} f^{(n)}(r, s)\, x^{r+n-2j} y^{s-n+2j}. \tag{1.9}$$

The polynomials $C_{j,k}^{(n)}(x, y)$ and $F_{r,s}^{(n)}(x, y)$ are closely related. Indeed

$$p!q!C_{p,q}^{(n)}(x, y) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} x^{p-r} y^{q-s} F_{r,s}^{(n)}(x, y) \tag{1.10}$$

and

$$F_{p,q}^{(n)}(x, y) = \sum_{r=0}^{p} \sum_{s=0}^{q} \binom{p}{r} \binom{q}{s} r!s! x^{p-r} y^{q-s} C_{r,s}^{(n)}(x, y). \tag{1.11}$$

The relations (1.10), (1.11) are equivalent to

$$p!q!c_{q-n+2j}(n, p, q) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} f_j^{(n)}(r, s) \tag{1.12}$$

and

$$f_j^{(n)}(p, q) = \sum_{r=0}^{p} \sum_{s=0}^{q} \binom{p}{r} \binom{q}{s} r!s!c_{s-n+2j}(n, r, s), \tag{1.13}$$

where the coefficients $c_k(n, p, q)$ are defined by

$$C_{p,q}^{(n)}(x, y) = \sum_{k=0}^{p+q} c_k(n, p, q)\, x^{p+q-k} y^k. \tag{1.14}$$

We shall show that the polynomials $C_{j,k}^{(n)}(x, y)$, $F_{r,s}^{(n)}(x, y)$ satisfy the generating relations

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y)\, u^j v^k = \exp\{(xu + yv)(\cosh z - 1) + (xv + yu \sinh z\} \tag{1.15}$$

and

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r,s=0}^{\infty} F_{r,s}^{(n)}(x, y) = \exp\{(xu + yv)\cosh z + (xv + yu)\sinh z\}, \tag{1.16}$$

respectively.

The generating functions (1.15), (1.16) imply the following combinatorial results. Let $A(n, j, k)$ denote the number of partitions of the set $z_n = \{1, 2,..., n\}$ into $j + k$ nonempty blocks of which j have even cardinality and k have odd cardinality. Then we have

$$c_k(n, j + k, 0) = A(n, j, k), \qquad c_k(n, 0, j + k) = A(n, k, j). \tag{1.17}$$

Moreover

$$c_k(n, p, q) = \sum_{\substack{a+b=k \\ r+s=n}} \binom{n}{r} A(r, p - a, a)\, A(s, b, q - s). \tag{1.18}$$

Next let $D(n, r, s)$ denote the number of ways of putting n numbered objecs into $r + s$ numbered boxes so that each of any r boxes contains an even number of objects while each of the remaining s boxes contains an odd number of objects; it is to be understood that all selections of the first r boxes are counted. Then we have

$$f_j^{(n)}(r, 0) = D(n, r + n - 2j, 2j - n), \quad f_j^{(n)}(0, s) = D(n, n - 2j, s - n + 2j) \tag{1.19}$$

and

$$f_k^{(n)}(r, s) = \sum_{\substack{a+b=n \\ i+j=k}} \binom{n}{a} D(a, r + a - 2i, 2i - a)\, D(b, b - 2j, s - b + 2j). \tag{1.20}$$

A number of related questions are suggested by (1.1). For example it would be of interest to determine the coefficients $B_{j,k}(x, y)$ in the expansion

$$(axE_y + byE_x)^n = \sum_{j+k\leqslant n} B_{j,k}^{(n)}(x, y)\, E_x{}^j E_y{}^k, \tag{1.21}$$

where the operators E_x, E_y are defined by

$$E_x f(x, y) = f(x + 1, y), \qquad E_y f(x, y) = f(x, y + 1).$$

Similar questions can be posed for other families of linear operators. Also one may seek the operators L_x, L_y such that

$$(xL_y + yL_x)^n = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y) L_x^{\,j} L_y^{\,k}, \tag{1.22}$$

where the $C_{j,k}^{(n)}(x, y)$ are the same as the coefficients in (1.1). This question was suggested by a recent paper by Al-Salam and Ismail [1] concerning the operational formula

$$(AC)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} C(C-1)\cdots(C-k+1)\, A^n,$$

where A, C are operators satisfying

$$AC - CA = A. \tag{1.23}$$

Since the coefficients of $C_{r,s}^{(n)}(x, y)$ and $F_{r,s}^{(n)}(x, y)$ are integers it is natural to look for arithmetic properties. It can for example be shown that, if p is an odd prime, then

$$\sum_{r+s=p} C_{r,s}^{(n)}(x, y)\, u_r v^s \equiv S(n, p)\, (u^p y^p + v^p x^p) \qquad (\text{mod}\, p), \tag{1.24}$$

where $S(n, p)$ denotes a Stirling number of the second kind.

For $F_{r,s}^{(n)}(x, y)$ we have the congruence of Kummer's type (compare [4, Ch. 14])

$$\sum_{j=0}^{t} (-1)^{jt} F_{r,s}^{(n+j(p-1))}(x, y) \equiv 0 \qquad (\text{mod}\, p^t), \tag{1.25}$$

where p is an arbitrary prime and $n \geqslant t \geqslant 1$.

We shall however not include the derivation of such results in the present paper.

2. Preliminaries

As above put

$$(xD_y + yD_x)^n = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y)\, D_x^{\,j} D_y^{\,k} \qquad (n = 0, 1, 2, \ldots). \tag{2.1}$$

As noted, $C_{j,k}^{(n)} = C_{j,k}^{(n)}(x, y)$ is homogeneous of degree $j + k$ in x, y and has nonnegative integral coefficients.

If we multiply both sides of (2.1) on the left by $xD_y + yD_x$ we get

$$(xD_y + yD_x)^{n+1} = (xD_y + yD_x) \sum_{y+k\leqslant n} C_{j,k}^{(n)} D_x{}^j D_y{}^k$$
$$= \sum_{j+k=n} \{x(D_y C_{j,k}^{(n)} + C_{j,k}^{(n)} C_y) + y(D_x C_{j,k}^{(n)} + C_{j,k}^{(n)} D_x)\} D_x{}^j D_y{}^k.$$

It follows that

$$C_{j,k}^{(n+1)} = (xD_y + yD_x)\, C_{j,k}^{(n)} + xC_{j,k-1}^{(n)} + yC_{j-1,k}^{(n)}\,. \tag{2.2}$$

Similarly, multiplying (2.1) on the right by $xD_y + yD_x$, we get

$$C_{j,k}^{(n+1)} = xC_{j,k-1}^{(n)} + yC_{j-1,k}^{(n)} + (j+1)\, C_{j+1,k-1}^{(n)} + (k+1)\, C_{j-1,k+1}^{(n)}\,. \tag{2.3}$$

Comparison of (2.3) with (2.2) gives

$$(j+1)\, C_{j+1,k-1}^{(n)} + (k+1)\, C_{j-1,k+1}^{(n)} = (xD_y + yD_x)\, C_{j,k}^{(n)}\,. \tag{2.4}$$

If we make use of the translation operators E_j, E_k defined by

$$E_j f(j, k) = f(j+1, k), \qquad E_k f(j, k) = f(j, k+1),$$

(2.2), (2.3), (2.4) become

$$C_{j,k}^{(n+1)} = (xD_y + yD_x + xE_k^{-1} + yE_j^{-1})\, C_{j,k}^{(n)}, \tag{2.5}$$

$$C_{j,k}^{(n+1)} = (xE_k^{-1} + yE_j^{-1} + (j+1)\, E_j E_k^{-1} + (k+1)\, E_j^{-1} E_k)\, C_{j,k}^{(n)}, \tag{2.6}$$

$$(xD_y + yD_x)\, C_{j,k}^{(n)} = ((j+1)\, E_j E_k^{-1} + (k+1)\, E_j^{-1} E_k)\, C_{j,k}^{(n)}, \tag{2.7}$$

respectively.

The first few values of $C_{j,k}^{(n)}$ can be computed either directly from (2.1) or by using one of the recurrences. We find that

$$(xD_y + yD_x)^2 = xD_x + yD_y + x^2 D_y{}^2 + 2xyD_x D_y + y^2 D_x{}^2,$$
$$(xD_y + yD_x)^3 = xD_y + yD_x + 3xyD_y{}^2 + 3(x^2 + y^2)\, D_x D_y + 3xyD_x{}^2$$
$$+ x^3 D_y{}^3 + 3x^2 y D_x D_y{}^2 + 3xy^2 D_x{}^2 D_y + y^3 D_x{}^3.$$

These special results suggest that

$$C_{n,0}^{(n)} = y^n, \qquad C_{0,n}^{(n)} = x^n \tag{2.8}$$

and

$$C_{1,0}^{(n)} = x \quad (n \text{ even}) \qquad = y \quad (n \text{ odd}), \tag{2.9}$$

$$C_{0,1}^{(n)} = y \quad (n \text{ even}) \qquad = x \quad (n \text{ odd}). \tag{2.10}$$

Formula (2.8) follows at once from (2.6) while (2.9) and (2.10) are implied by (2.2).

We have also by induction, using (2.2),

$$C^{(n)}_{n-k,k} = \binom{n}{k} x^k y^{n-k} \qquad (0 \leqslant k \leqslant n). \tag{2.11}$$

3. The Polynomials $F^{(n)}_{r,s}(x, y)$

We define the polynomial $F^{(n)}_{r,s}(x, y)$ and the coefficients $f^{(n)}_j(r,s)$ by means of

$$\begin{aligned} F^{(n)}_{r,s}(x, y) &\equiv (xD_y + yD_x)^n x^r y^s \\ &= \sum_{j=0}^{n} f^{(n)}_j(r, s)\, x^{r+n-2j} y^{s-n+2j}. \end{aligned} \tag{3.1}$$

That the extreme right member has the stated appearance follows by induction. Moreover we get the recurrence

$$f^{(n+1)}_j(r, s) = (s - n + 2j) f^{(n)}_j(r, s) + (r + n - 2j + 2) f^{(n)}_{j-1}(r, s). \tag{3.2}$$

Also, taking

$$(xD_y + yD_x)^{n+1} = (xD_y + yD_x)^n (xD_y + yD_x),$$

we get

$$f^{(n+1)}_j(r, s) = s f^{(n)}_j(r + 1, s - 1) + r f^{(n)}_{j-1}(r - 1, s + 1). \tag{3.3}$$

The first few values of $f^{(n)}_j(r, s)$, with r, s fixed, follow.

$f^{(n)}_j(r, s)$:

$n \backslash j$	0	1	2	3
0	1			
1	s	r		
2	$s(s-1)$	$2rs + r + s$	$r(r-1)$	
3	$s(s-1)(s-2)$	$s(3rs + 3s - 2)$	$r(3rs + 3r - 2)$	$r(r-1)(r-2)$

Clearly $f_j^{(n)}(r, s)$ is a polynomial of degree n in r, s with rational coefficients and

$$f_{n-j}^{(n)}(r, s) = f_j^{(n)}(s, r) \qquad (0 \leqslant j \leqslant n). \tag{3.4}$$

Also it follows from (3.3) that

$$\begin{aligned} f_0^{(n)}(r, s) &= s(s-1) \cdots (s-n+1), \\ f_n^{(n)}(r, s) &= r(r-1) \cdots (r-n+1). \end{aligned} \tag{3.5}$$

In the next place, by (3.3),

$$\sum_{j=0}^{n+1} f_j^{(n+1)}(r, s) = s \sum_{j=0}^{n} f_j^{(n)}(r+1, s-1) + r \sum_{j=0}^{n} f_j^{(n)}(r-1, s+1).$$

Since

$$f_0^{(1)}(r, s) + f_1^{(1)}(r, s) = r + s,$$

we get

$$F_{r,s}^{(n)}(1, 1) = \sum_{j=0}^{n} f_j^{(n)}(r, s) = (r+s)^n \qquad (n = 0, 1, 2,...). \tag{3.6}$$

By (3.1) and (3.3), we have

$$F_{r,s}^{(n+1)}(x, y) = \sum_{j=0}^{n} \{sf_j^{(n)}(r+1, s-1) + rf_{j-1}^{(n)}(r-1, s+1)\} \, x^{r+n+1-2j} y^{s-n-1-2j},$$

which yields

$$F_{r,s}^{(n+1)}(x, y) = sF_{r+1,s-1}^{(n)}(x, y) + rF_{r-1,s+1}^{(n)}(x, y). \tag{3.7}$$

Iteration of (3.7) gives

$$\begin{aligned} F_{r,s}^{(n+2)}(x, y) &= s(s-1) F_{r+2,s-2}^{(n)}(x, y) + (2rs + r + s) F_{r,s}^{(n)}(x, y) \\ &\quad + r(r-1) F_{r-2,s+2}^{(n)}(x, y), \end{aligned}$$

$$\begin{aligned} F_{r,s}^{(n+3)}(x, y) &= s(s-1)(s-2) F_{r+3,s-3}^{(n)}(x, y) + s(3rs + 3s - 2) F_{r+1,s-1}^{(n)}(x, y) \\ &\quad + r(3rs + 3r - 2) F_{r-1,s+1}^{(n)}(x, y) + r(r-1)(r-2) F_{r-3,s+3}^{(n)}(x, y). \end{aligned}$$

These results suggest that

$$F_{r,s}^{(m+n)}(x, y) = \sum_{j=0}^{m} f_j^{(m)}(r, s) \, F_{r+m-2j,s-m+2j}^{(n)}(x, y). \tag{3.8}$$

The proof of (3.8) is by induction on m and will be omitted.

A formula equivalent to (3.8) is

$$f_k^{(m+n)}(r, s) = \sum_{j=0}^{n} f_j^{(m)}(r, s) f_{k-j}^{(n)}(r + m - 2j, s - m + 2j). \tag{3.9}$$

It follows from (2.1) and (3.1) that

$$F_{r,s}^{(n)}(x, y) = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y) \frac{r!}{(r-j)!} \frac{s!}{(s-k)!} x^{r-j} y^{s-k}. \tag{3.10}$$

Multiply both sides of (3.10) by

$$(-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} x^{p-r} y^{q-s}$$

and sum over r, s. Since

$$\sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} x^{p-r} y^{q-s} \cdot \sum_{j+k\leqslant n} j!k! \binom{r}{j} \binom{s}{k} C_{j,k}^{(n)}(x, y) x^{r-j} y^{s-k}$$

$$= \sum_{j=0}^{p} \sum_{k=0}^{q} j!k! x^{p-j} y^{q-k} C_{j,k}^{(n)}(x, y) \cdot \sum_{r=j}^{p} (-1)^{p-r} \binom{p}{r} \binom{r}{j} \sum_{s=k}^{q} (-1)^{q-s} \binom{q}{s} \binom{s}{k}$$

$$= p!q! C_{p,q}^{(n)}(x, y),$$

it follows that

$$p!q! C_{p,q}^{(n)}(x, y) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} x^{p-r} y^{q-s} F_{r,s}^{(n)}(x, y). \tag{3.11}$$

In particular, for $y = x$, we get using (3.6),

$$p!q! C_{p,q}^{(n)}(x, x) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} (r + s)^n x^{p+q}.$$

Since

$$\sum_{r+s=k} \binom{p}{r} \binom{q}{s} = \binom{p+q}{k}$$

and

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n = m! S(n, m), \tag{3.12}$$

where $S(n, m)$ is a Stirling number of the second kind, we get

$$C_{p,q}^{(n)}(x, x) = \binom{p+q}{p} S(n, p + q) x^{p+q}. \tag{3.13}$$

Since $C_{p,q}^{(n)}(x, y)$ is homogeneous of degree $p + q$ in x, y, we may put

$$C_{p,q}^{(n)}(x, y) = \sum_{k=0}^{p+q} c_k(n, p, q)\, x^{p+q-k} y^k, \tag{3.14}$$

where the $c_k(n, p, q)$ are nonnegative integers. Thus by (3.11) and (3.14), we get

$$p!q!c_{q-n+2j}(n, p, q) = \sum_{r=0}^{p} \sum_{s=0}^{q} (-1)^{p+q-r-s} \binom{p}{r} \binom{q}{s} f_j^{(n)}(r, s) \qquad (0 \leqslant j \leqslant n). \tag{3.15}$$

An equivalent result is

$$f_j^{(n)}(p, q) = \sum_{r=0}^{p} \sum_{s=0}^{q} r!s! \binom{p}{r} \binom{q}{s} c_{s-n+2j}(n, r, s) \qquad (0 \leqslant j \leqslant n). \tag{3.16}$$

4. Evaluation of $C_{j,k}^{(n)}(x, y)$

Consider functions $F = F(x, y)$ such that

$$(xD_y + yD_x) F = F. \tag{4.1}$$

The general solution of this differential equation is given by

$$F = (x + y)\, \phi(x^2 - y^2), \tag{4.2}$$

where $\phi(z)$ is an arbitrary function of z. More generally, the general solution of

$$(xD_y + yDx) F = \lambda F, \tag{4.3}$$

where λ is constant, is given by

$$F = (x + y)^\lambda\, \phi(x^2 - y^2), \tag{4.4}$$

where again ϕ is arbitrary.

We first take

$$F = (x + y)^p \qquad (p = 0, 1, 2, \ldots). \tag{4.5}$$

Since

$$(xD_y + yD_x)\,(x + y)^p = p(x + y)^p,$$

it follows that

$$(xD_y + yD_x)^n\,(x + y)^p = p^n(x + y)^p. \tag{4.6}$$

On the other hand,

$$\sum_{j+k\leqslant n} C_{j,k}^{(n)} D_x{}^j D_y{}^k (x + y)^p = \sum_{j+k\leqslant n} C_{j,k}^{(n)} \frac{p!}{(p-j-k)!} (x + y)^{p-j-k}$$

$$= \sum_{m=0}^{p} \frac{p!}{(p-m)!} (x + y)^{p-m} \sum_{j+k=m} C_{j,k}^{(n)},$$

so that

$$p^n(x+y)^p = \sum_{m=0}^{p} \frac{p!}{(p-m)!}(x+y)^{p-m} \sum_{j+k=m} C_{j,k}^{(n)}. \tag{4.7}$$

Multiplying both sides of (4.7) by

$$(-1)^{r-p}\binom{r}{p}(x+y)^{r-p}$$

and summing over p, we get, after a little manipulation,

$$\sum_{j+k=r} C_{j,k}^{(n)}(x,y) = (x+y)^r S(n,r), \tag{4.8}$$

where, as in (3.13), $S(n, r)$ is a Stirling number of the second kind.

To improve this result we take

$$F = (x+y)^p (x-y)^q, \tag{4.9}$$

where p, q are nonnegative integers. It is easily verified that

$$(XD_y + yD_x)^n (x+y)^p (x-y)^q = (p-q)^n (x+y)^p (x-y)^q. \tag{4.10}$$

Also

$$D_x{}^j D_y{}^k (x+y)^p (x-y)^q$$

$$= D_x{}^j \sum_{b=0}^{k} (-1)^b \binom{k}{b} \frac{p!}{(p-k+b)!} \frac{q!}{(q-b)!} (x+y)^{p-k+b} (x-y)^{q-b}$$

$$= \sum_{b=0}^{k} (-1)^b \binom{k}{b} \frac{p!}{(p-k+b)!} \frac{q!}{(q-b)!}$$

$$\cdot \sum_{a=0}^{j} \binom{j}{a} \frac{(p-k+b)!}{(p-j-k+a+b)!} \frac{(q-b)!}{(q-a-b)!} (x+y)^{p-j-k+a+b} (x-y)^{q-a-b}.$$

It follows that

$$(p-q)^n (x+y)^p (x-y)^q$$

$$= \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x,y) \sum_{a=0}^{j} \sum_{b=0}^{k} (-1)^b \binom{j}{a}\binom{k}{b}$$

$$\cdot \frac{p!}{(p-j-k+a+b)!} \frac{q!}{(q-a-b)!} (x+y)^{p-j-k+a+b} (x-y)^{q-a-b}. \tag{4.11}$$

Let

$$a_m(j,k) = \sum_{a+b=m} (-1)^b \binom{j}{a}\binom{k}{b}, \tag{4.12}$$

so that (4.11) becomes

$$(p-q)^n (x+y)^p (x-y)^q$$
$$= \sum_{j+k\leqslant n} C^{(n)}_{j,k}(x,y) \sum_{m\leqslant j+k} \frac{p!}{(p-j-k+m)!}\,\frac{q!}{(q-m)!}\, a_m(j,k) \tag{4.13}$$
$$\cdot\, (x+y)^{p-j-k+m} (x-y)^{q-m}.$$

Now multiply both sides of (4.13) by

$$(-1)^{r+s-p-q} \binom{r}{p}\binom{s}{q} (x+y)^{r-p} (x-y)^{s-q}$$

and sum over p, q. We get

$$\sum_{p,q} (-1)^{r+s-q} \binom{r}{p}\binom{s}{q} (x+y)^{r-p} (x-y)^{s-q}$$
$$\cdot \sum_{j+k\leqslant n} C^{(n)}_{j,k}(x,y) \sum_{m\leqslant j+k} \frac{p!}{(p-j-k+m)!}\,\frac{q!}{(q-m)!}\, a_m(j,k)$$
$$\cdot\, (x+y)^{p-j-k+m} (x-y)^{q-m}$$
$$= \sum_{j+k\leqslant n} C^{(n)}_{j,k}(x,y) \sum_{m\leqslant j+k} (j+k-m)!\, m! a_m(j,k)$$
$$\cdot\, (x+y)^{r-j-k+m} (x-y)^{s-m} \sum_{p} (-1)^{r-p} \binom{r}{p}\binom{p}{j+k-m} \sum_{q} (-1)^{s-q} \binom{s}{q}\binom{q}{m}. \tag{4.14}$$

The inner sums on the extreme right vanish unless $j+k-m=r$ and $s=m$. Thus the right side of (4.14) reduces to

$$r!s! \sum_{j+k=r+s} a_s(j,k)\, C^{(n)}_{j,k}(x,y)$$

and therefore

$$\sum_{p=0}^{r} \sum_{q=0}^{s} (-1)^{r+s-p-q} \binom{r}{p}\binom{s}{q} (p-q)^n (x+y)^r (x-y)^s$$
$$= r!s! \sum_{j+k=r+s} a_s(j,k)\, C^{(n)}_{j,k}(x,y). \tag{4.15}$$

It follows from (4.12) that

$$\sum_{s=0}^{p+q} a_s(p, q)\, u^{p+q-s} v^s = (u + v)^p\, (u - v)^q. \tag{4.16}$$

Thus (4.15) yields

$$\begin{aligned} m! \sum_{j+k=m} & C^{(n)}_{j,k}(x, y)\, (u + v)^j\, (u - v)^k \\ &= \sum_{r+s=m} \binom{m}{s} u^{r+s-m} v^m \sum_{p=0}^{r} \sum_{q=0}^{s} (-1)^{r+s-p-q}\, (p - q)^n\, (x + y)^r\, (x - y)^s \\ &= \sum_{p+q\leqslant m} (-1)^{m-p-q}\, (p - q)^n\, (x + y)^p\, (x -)^q \\ &\quad \cdot \sum_{r+s=m} \frac{m!}{p!q!(r - p)!\,(s - q)!} (x + y)^{r-p}\, (x - y)^{s-q}\, u^r v^s \\ &= \sum_{p+q\leqslant m} (-1)^{m-p-q} \frac{m!}{p!q!(m - p - q)!} (p - q)^n\, (x + y)^p\, (x - y)^q\, u^p v^q \\ &\quad \cdot \sum_{r+s=m} \binom{m - p - q}{s - q} (x + y)^{r-p}\, (x - y)^{s-q}\, u^{r-p} v^{s-q}. \end{aligned}$$

We have therefore

$$\begin{aligned} m! \sum_{j+k=m} & C^{(n)}_{j,k}(x, y)\, (u + v)^j\, (u - v)^k \\ &= \sum_{p+q\leqslant m} (-1)^{m-p-q} \frac{m!}{p!q!(m - p - q)!} (p - q)^n\, (x + y)^p\, (x - y)^q\, p^p v^q \\ &\quad \cdot ((x + y)\, u + (x - y)\, v)^{m-p-q}. \end{aligned} \tag{4.17}$$

We now put

$$\bar{u} = u + v, \qquad \bar{v} = u - v,$$

so that

$$u = \tfrac{1}{2}(\bar{u} + \bar{v}), \qquad v = \tfrac{1}{2}(\bar{u} - \bar{v}).$$

Then (4.17) becomes (after dropping bars)

$$\begin{aligned} m! \sum_{j+k=m} & C^{(n)}_{j,k}(x, y)\, u^j v^k \\ &= \sum_{p+q\leqslant m} (-1)^{m-p-q} \frac{m!}{(m - p - q)!} (p - q)^n\, (x + y)^p\, (x - y)^q \\ &\quad \cdot 2^{-p-q}(u + v)^p\, (u - v)^q\, (xu + yv)^{m-p-q}. \end{aligned} \tag{4.18}$$

Comparing coefficients of $u^j v^k$ on both sides of (4.18), we get the explicit formula

$$m! C^{(n)}_{m-k,k}(x, y)$$

$$= \sum_{p+q \leq m} (-1)^{m-p-q} \frac{m!}{p!q!(m-p-q)!} (p-q)^n (x+y)^p (x-y)^q \qquad (4.19)$$

$$\cdot\, 2^{-p-q} \sum_{t=0}^{k} \binom{m-p-q}{t} a_{k-t}(p, q)\, x^{m-p-q-t} y^t.$$

5. Generating Functions

In first defining $C^{(n)}_{j,k}(x, y)$ we have $j + k \leq n$. We now change our point of view and define $C^{(n)}_{j,k}(x, y)$ by means of (4.18). It will soon be apparent, with this definition, that

$$C^{(n)}_{j,k}(x, y) = 0 \qquad (j + k > n), \qquad (5.1)$$

so that the new definition is in fact in agreement with the old.

It follows from (4.18) that

$$m! \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k=m} C^{(n)}_{j,k}(x, y)\, u^j v^k$$

$$= \sum_{p+q \leq m} (-1)^{m-p-q} \frac{m!}{p!q!(m-p-q)!} e^{(p-q)z}(x+y)^p (x-y)^q$$

$$\cdot\, 2^{-p-q}(u+v)^p (u-v)^q (xu+yv)^{m-p-q}$$

$$= \left\{ e^z(x+y)\frac{u+v}{2} + e^{-z}(x-y)\frac{u-v}{2} - xu - yv \right\}^m$$

$$= \{u[\tfrac{1}{2}x(e^z + e^{-z}) + y(e^z - e^{-z}) - x] + v[\tfrac{1}{2}x(e^z - e^{-z}) + y(e^z + e^{-z} - y)]\}^m.$$

We have therefore

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k=m} C^{(n)}_{j,k}(x, y)\, u^j v^k$$

$$= \frac{1}{m!} \{u[x(\cosh z - 1) + y \sinh z] + v[x \sinh z + y(\cosh z - 1)]\}^m. \qquad (5.2)$$

The expansion of the right hand side of (5.2) begins with the term $(yu + xv)^m/m!$, which evidently implies (5.1). Hence, as remarked above, the two definitions of $C^{(n)}_{j,k}(x, y)$ are equivalent.

Summing over m, (5.2) gives

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j,k=0}^{\infty} C_{j,k}^{(n)}(x, y)\, u^j v^k$$

$$= \exp\{u[x(\cosh z - 1) + y \sinh z] + v[y(\cosh z - 1) + x \sinh z]\}. \quad (5.3)$$

Thus we have obtained a generating function for $C_{j,k}^{(n)}(x, y)$.

If we take $u = v = 1$ in (5.2), we get

$$\sum_{n=0} \frac{z^n}{n!} \sum_{j+k=m} C_{j,k}^{(n)}(x, y) = \frac{1}{m!} \{(x + y)(\cosh z - 1 + \sinh z)\}^m$$

$$= \frac{1}{m!} (x + y)^m (e^z - 1)^m.$$

Hence

$$\sum_{j+k=m} C_{j,k}^{(n)}(x, y) = (x + y)^m S(n, m), \quad (5.4)$$

in agreement with (4.8).

For $u = 1$, $v = -1$, (5.2) reduces to

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k=m} (-1)^n C_{j,k}^{(n)}(x, y)$$

$$= \frac{1}{m!} \{x(\cosh z - 1 - \sinh z) + y(\sinh z - \cosh z + 1)\}^m$$

$$= \frac{1}{m!} (x - y)^m (e^{-z} - 1)^m.$$

It follows that

$$\sum_{j+k=m} (-1)^k C_{j,k}^{(n)}(x, y) = (-1)^n (x - y)^m S(n, m). \quad (5.5)$$

For $u = 1$, $v = 0$ and $u = 0$, $v = 1$, we get

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} C_{m,0}^{(n)}(x, y) = \frac{1}{m!} \{x(\cosh z - 1) + y \sinh z\}^m \quad (5.6)$$

and

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} C_{0,m}^{(n)}(x, y) = \frac{1}{m!} \{y(\cosh z - 1) + x \sinh z\}^m, \quad (5.7)$$

respectively. It accordingly follows that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} C_{j,k}^{(n)}(x, y) = \sum_{r=0}^{\infty} \frac{z^r}{r!} C_{j,0}^{(r)}(x, y) \sum_{s=0}^{\infty} \frac{z^s}{s!} C_{0,k}^{(s)}(x, y)$$
$$= \sum_{r=0}^{\infty} \frac{z^r}{r!} C_{j,0}^{(r)}(x, y) \sum_{s=0}^{\infty} \frac{z^s}{s!} C_{k,0}^{(s)}(y, x). \tag{5.8}$$

Hence

$$C_{p,q}^{(n)}(x, y) = \sum_{r+s=n} \binom{n}{r} C_{p,0}^{(r)}(x, y)\, C_{0,q}^{(s)}(x, y)$$
$$= \sum_{r+s=n} \binom{n}{r} C_{p,0}^{(r)}(x, y)\, C_{q,0}^{(s)}(y, x). \tag{5.9}$$

In terms of the $c_k(n, p, q)$ defined by

$$C_{p,q}^{(n)}(x, y) = \sum_{k=0}^{p+q} c_k(n, p, q)\, x^{p+q-k} y^k,$$

(5.9) gives

$$c_k(n, p, q) = \sum_{\substack{a+b=k \\ r+s=n}} \binom{n}{r} c_a(r, p, 0)\, c_b(s, 0, q). \tag{5.10}$$

Turning next to the polynomial

$$F_{r,s}^{(n)}(x, y) = \sum_{j=0}^{n} f_j^{(n)}(r, s)\, x^{r+n-2j} y^{s-n+2j},$$

we recall that, by (3.10),

$$F_{r,s}^{(n)}(x, y) = \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y) \frac{r!}{(r-j)!} \frac{s!}{(s-k)!} x^{r-j} y^{s-k}.$$

Thus we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r,s=0}^{\infty} F_{r,s}^{(n)}(x, y) \frac{u^r v^s}{r!s!}$$
$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y) \sum_{r,s=0}^{\infty} \frac{x^{r-j} y^{s-k}}{(r-j)\,(s-k)!} u^r v^s$$
$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k\leqslant n} C_{j,k}^{(n)}(x, y)\, u^j v^k \sum_{r,s=0}^{\infty} \frac{(xu)^r\,(yv)^s}{r!s!}.$$

Hence, by (5.3),

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r,s=0}^{\infty} F_{r,s}^{(n)}(x, y) \frac{u^r v^s}{r!s!} = \exp\{u(x \cosh z + y \sinh z) + v(y \cosh z + \sinh z)\}. \tag{5.11}$$

In particular we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{\infty} F_{r,0}^{(n)}(x, y) \frac{u^r}{r!} = \exp\{u(x \cosh z + y \sinh z)\}, \tag{5.12}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{s=0}^{\infty} F_{0,s}^{(n)}(x, y) \frac{v^s}{s!} = \exp\{v(y \cosh z + x \sinh z)\}. \tag{5.13}$$

It therefore follows from (5.11), (5.12) and (5.13) that

$$F_{r,s}^{(n)}(x, y) = \sum_{j+k=n} \binom{n}{k} F_{r,0}^{(j)}(x, y) F_{0,s}^{(k)}(x, y). \tag{5.14}$$

Moreover we have

$$F_{r,0}^{(n)}(x, y) = 2^{-r} \sum_{j=0}^{r} \binom{r}{j} (r - 2j)^n (x + y)^{r-j} (x - y)^j, \tag{5.15}$$

$$F_{0,s}^{(n)}(x, y) = 2^{-s} \sum_{k=0}^{s} (-1)^k \binom{s}{k} (s - 2k)^n (x + y)^{s-k} (x - y)^k \tag{5.16}$$

and

$$F_{r,s}^{(n)}(x, y) = 2^{-r-s} \sum_{j+k=r+s} \frac{r!s!}{j!k!} (j - k)^n a_s(j, k) (x + y)^j (x - y)^k, \tag{5.17}$$

where

$$(u + v)^j (u - v)^k = \sum_{s=0}^{j+k} a_s(j, k) u^{j+k-s} v^s.$$

It follows from (5.14) that

$$f_k^{(n)}(p, q) = \sum_{\substack{a+b=k \\ r+s=n}} \binom{n}{r} f_a^{(r)}(p, 0) f_b^{(s)}(0, q). \tag{5.18}$$

This result may be compared with (5.10).

6. Combinatorial Interpretation

Put

$$\exp\{x(\cosh z - 1) + y \sinh z\} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j,k} A(n, j, k)\, x^j y^k. \tag{6.1}$$

Then $A(n, j, k)$ is equal to the number of partitions of the set $z_n = \{1, 2,..., n\}$ into $j + k$ non-vacuous blocks of which j are of even cardinality and k of odd cardinality. The enumerant $A(n, j, k)$ is discussed in some detail in [2]; see also [5, Chap. 4].

It follows from (6.1) that

$$\frac{1}{m!} \{x(\cosh z - 1) + y \sinh z\}^m = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j+k=m} A(n, j, k)\, x^j y^k.$$

On the other hand, by (5.6),

$$\frac{1}{m!} \{x(\cosh z - 1) + y \sinh z\}^m = \sum_{n=0}^{\infty} \frac{z^n}{n!} C_{m,0}^{(n)}(x, y).$$

Hence we have

$$C_{m,0}^{(n)}(x, y) = \sum_{j+k=m} A(n, j, k)\, x^j y^k. \tag{6.2}$$

Since

$$C_{m,0}^{(n)}(x, y) = \sum_{k=0}^{m} c_k(n, m, 0)\, x^{m-k} y^k,$$

it follows that

$$c_k(n, j + k, 0) = A(n, j, k). \tag{6.3}$$

Similarly

$$c_k(n, 0, j + k) = A(n, k, j). \tag{6.4}$$

Thus we have simple combinatorial interpretations of $c_k(n, m, 0)$ and $c_k(n, 0, m)$.

By (5.10), (6.3) and (6.4) we have

$$c_k(n, p, q) = \sum_{\substack{a+b=k \\ r+s=n}} \binom{n}{r} A(r, p - a, a)\, A(s, b, q - b). \tag{6.5}$$

We can also express $f_k^{(n)}(r, s)$ in terms of $A(n, p, q)$. We find that

$$f_j^{(n)}(r, 0) = \sum_t \frac{r!}{(r - t)!} A(n, n - 2j + t, -n + 2j) \tag{6.6}$$

and

$$f_j^{(n)}(0, s) = \sum_t \frac{s!}{(s-t)!} A(n, n-2j, -n+2j+t). \tag{6.7}$$

Applying (5.18) we can then evaluate $f_k^{(n)}(r, s)$.

A simpler interpretation of $f_j^{(n)}(r, 0)$ and $f_j^{(n)}(0, s)$ will now be obtained. Put

$$\exp(x \cosh z + y \sinh z) = \sum_{n=0}^{\infty} \frac{z_n}{n!} \sum_{j,k} \bar{D}(n, j, k) \frac{x_j y^k}{j!k!}. \tag{6.8}$$

Then $\bar{D}(n, j, k)$ is the number of ways of putting n numbered objects into $r+s$ numbered boxes so that each of the first r boxes contains an even number of objects and each of the remaining s boxes contains an odd number of objects. (For a detailed discussion of similar enumerants see [3, Vol. I; 5, Chap. 5].)

We now define $D(n, r, s)$ as the number of ways of putting n numbered objects into $r+s$ numbered boxes so that each of any r boxes contains an even number of objects while each of remaining s boxes contains an odd number of objects. It follows at once that

$$D(n, r, s) = \binom{r+s}{r} \bar{D}(n, r, s). \tag{6.9}$$

By (5.11) we have

$$\frac{1}{m!}(x \cosh z + y \sinh z)^m = \sum_{n=0}^{\infty} \frac{z^n}{n!} F_{m,0}^{(n)}(x, y)$$

and

$$\frac{1}{m!}(y \cosh z + x \sinh z)^m = \sum_{n=0}^{\infty} \frac{z^n}{n!} F_{0,m}^{(n)}(x, y).$$

Hence, by comparison with (6.8), we get

$$\frac{1}{m!} F_{m,0}^{(n)}(x, y) = \sum_{j,k} \bar{D}(n, j, k) \frac{x^j y^k}{j!k!} \tag{6.10}$$

and

$$\frac{1}{m!} F_{0,m}^{(n)}(x, y) = \sum_{j,k} \bar{D}(n, k, j) \frac{x^j y^k}{j!k!}. \tag{6.11}$$

Since

$$F_{m,0}^{(n)}(x, y) = \sum f_k^{(n)}(m, 0)\, x^{m+n-2k} y^{-n+2k},$$

it follows that

$$f_k^{(n)}(m, 0) = \binom{m}{2k - n} \bar{D}(n, m + n - 2k, 2k - n).$$

In view of (6.9), this gives

$$f_k^{(n)}(m, 0) = D(n, m + n - 2k, 2k - n) \tag{6.12}$$

and similarly

$$f_k^{(n)}(0, m) = D(n, n - 2k, m - n + 2k). \tag{6.13}$$

Finally, by (5.18),

$$f_k^{(n)}(p, q) = \sum_{\substack{a+b=k \\ r+s=n}} \binom{n}{r} D(r, p + r - 2a, 2a - p)\, D(s, s - 2b, q - s + 2b). \tag{6.14}$$

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