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EXISTENCE AND REPRESENTATION OF DIOPHANTINE AND MIXED DIOPHANTINE SOLUTIONS TO LINEAR EQUATIONS AND INEQUALITIES*

A. CHARNES
The University of Texas, Austin, Texas 78712, USA

and

F. GRANOT
Department of Mathematics, Dalhousie University, Halifax, N.S., Canada

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In this paper we present necessary and sufficient conditions for the existence of solutions to more general systems of linear diophantine equations and inequalities than have previously been considered. We do this in terms of variants and extensions of generalized inverse concepts which also permit us to give representations of the set of all solutions to the systems. The results are further extended to mixed integer systems.

1. Introduction

We present in this paper necessary and sufficient conditions for the existence of solutions to the general system of linear diophantine equations

$$AXB = C, \quad X \text{ integer},$$

and to the linear diophantine system of inequalities

$$D \leqslant AXB \leqslant C, \quad X \text{ integer},$$

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where A, B, C and D are matrices of rational numbers. If solvable, the paper gives a representation to the set of all solutions of the corresponding systems.

The results of the first part of this paper are then extended to mixed integer systems.

2. Preliminaries and notations

We denote by

$\mathbf{R}^n$ the n-dimensional real vector space,

I the identity matrix with dimension as needed.

For any two $m \times n$ real matrices A and B,

$A \geqslant B$ denotes $a_{ij} \geqslant b_{ij}$, $i = 1, ..., m$, $j = 1, ..., n$;

A integer means a_{ij} integer, $i = 1, ..., m$, $j = 1, ..., n$;

A^T denotes the transpose of A;

$R(A)$ denotes the range space of A;

$N(A)$ denotes the null space of A.

For a fixed $m \times n$ real matrix A, consider the four matrix equations

(1) $$AXA = A\,,$$

(2) $$XAX = X\,,$$

(3) $$(AX)^T = AX\,,$$

(4) $$(XA)^T = XA\,.$$

We denote (following the notation of [1]) by $A\{i, j, ..., k\}$ the set of $n \times m$ real matrices X satisfying equations (i), (j), ..., (k) ($1 \leqslant i, j, ..., k \leqslant 4$). These sets $A\{i, j, ..., k\}$ ($1 \leqslant i, j, ..., k \leqslant 4$) are nonempty because $A\{1, 2, 3, 4\}$ is nonempty [13].

A matrix $X \in A\{i, j, ..., k\}$ is called an "$\{i, j, ..., k\}$-g.i." (generalized inverse) of A. The $\{1, 2, 3, 4\}$-g.i. of A is unique, and is the Moore–Penrose generalized inverse denoted by A^+ (see [12,13]).

For many applications a weaker g.i. is sufficient. Thus for solving linear equations "{1}-g.i." are sufficient as shown by the following:

Lemma 2.1 (see [4,14]). *The linear equations*

(5) $Ax = b$

are solvable if and only if for any $T \in A\{1\}$,

(6) $ATb = b$

in which case the general solution of (5) *is*

(7) $x = Tb + (I - TA)y$, y *arbitrary*.

The set $A\{1\}$ is represented in terms of one of its elements as follows:

Lemma 2.2 (see [4]). *Let R be any* {1}-g.i. *of A. Then*

(8) $A\{1\} = \{RAR + Y - RAYAR;\ Y \text{ arbitrary } n \times m \text{ real matrix}\}$.

3. Existence of solutions to linear diophantine equations

Definition 3.1. If $T \in A\{i, j, \ldots, k\}$ $(1 \leqslant i, j, \ldots, k \leqslant 4)$ and TA is integer, then T is called a "left integer $\{i, j, \ldots, k\}$-g.i." of A and will be denoted $\{i, j, \ldots, k\}$-l.i.g.i. of A. (This generalizes a definition of Bowman and Burdet [7].)

Since for every matrix A there exists a unique $\{1, 2, 3, 4\}$-g.i., either A^+A is integer and then A^+ is the $\{1, 2, 3, 4\}$-l.i.g.i. of A, or there is no $\{1, 2, 3, 4\}$-l.i.g.i. of A.

The set of all $\{i, j, \ldots, k\}$-l.i.g.i. of A will be denoted by $A_\mathrm{I}^\mathrm{L}\{i, j, \ldots, k\}$.

In the same way, we define a r.i.g.i. of A to be a matrix S such that AS is integer and S is $\{i, j, \ldots, k\}$-g.i. of A.

The set of all $\{i, j, \ldots, k\}$-r.i.g.i. of A will be denoted by $A_\mathrm{I}^\mathrm{R}\{i, j, \ldots, k\}$.

Theorem 3.2. *The linear equations*

(9) $$\begin{cases} AXB = C, \\ X \quad \text{integer} \end{cases}$$

are solvable if and only if for any $T \in A_\mathrm{I}^\mathrm{L}\{1\}$ *and* $S \in B_\mathrm{I}^\mathrm{R}\{1\}$,

(i) TCS *is integer, and*

(ii) $ATCSB = C$

in which case the general solution of (9) *is*

(10) $X = TCS + Y - TAYBS$, Y *integer*.

Proof. Suppose X satisfies (9), then

$$C = AXB = ATAXBSB = ATCSB,$$

hence (ii) holds, and

$$TCS = TAXBA$$

is integer since TA, X and BS are all integer, hence (i) holds.

Conversely. If $C = ATCSB$, then TCS is a particular solution of (9) because of proposition (i). Now in order to get the general solution of (9) we have to solve $AXB = 0$, X integer. But any expression of the form $X = Y - TAYBS$, Y integer, satisfies $AXB = 0$, X integer, and conversely if $AXB = 0$, X integer, then $X = X - TAXBS$ which completes the proof.

In order to find a representation for $A_{\mathrm{I}}^{\mathrm{L}}\{1\}$ we consider now that

(11) $X \in A_{\mathrm{I}}^{\mathrm{L}}\{1\} \Leftrightarrow \begin{cases} AXA = A, \\ XA \quad \text{integer}. \end{cases}$

Substituting

(12) $Y = XA$

in (11), we obtain

(13) $\begin{cases} AY = AYI = A, \\ Y \quad \text{integer}. \end{cases}$

The system (13) has a solution (we assume it); hence the general solution of (13) can be written as:

(14) $Y = TAR + Z - TAZIR$,

Z integer, $T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}$, $R = I \in I_{\mathrm{I}}^{\mathrm{R}}\{1\}$.

From (12) and (14) we conclude that in order to find an X which solves (11), we have to solve the following:

$$(15)\qquad \begin{cases} IXA = TA + Z - TAZ\,, \\ X \quad \text{integer}\,. \end{cases}$$

If we assume again that (15) has a solution, then its general solution can be written as:

$$(16)\qquad X = [TA + Z - TAZ]S + V - VAS\,,$$

$$T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}, \qquad S \in A_{\mathrm{I}}^{\mathrm{R}}\{1\}, \qquad Z, V \quad \text{integers}\,,$$

or by changing order we obtain

$$(17)\qquad X = TAS + (I - TA)ZS + V(I - AS)\,,$$

$$Z, V \quad \text{integers}\,.$$

Lemma 3.3. *Given any pair of matrices T and S such that $T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}$, $S \in A_{\mathrm{I}}^{\mathrm{R}}\{1\}$, then the set $A_{\mathrm{I}}^{\mathrm{L}}\{1\}$ can be expressed as*

$$(18)\qquad A_{\mathrm{I}}^{\mathrm{L}}\{1\} = \{X : X = TAS + (I - TA)ZS + V(I - AS)\,,\ Z, V \text{ integers}\}\,.$$

Proof. Follows immediately from the procedure above and the fact that

$$(19)\qquad X \in A_{\mathrm{I}}^{\mathrm{L}}\{1\} \Leftrightarrow \begin{cases} AXA = A\,, \\ X, A \quad \text{integers}\,. \end{cases}$$

In a similar way we can find the representation of the set $A_{\mathrm{I}}^{\mathrm{R}}\{1\}$ in terms of a given pair T, S satisfying $T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}$ and $S \in A_{\mathrm{I}}^{\mathrm{R}}\{1\}$.

Lemma 3.4. *Given any pair of matrices T and S such that $T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}$, $S \in A_{\mathrm{I}}^{\mathrm{R}}\{1\}$, then the set $A_{\mathrm{I}}^{\mathrm{R}}\{1\}$ can be expressed as*

$$(20)\qquad A_{\mathrm{I}}^{\mathrm{R}}\{1\} = \{X : X = TAS + TZ(I - AS) + (I - TA)V\,,\ Z, V \text{ integers}\}\,.$$

Proof. Follows in a similar way to Lemma 3.3 above.

Remark 3.5. We shall sometimes use A_I^L and A_I^R to denote elements of $A_I^L\{1\}$ and $A_I^R\{1\}$, respectively.

Corollary 3.6. *The general solution of the vector equation*

$$(21)\qquad \begin{cases} Ax = b, \\ x \quad \text{integer} \end{cases}$$

is

$$x = Tb + (I - TA)y,$$

where y is integer and $T \in A_I^L\{1\}$, provided that the system (21) has a solution.

Corollary 3.7. *A necessary and sufficient condition for the equations*

$$(22)\qquad \begin{cases} AX = C, \\ XB = D, \\ X \quad \text{integer} \end{cases}$$

to have a common solution is that each of the systems

$$(23)\qquad \begin{cases} AX = C, \\ X \quad \text{integer} \end{cases}$$

and

$$(24)\qquad \begin{cases} XB = D, \\ X \quad \text{integer} \end{cases}$$

will individually have a solution and that

$$(25)\qquad AD = CB.$$

Proof. The condition is obviously necessary. To show that it is sufficient let us substitute for X in (22) the following:

$$(26)\qquad X = A_I^L C + D B_I^R - A_I^L A D B_I^R.$$

Using now the required conditions

$$(27)\qquad \begin{cases} A A_1^{\mathrm{L}} C = C & \text{(solvability of (23))}, \\ D B_1^{\mathrm{R}} B = D & \text{(solvability of (24))}, \\ A D = C B & \text{((25))} \end{cases}$$

we obtain

$$A X = A A_1^{\mathrm{L}} C + A D B_1^{\mathrm{R}} - A A_1^{\mathrm{L}} A D B_1^{\mathrm{R}}$$
$$= C + A D B_1^{\mathrm{R}} - A D B_1^{\mathrm{R}} = C ,$$

$$X B = A_1^{\mathrm{L}} C B + D B_1^{\mathrm{R}} B - A_1^{\mathrm{L}} A D B_1^{\mathrm{R}} B$$
$$= A_1^{\mathrm{L}} C B + D - A_1^{\mathrm{L}} C B = D ,$$

$$X = A_1^{\mathrm{L}} C + D B_1^{\mathrm{R}} - A_1^{\mathrm{L}} A D B_1^{\mathrm{R}}$$

is integer as a sum of integer terms, which completes the proof.

4. Existence of solutions to linear diophantine inequalities

We now turn to consideration of systems of linear diophantine inequalities and obtain analogous results to the above.

Consider the set of inequalities

$$(28)\qquad a \leqslant Ax \leqslant b ,$$

where A is an $m \times n$ real matrix and a and b are real vectors.

Lemma 4.1. *The set of linear inequalities* (28) *has a solution if and only if there exists a vector d, $a \leqslant d \leqslant b$, such that for $T \in A\{1\}$,*

$$(29)\qquad A T d = d$$

in which case the set of all solutions to (28) *can be expressed as*

$$(30)\qquad S = \left\{ \begin{matrix} x : x = Td + (I - TA)y,\ y \text{ arbitrary},\ d \text{ is any vector such} \\ \text{that } a \leqslant d \leqslant b, \text{ and } A T d = d \end{matrix} \right\} .$$

Proof. Follows immediately from Lemma 2.1.

Theorem 4.2. *The diophantine set of inequalities*

$$(31) \qquad \begin{cases} D \leqslant A\,X\,B \leqslant C\,, \\ X \quad \text{integer} \end{cases}$$

has a non-empty set of solutions if and only if there exists a matrix E, $D \leqslant E \leqslant C$, *such that for* $T \in A_{\mathrm{I}}^{\mathrm{L}}\{1\}$ *and* $S \in A_{\mathrm{I}}^{\mathrm{R}}\{1\}$,

(i) $T\,E\,S$ *is integer,*

(ii) $A\;T\,E\,S\,B = E$.

In which case the set of all solutions to (31) *can be expressed as*

$$(32) \qquad R = \left\{ \begin{matrix} X : X = T\,E\,S + Y - T\,A\;Y\,B\,S,\ Y \text{ arbitrary}, \\ D \leqslant E \leqslant C \text{ such that (i) and (ii) above are satisfied} \end{matrix} \right\}.$$

Proof. Similar to that of Theorem 3.2 hence omitted.

Corollary 4.3. *The set of diophantine inequalities*

$$(33) \qquad \begin{cases} a \leqslant Ax \leqslant b\,, \\ x \quad \text{integer} \end{cases}$$

has a non-empty set of solutions if and only if there exists a vector d, $a \leqslant d \leqslant b$, *and the system* $Ax = d$, x *integer, is solvable, in which case the set of all the solutions to* (33) *is given by*

$$(34) \qquad R = \left\{ \begin{matrix} x : x = A_{\mathrm{I}}^{\mathrm{L}}d + y - A_{\mathrm{I}}^{\mathrm{L}}Ay,\ y \text{ arbitrary integer} \\ a \leqslant d \leqslant b,\ A_{\mathrm{I}}^{\mathrm{L}}d \text{ integer and } A\;A_{\mathrm{I}}^{\mathrm{L}}d = d \end{matrix} \right\}.$$

5. The set of all solutions represented in terms of special l.i.g.i. and r.i.g.i.

In the following we shall represent the set of all the solutions to the diophantine system of inequalities in terms of special l.i.g.i. and r.i.g.i.

Let H be any unimodular non-singular integer matrix such that

(35) $$AH = K = \begin{pmatrix} K_1 & 0 \\ K_2 & 0 \end{pmatrix}.$$

where K_1 is an $r \times r$ matrix, and r is the rank of A. It is easy to see that

(36) $$A_{\text{I}}^{\text{L}} = H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

since

(37) $$H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} KH^{-1} = H \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} H^{-1} = \text{integer}$$

and

(38) $$H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in A\{1\}.$$

Similarly let E be a unimodular non-singular integer matrix such that

(39) $$EA = L = \begin{pmatrix} L_1 & L_2 \\ 0 & 0 \end{pmatrix}.$$

where L_1 is an $r \times r$ matrix. Thus

(40) $$A_{\text{I}}^{\text{R}} = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} E.$$

Since

(41) $$AA_{\text{I}}^{\text{R}} = E^{-1}L \begin{pmatrix} L_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} E = E^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} E \quad \text{is integer}.$$

and

(42) $$\begin{pmatrix} L_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} E \in A\{1\}.$$

$A_{\text{I}}^{\text{L}}\{1\}$ and $A_{\text{I}}^{\text{R}}\{1\}$ can now be expressed in terms of H, E, K_1 and L_1.

Another representation of the set R of all the solutions to (33) can be obtained from (36) and (34). The requirement that $A_{\text{I}}^{\text{L}}d$ should be integer turns out in this case to be

(43) $$H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} d = \text{integer}.$$

Since H is a unimodular integer matrix, (43) is equivalent to

$$\begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} d = \text{integer}\,. \tag{44}$$

The requirement $AA_I^{\dagger}d = d$ can be written in the above case as

$$\begin{pmatrix} K_1 & 0 \\ K_2 & 0 \end{pmatrix} H^{-1}H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} d = d\,. \tag{45}$$

If d is partitioned to $d = \binom{d_1}{d_2}$, where d_1 is an r column vector, then (45) is equal to

$$\begin{cases} d_1 = d_1\,, \\ K_2 K_1^{-1} d_1 = d_2 \end{cases} \tag{46}$$

and hence (34) for the above case can be written as

$$R = \left\{ \begin{array}{l} x : x = H \begin{pmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} d + y - H \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} H^{-1} y\,, \\ y \text{ arbitrary integer, and } a \leqslant d \leqslant b \text{ such that } K_1^{-1} d_1 \\ \text{is integer and } K_2 K_1^{-1} d_1 = d_2 \end{array} \right\}. \tag{47}$$

6. Characterization of solutions to mixed integer systems

Consider the mixed integer system

$$\begin{cases} a \leqslant Ax + By \leqslant b\,, \\ x \quad \text{integer}\,. \end{cases} \tag{48}$$

The set of all the solutions to (48) will be obtained by solving the system

$$\begin{cases} Ax + By = d\,, \\ x \quad \text{integer}\,, \end{cases} \tag{49}$$

for any vector d such that

$$a \leqslant d \leqslant b\,. \tag{50}$$

Let us denote

(51) $Ax = q$

and solve first

(52) $By = d - q$.

If T is any matrix such that $T \in B\{1\}$, then (52) is solvable if and only if

(53) $BT(d - q) = d - q$.

Thus for every $a \leqslant d \leqslant b$ only the values of q satisfying

(54) $(I - BT)q = (I - BT)d$

are acceptable. By substituting (51) in (54), we obtain

(55) $(I - BT)Ax = (I - BT)d$.

Thus we conclude:

Lemma 6.1. *The mixed integer system* (48) *has a solution if and only if there exists a vector d, $a \leqslant d \leqslant b$, such that the diophantine system*

(56) $$\begin{cases} (I - BT)Ax = (I - BT)d, \\ x \quad \text{integer} \end{cases}$$

has a solution, or more precisely, if there exists a vector d, $a \leqslant d \leqslant b$, such that

(i) $[(I - BT)A]_\mathrm{I}^\mathrm{L}[I - BT]d$ *is integer,*

(ii) $(I - BT)A[(I - BT)A]_\mathrm{I}^\mathrm{L}(I - BT)d = (I - BT)d$,

in which case the set of all the solutions to (48) *is given by*

(57) $$R = \left\{ \begin{array}{l} \begin{pmatrix} x \\ y \end{pmatrix} : x = [(I - BT)A]_\mathrm{I}^\mathrm{L}(I - BT)d \\ \qquad + [I - [(I - BT)A]_\mathrm{I}^\mathrm{L}(I - BT)A]w, \\ \qquad y = Td - TA[(I - BT)A]_\mathrm{I}^\mathrm{L}(I - BT)A\,d \\ \qquad - TA[I - [(I - BT)A]_\mathrm{I}^\mathrm{L}(I - BT)A]w - (I - TB)z, \\ z \text{ arbitrary, } w \text{ integer and } d \text{ any vector,} \\ a \leqslant d \leqslant b, \text{ which satisfies (i) and (ii)} \end{array} \right\}$$

Proof. Follows from (30) and (34).

Example 6.2. Consider the system of mixed integer inequalities:

$$(58)\qquad \left\{\begin{array}{l} 0 \leqslant x_1 + 2x_2 + y_1 + 2y_2 - y_3 \leqslant 3 \\ -2 \leqslant x_1 + x_3 - y_1 + y_3 - y_4 \leqslant 2 \\ -3 \leqslant x_2 + 2x_3 + 2y_1 + y_2 - 3y_3 + y_4 \leqslant 1 \\ x_1, x_2, x_3 \text{ integer} \end{array}\right\}.$$

Thus

$$(59)\qquad A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

and

$$(60)\qquad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \qquad a = \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}.$$

Let $T \in B\{1\}$, then T can be of the form:

$$(61)\qquad T = \begin{bmatrix} \frac{1}{2} & -\frac{5}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{3}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(62)\qquad BT = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{5}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{3}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(63)\qquad TB = \begin{bmatrix} \frac{1}{2} & -\frac{5}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{3}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By substituting (62) in (55) we obtain

$$(64)\qquad (I - BT)Ax = 0 = (I - BT)d = 0.$$

Hence (55) is always satisfied and the general solution to (58) is

(65) $x = w$,

(66) $$y = Td - (I - BT)z = \begin{bmatrix} \frac{1}{2}d_1 - \frac{5}{2}d_2 - d_3 + \frac{3}{2}z_4 \\ \frac{1}{2}d_1 + \frac{1}{2}d_2 \quad - \frac{1}{2}z_4 \\ \frac{1}{2}d_1 - \frac{3}{2}d_2 - d_3 + \frac{1}{2}z_4 \\ 0 \quad - z_4 \end{bmatrix}$$

z_4 is arbitrary, w integer, d any vector such that

$$\begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix} \leqslant \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \leqslant \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} .$$

Example 6.3. Consider the system of mixed integer inequalities:

(67) $$\begin{Bmatrix} 0 \leqslant 3x_1 + x_2 + x_3 + y_1 - y_2 - y_3 \leqslant 2 \\ 1 \leqslant -2x_2 - 3x_3 - y_1 + 2y_3 + y_4 \leqslant 3 \\ 2 \leqslant -5x_1 + x_3 - y_1 + 2y_2 - y_4 \leqslant 5 \\ x_1, x_2, x_3 \text{ integers} \end{Bmatrix} .$$

Thus

(68) $$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -2 & -3 \\ -5 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix}$$

and

(69) $$b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \qquad a = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} .$$

Let $T \in B\{1\}$, then T can be of the form:

(70) $$T = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

(71) $$BT = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix},$$

(72) $$TB = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By substituting (71) in (55) we obtain

(73) $$(I - BT)Ax = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & -2 & -3 \\ -5 & 0 & 1 \end{bmatrix} x$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} d.$$

A l.i.g.i. of the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is

(74) $$[(I - BT)A]_I^L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence the general solution of (67) is:

(75) $$x = \begin{bmatrix} 2d_1 + d_2 + d_3 \\ w_2 \\ w_3 \end{bmatrix}.$$

(76)
$$y = \begin{bmatrix} -d_2 \\ 2d_1 - d_2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2w_2 + 3w_3 \\ w_2 + w_3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2z_3 + z_4 \\ z_3 + z_4 \\ z_3 \\ z_4 \end{bmatrix}$$

$$= \begin{bmatrix} -d_2 - 2w_2 - 3w_3 - 2z_3 - z_4 \\ 2d_1 - d_2 - w_2 - 2w_3 - z_3 - z_4 \\ -z_3 \\ -z_4 \end{bmatrix},$$

where z_3, z_4 arbitrary, w_2, w_3 integers, and d_1, d_2, d_3 satisfy

(77)
$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \leqslant \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \leqslant \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

and

(78)
$$2d_1 + d_2 + d_3 = \text{integer}.$$

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