Meixner polynomials and nonvanishing holomorphic functions

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Abstract

We consider a class of holomorphic nonvanishing functions $f(z) = 1 + a_1 z + \cdots$ in the unit disk $|z| < 1$ which is defined by subordination to some majorant $G(t, \tau; z)$ which has connection with Meixner polynomials. © 2001 Elsevier Science B.V. All rights reserved.

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1. Let $\beta$ and $c$ be fixed complex numbers such that $c \neq 0, 1$ and $D = \{z \in \mathbb{C}: |z| < 1\}$.

The Meixner polynomials of the first kind: $m_n(x; \beta, c)$ of a real variable $x$ can be defined by the following generating function [1]:

$$
\frac{(1 - z^x)}{(1 - z)^{1+\beta}} = \sum_{n=0}^{\infty} m_n(x; \beta, c)z^n, \quad |z| < \min(1, |c|).
$$

(1)

Putting $x/(1 - c)$ instead of $x$ and taking the limit $c \to 1$ we obtain

$$
\lim_{c \to 1} m_n \left( \frac{x}{1 - c}; \beta, c \right) = L_n^{(\beta-1)}(x),
$$

(2)

where $L_n^{(\beta)}(x)$ denotes the Laguerre polynomials defined by the generating function

$$
\frac{1}{(1 - z)^{1+\beta}} \exp \left( -\frac{xz}{1 - z} \right) = \sum_{n=0}^{\infty} L_n^{(\beta)}(x)z^n, \quad |z| < 1.
$$

(3)

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In what follows, we will be interested in the case \( \beta = 0 \) in (1) and (2) and \( \alpha = -1 \) in (3). The Laguerre polynomials are not orthogonal if \( \alpha = -1 \); however, this case is very interesting in the framework of the so-called Krzyż conjecture (e.g., [2]).

For fixed \( t > 0 \) and \( \tau \in (0,1) \) let us consider the function
\[
G(t, \tau; z) = \left( \frac{1 - z}{1 - \tau z} \right)^{2t} = \sum_{n=0}^{\infty} \gamma_n(t, \tau) z^n, \quad z \in D. \tag{4}
\]
Comparing (4) and (1), we easily see that
\[
\gamma_n(t, \tau) = m_n \left( -2t, 0, \frac{1}{\tau} \right), \quad n = 0, 1, \ldots. \tag{5}
\]

The fractional function
\[
h_{t}(z) = \frac{1 - z}{1 - \tau z}, \quad z \in D, \quad \tau \in (0,1)
\]
maps the disk \( |z| \leq r < 1 \) onto the disk which has as the diameter the segment of the real axis: \( \left[ (1 - r)/(1 - \tau r), (1 + r)/(1 + \tau r) \right] \).

Therefore, for every \( z \in D \), the function \( h_{t}(z) \) satisfies the condition
\[
\left| h_{t}(z) - \frac{1}{1 + \tau} \right| < \frac{1}{1 + \tau}, \quad z \in D, \tag{6}
\]
which implies that for every \( z \in D \)
\[
0 < |G(t, \tau; z)| < \left( \frac{2}{1 + \tau} \right)^{2t}. \tag{7}
\]

By further analysis, one can observe the limit relation:
\[
\lim_{\tau \to 1} \left( \frac{1 + \tau}{2} \right)^{2t(1-\tau)} G \left( \frac{t}{1 - \tau}; \tau, z \right) = \exp \left( -t \frac{1 + z}{1 - z} \right) = F(t; z). \tag{8}
\]

Let \( \mathcal{M}_{t,\tau}, \ t > 0, \ \tau \in (0,1) \) denote the class of holomorphic functions \( f \) in \( D \) which has the form
\[
f(z) = 1 + a_1 z + a_2 z^2 + \cdots, \quad z \in D \tag{9}
\]
and satisfy the relation
\[
f(z) = \left( \frac{1 - \omega(z)}{1 - \tau \omega(z)} \right)^{2t}, \quad z \in D, \tag{10}
\]
where
\[
\omega(z) = c_1 z + c_2 z^2 + \cdots, \quad z \in D \tag{11}
\]
is a holomorphic function in \( D \) and satisfies the Schwarz Lemma conditions: \( \omega(0) = 0, \ |\omega(z)| < 1, \ z \in D \) (we denote such a class of functions \( \omega(z) \) by \( \Omega \)). Relation (10) means that \( f(z) \) is subordinate to \( G(t, \tau; z) \) in the unit disk \( D \).

In this note we prove the sharp estimate for \( |a_n|, n = 1, 2, 3 \) if \( f \in \mathcal{M}_{t,\tau} \). We will use appropriate inequalities for the coefficients of \( \omega \) [3]. When \( \tau \to 1 \) from our results we obtain the known sharp bounds for the corresponding coefficients in the class \( B_0 \) of holomorphic functions in \( D \):
\[
f(z) = e^{-t} + a_1 z + a_2 z^2 + \cdots, \quad z \in D, \ t > 0,
\]
which satisfy the conditions: $0 < |f(z)| < 1$, $z \in D$ \cite{4}. As a corollary we will also get some bounds for special cases of Meixner polynomials.

2. The following two lemmas will be useful:

**Lemma 1.** If $\omega \in \Omega$, then

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2.$$  \hspace{1cm} (12)

**Lemma 2** (Prokhorov and Szynal \cite{3}). If $\omega \in \Omega$, then for any real numbers $p$, $q$ the following sharp estimate holds:

$$|c_3 + pc_1c_2 + qc_1^3| \leq H(p, q),$$

where

$$H(p, q) = \begin{cases} 1 & \text{for } (p, q) \in D_1 \cup D_2, \\ |q| & \text{for } (p, q) \in \bigcup_{k=3}^7 D_k, \\ \frac{1}{2}(|p| + 1) \left( \frac{|p| + 1}{3(|p| + 1 + q)} \right)^{1/2} & \text{for } (p, q) \in D_8 \cup D_9, \\ \frac{1}{2}q \left( \frac{p^2 - 4}{p^2 - 4q} \right) \left( \frac{p^2 - 4}{3(q - 1)} \right)^{1/2} & \text{for } (p, q) \in D_{10} \cup D_{11} \setminus \{ \pm 2, 1 \}, \\ \frac{1}{2}(|p| - 1) \left( \frac{|p| - 1}{3(|p| - 1 - q)} \right)^{1/2} & \text{for } (p, q) \in D_{12}. \end{cases}$$  \hspace{1cm} (13)

The sets $D_1, \ldots, D_{12}$ are defined as follows:

- $D_1 := \{(p, q): |p| \leq \frac{1}{2}, \quad |q| \leq 1\}$,
- $D_2 := \{(p, q): \frac{1}{2} \leq |p| \leq 2, \quad \frac{1}{27}(|p| + 1)^3 - (|p| + 1) \leq q \leq 1\}$,
- $D_3 := \{(p, q): |p| \leq \frac{1}{2}, \quad q \leq -1\}$,
- $D_4 := \{(p, q): |p| \geq \frac{1}{2}, \quad q \leq -\frac{1}{2}(|p| + 1)\}$,
- $D_5 := \{(p, q): |p| \leq 2, \quad q \geq 1\}$,
- $D_6 := \{(p, q): 2 \leq |p| \leq 4, \quad q \geq \frac{1}{12}(p^2 + 8)\}$,
- $D_7 := \{(p, q): |p| \geq 4, \quad q \geq \frac{2}{3}(|p| - 1)\}$,
- $D_8 := \{(p, q): \frac{1}{2} \leq |p| \leq 2, \quad -\frac{2}{3}(|p| + 1) \leq q \leq \frac{4}{27}(|p| + 1)^3 - (|p| + 1)\}$,
- $D_9 := \{(p, q): |p| \geq 2, \quad -\frac{1}{2}(|p| + 1) \leq q \leq \frac{2|p|(|p| + 1)}{p^2 + 2|p| + 4}\}$,
\[ D_{10}:= \left\{ (p, q) : 2 \leq |p| \leq 4, \quad \frac{2|p|(|p| + 1)}{p^2 + 2|p| + 4} \leq q \leq \frac{1}{12}(p^2 + 8) \right\}, \]

\[ D_{11}:= \left\{ (p, q) : |p| \geq 4, \quad \frac{2|p|(|p| + 1)}{p^2 + 2|p| + 4} \leq q \leq \frac{2}{3}(|p| - 1) \right\}. \]

\[ D_{12}:= \left\{ (p, q) : |p| \geq 4, \quad \frac{2|p|(|p| - 1)}{p^2 - 2|p| + 4} \leq q \leq \frac{2}{3}(|p| - 1) \right\}. \]

Denote: \( p = (1 + \tau) - (1 - \tau)x \) and \( x = 2t > 0, \tau \in (0, 1) \). We have the following:

**Theorem 1.** If \( f \in \mathfrak{M}_{\tau}, \) then the following sharp bounds hold:

\[ |a_1| \leq (1 - \tau)x \]

\[ |a_2| \leq \begin{cases} 
(1 - \tau)x & \text{if } 0 < x \leq \frac{3 + \tau}{1 - \tau}, \\
\frac{1}{2}(1 - \tau)x[(1 - \tau)x - (1 + \tau)] & \text{if } x \geq \frac{3 + \tau}{1 - \tau}, 
\end{cases} \]

\[ |a_3| \leq \begin{cases} 
\frac{1}{6}(1 + \tau - p)(p^2 + (1 + \tau)p - 2\tau) & \text{if } -\infty \leq p \leq p_1(\tau), \\
\frac{2\sqrt{2}}{3}(1 + \tau - p)(-p - 1)\left(\frac{p + 1}{p^2 + (7 + \tau)p + 6 + 2\tau}\right)^{1/2} & \text{if } p_1(\tau) \leq p \leq p_2(\tau), \\
\frac{\sqrt{2}}{6}(1 + \tau - p)(p^2 + (1 + \tau)p - 2\tau)(p^2 - 4)^{3/2} & \text{if } p_2(\tau) \leq p \leq p_3(\tau), \\
(p^2 - 2(1 + \tau)p + 4\tau)^{-1}(p^2 + (1 + \tau)p - 2\tau - 6)^{-1/2} & \text{if } p_3(\tau) \leq p \leq p_4(\tau), \\
\frac{2\sqrt{2}}{3}(1 + \tau - p)(1 - p)^{3/2}(p^2 + (\tau - 5)p + 6 - 2\tau)^{-1/2} & \text{if } p_4(\tau) \leq p \leq 1 + \tau, \\
(1 + \tau - p) & \text{if } p_4(\tau) \leq p \leq 1 + \tau,
\end{cases} \]

where \( p_1(\tau), p_2(\tau), p_3(\tau), p_4(\tau) \) are the unique negative roots of the following equations

\[ p^2 + (5 + \tau)p - 2(\tau - 2) = 0, \]

\[ p^3 + (5 + \tau)p^2 + 2(\tau + 2)p + 4\tau = 0, \]

\[ p^3 + (1 + \tau)p^2 - 2(\tau + 4)p + 4\tau = 0, \]

\[ 8p^3 - 15p^2 + 3(3\tau - 7)p + 46 - 18\tau = 0, \]

respectively.
Proof. Putting expansion (11) of \( \omega \in \Omega \) into the relation (10) and comparing the coefficients we obtain

\[ a_1 = -(1 - \tau)xc_1, \]
\[ a_2 = -(1 - \tau)x \{ c_2 - \frac{1}{2}[(1 - \tau)x - (1 + \tau)]c_1^2 \}, \]
\[ a_3 = -(1 - \tau)x \{ c_3 + [(1 + \tau) - (1 - \tau)x]c_1c_2 \]
\[ + \frac{1}{6}[(1 - \tau)^2x^2 - 3(1 - \tau^2)x + 2(1 + \tau + \tau^2)]c_1^3 \}. \]  

(18)

Direct application of Lemma 1 gives bounds (13) and (14). In order to get the bound for \( |a_3| \) we have to apply Lemma 2 to

\[ p = (1 + \tau) - (1 - \tau)x, \]
\[ q = \frac{1}{2}[(1 - \tau)^2x^2 - 3(1 - \tau^2)x + 2(1 + \tau + \tau^2)]. \]  

(19)

The above equations give the family of arcs of parabola depending on the parameter \( \tau \in (0, 1) \):

\[ q_1(p) = \frac{2}{3}(p^2 + (1 + \tau)p - 2\tau), \quad p \leq 1 + \tau. \]  

(20)

Parabola (20) intersects the boundary curves of the sets \( D_k \) given by Eqs. (14) at the points: \( p_4(\tau), p_3(\tau), p_2(\tau), p_1(\tau) \) given by (17) which lie on the curves:

\[ q(p) = \frac{4}{27}(-p + 1)^3 - (-p + 1), \]
\[ q(p) = \frac{2p(p - 1)}{p^2 - 2p + 4}, \]
\[ q(p) = \frac{2p(p + 1)}{p^2 + 2p + 4}, \]
\[ q(p) = \frac{2}{3}(-p - 1), \]  

(21)

respectively.

Corollary. (1) If we consider a function \( f_1(z) = ((1 + \tau)/2)^{2\nu(1-\nu)}f(z) \) where \( f(z) \in \mathcal{H}_0((1-\nu), \tau) \), then

\[ f_1(z) \prec \left( \frac{1 + \tau}{2} \right)^{2\nu(1-\nu)} G(t; \tau; z) \]

and \( f_1(z) \) satisfies the conditions:

\[ 0 < |f_1(z)| < 1 \quad \text{for } z \in D. \]

Using the results of Theorem 1 and taking into account the limit (8) we obtain the corresponding sharp bounds for the \( |a_k| \), \( k = 1, 2, 3 \) if \( f_1 \in B_0 \) [4].

(2) Because a bounded holomorphic function in \( D \) has bounded coefficients, by (7) we have:

\[ |\gamma_n(t, \tau)| < \left( \frac{2}{1 + \tau} \right)^{2^n}, \quad n = 0, 1, 2, \ldots \]
or 

\[ |m_n(-x; 0, \frac{1}{\tau})| \leq \left( \frac{2}{1 + \tau} \right)^x \]

if \( x > 0 \) and \( \tau \in (0, 1) \).

References