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Multiple closed geodesics on bumpy Finsler n-spheres

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Abstract

In this paper we prove that for every bumpy Finsler metric F on every rationally homological n-dimensional sphere S^n with $n \geqslant 2$, there exist always at least two distinct prime closed geodesics.

1. Introduction and the main result

Let us recall firstly the definition of the Finsler metric.

Definition 1.1. (Cf. [3] and [28].) Let M be a finite-dimensional manifold and TM be its tangent bundle. A function $F : TM \to [0, +\infty)$ is a Finsler metric if it satisfies the following properties:

(F_1) F is C^∞ on $TM \setminus \{0\}$.
(F_2) $F(\lambda y) = \lambda F(y)$ for all $\lambda > 0$ and $y \in TM$.
(F_3) For any $y \in TM \setminus \{0\}$, the symmetric bilinear form g_y on TM is positive definite, where

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[F^2(y + su + tv)\right]\Big|_{s=t=0}.$$

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The pair (M, F) is called a Finsler manifold. A Finsler metric F is reversible if $F(-v) = F(v)$ for all $v \in TM$.

For the definition of closed geodesics on a Finsler manifold, we refer readers to [3] and [28]. As usual, on any Finsler manifold $M = (M, F)$ a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime*, if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the mth iteration c^m of c is defined by $c^m(t) = c(mt)$ for $m \in \mathbf{N}$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$. We shall omit the word "distinct" for short when we talk about more than one prime closed geodesics. A closed geodesic c on (M, F) is non-degenerate, if its linearized Poincaré map P_c has no eigenvalue 1. A Finsler metric F on M is *bumpy* if all closed geodesics and their iterates on (M, F) are non-degenerate. For studies about closed geodesics on Riemannian manifolds, we refer readers to excellent papers [4,5] of Bangert and [10] of Franks and references therein.

In recent years, geodesics and closed geodesics on Finsler manifolds have got more attentions. We refer readers to [7] of Bao, Robles and Shen, [27] of Robles, and [20] of Long and the references therein for recent progress in this area.

Note that by the classical theorem of Lyusternik–Fet [22] in 1951, there exists at least one closed geodesic on every compact Riemannian manifold. Because the proof is variational, this result works also for compact Finsler manifolds. In [26] of 2005, Rademacher obtained existence of closed geodesics on n-dimensional Finsler spheres under pinching conditions which generalizes results in [14] of Klingenberg in 1969, [1] and [2] of Ballmann, Thorbergsson and Ziller in 1982–1983 on Riemannian manifolds.

We are only aware of a few results on the existence of multiple closed geodesics on Finsler spheres without pinching conditions. In [9] of 1965, Fet proved that there exist at least two distinct closed geodesics on every reversible bumpy Finsler manifold (M, F). In [24] of 1989, Rademacher proved that there exist at least two elliptic closed geodesics on every bumpy Finsler 2-sphere. In [13] of 2003, Hofer, Wysocki and Zehnder proved that there exist either two or infinitely many distinct closed geodesics on every bumpy Finsler 2-sphere if the stable and unstable manifolds of every hyperbolic closed geodesics intersect transversally. In [6] of 2005, Bangert and Long proved that there exist at least two distinct prime closed geodesics on every Finsler 2-sphere (S^2, F).

The aim of this paper is to prove the following main result, specially for bumpy irreversible Finsler rationally homological n-spheres without pinching conditions.

Theorem 1.2. *For every bumpy Finsler metric F on every rationally homological n-sphere S^n with $n \geqslant 2$, there exist at least two distinct prime closed geodesics.*

Note that our proof of Theorem 1.2 uses only the $\mathbf{Q}$-homological properties of the Finsler manifold, thus we shall carry out our proof of this theorem below just for n-dimensional spheres.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of positive integers, non-negative integers, rational numbers, real numbers and complex numbers, respectively. We denote by $[a] = \max\{k \in \mathbf{Z} \mid k \leqslant a\}$ for any $a \in \mathbf{R}$. We use only singular homology modules with $\mathbf{Q}$-coefficients.

2. Critical modules of iterations of closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold (M, F), the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 =$

$\mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [15, Chapters 1 and 2], and [16]. This action is defined by $(s \cdot \gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2}\int_{S^1} F\big(\gamma(t), \dot\gamma(t)\big)^2\, dt. \tag{2.1}$$

It is of class $C^{1,1}$ (cf. [23]) and invariant under the S^1-action. The critical points of E of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional E is well defined along any closed geodesic c on M, which we denote by $E''(c)$ (cf. [28]). As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of E at c. In the following, we denote by

$$\Lambda^\kappa = \big\{d \in \Lambda \bigm| E(d) \leqslant \kappa\big\}, \qquad \Lambda^{\kappa -} = \big\{d \in \Lambda \bigm| E(d) < \kappa\big\}, \quad \forall \kappa \geqslant 0. \tag{2.2}$$

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda$, $t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. For a closed geodesic c, recall that the mean index $\hat{i}(c)$ is defined by

$$\hat{i}(c) = \lim_{m\to\infty} \frac{i(c^m)}{m}. \tag{2.3}$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is prime. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde\gamma \in \Lambda$ such that $\gamma = \tilde\gamma^m$.

For a closed geodesic c we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$. If $A \subseteq \Lambda$ is invariant under the action of some subgroup Γ of S^1, we denote by A/Γ the quotient space of A module the action of Γ.

Using singular homology with rational coefficients we will consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_*\big(\big(\Lambda(c) \cup S^1 \cdot c\big)/S^1, \Lambda(c)/S^1\big). \tag{2.4}$$

In order to apply the results of Gromoll and Meyer in [11] and [12], following [25], Section 6.2, we introduce finite-dimensional approximations to Λ. We choose an arbitrary energy value $a > 0$ and $k \in \mathbf{N}$ such that every geodesic segment of length $< \sqrt{2a/k}$ is minimal. Then

$$\Lambda(k, a) = \big\{\gamma \in \Lambda \bigm| E(\gamma) < a \text{ and } \gamma|_{[i/k,(i+1)/k]} \text{ is a geodesic segment for } i = 0, \ldots, k-1\big\}$$

is a $(k \cdot \dim M)$-dimensional submanifold of Λ consisting of closed geodesic polygons with k vertices. The set $\Lambda(k, a)$ is invariant under the action of the subgroup $\mathbf{Z}_k$ of S^1. Closed geodesics in $\Lambda^{a-} = \{\gamma \in \Lambda \mid E(\gamma) < a\}$ are precisely the critical points of $E|_{\Lambda(k,a)}$, and for every closed geodesic $c \in \Lambda(k, a)$ the index of $(E|_{\Lambda(k,a)})''(c)$ equals $i(c)$ and the null space of $(E|_{\Lambda(k,a)})''(c)$ coincides with the null space of $E''(c)$, cf. [25, p. 51].

We call a closed geodesic satisfying the isolation condition, if the following holds:

(Iso) *The orbit $S^1 \cdot c^m$ is an isolated critical orbit of E for all $m \in \mathbf{N}$.*

Since our aim is to prove the existence of more than one closed geodesic for every bumpy Finsler metric on S^n, the condition (Iso) does not restrict generality.

Now we can apply the results by Gromoll and Meyer [11] to a given closed geodesic c satisfying (Iso). If $m=m(c)$ is the multiplicity of c, we choose a finite-dimensional approximation $\Lambda(k,a)\subseteq\Lambda$ containing c such that m divides k. Then the isotropy subgroup $\mathbf{Z}_m\subseteq S^1$ of c acts on $\Lambda(k,a)$ by isometries. Recall that the $\mathbf{Z}_m$-action is defined by $\frac{i}{m}\cdot g(t)=g(t+\frac{i}{m})$ for all $g\in\Lambda(k,a)$ and $\frac{i}{m}\in\mathbf{Z}_m$ with $1\leqslant i\leqslant m$. Let D be a $\mathbf{Z}_m$-invariant local hypersurface transverse to $S^1\cdot c$ in $c\in\Lambda(k,a)$. Such a D can be obtained by applying the exponential map of $\Lambda(k,a)$ at c to the normal space to $S^1\cdot c$ at c. We denote by

$$T_cD=V_+\oplus V_-\oplus V_0, \tag{2.5}$$

the orthogonal decomposition of T_cD into the positive, negative and null eigenspace of the endomorphism of T_cD associated to $(E|_D)''(c)$ by the Riemannian metric. In particular, we have $\dim V_-=i(c)$ and $\dim V_0=\nu(c)$. According to [11, Lemma 1], for every such a D there exist balls $B_+\subseteq V_+$, $B_-\subseteq V_-$ and $B_0\subseteq V_0$ centered at the origins, a diffeomorphism

$$\psi: B=B_+\times B_-\times B_0\to\psi(B_+\times B_-\times B_0)\subseteq D$$

with $\psi(0)=c$, ψ_{*0} preserving the splitting (2.5), and a smooth function $f:B_0\to\mathbf{R}$ satisfying

$$f'(0)=0\quad\text{and}\quad f''(0)=0, \tag{2.6}$$

$$E\circ\psi(x_+,x_-,x_0)=|x_+|^2-|x_-|^2+f(x_0), \tag{2.7}$$

for $(x_+,x_-,x_0)\in B_+\times B_-\times B_0$. Since the $\mathbf{Z}_m$-action is isometric and E is $\mathbf{Z}_m$-invariant, the tangential map $(\frac{i}{m}|_D)_{*c}$ of $\frac{i}{m}\in\mathbf{Z}_m$ restricted to D at c preserves the above splitting (2.5). It follows from the construction of ψ that ψ is equivariant with respect to the $\mathbf{Z}_m$-action, i.e., $\frac{i}{m}\cdot\psi=\psi\circ(\frac{i}{m}|_D)_{*c}\cdot$ for $\frac{i}{m}\in\mathbf{Z}_m$, cf. [12, p. 501].

As in [11] and [12], we call $N=\{\psi(0,0,x_0)\mid x_0\in B_0\}$ a local characteristic manifold at c, $U=\{\psi(0,x_-,0)\mid x_-\in B_-\}$ a local negative disk at c. Note that N and U are $\mathbf{Z}_m$-invariant. It follows from (2.7) that c is an isolated critical point of $E|_N$. We set $N^-=N\cap\Lambda(c)$, $U^-=U\cap\Lambda(c)=U\setminus\{c\}$ and $D^-=D\cap\Lambda(c)$. Using (2.7), the fact that c is an isolated critical point of $E|_N$, and the Künneth formula, we obtain

$$H_*\big(D^-\cup\{c\},D^-\big)=H_*\big(U^-\cup\{c\},U^-\big)\otimes H_*\big(N^-\cup\{c\},N^-\big), \tag{2.8}$$

$$H_q\big(U^-\cup\{c\},U^-\big)=H_q\big(U,U\setminus\{c\}\big)=\begin{cases}\mathbf{Q}, & \text{if } q=i(c),\\ 0, & \text{otherwise},\end{cases} \tag{2.9}$$

cf. [25, Lemma 6.4] and its proof. As in [25, p. 59], for all $m\in\mathbf{N}$, let respectively

$$H_*(X,A)^{\pm\mathbf{Z}_m}=\big\{[\xi]\in H_*(X,A)\ \big|\ T_*[\xi]=\pm[\xi]\big\}, \tag{2.10}$$

where T is a generator of the $\mathbf{Z}_m$ action.

Now we have the following propositions.

Proposition 2.1. (Cf. Satz 6.11 of [25].) *Let c be a prime closed geodesic on a Finsler manifold (M,F) satisfying* (Iso). *Then we have*

$$\overline{C}_q\big(E,c^m\big) \equiv H_q\big(\big(\Lambda\big(c^m\big) \cup S^1\cdot c^m\big)/S^1, \Lambda\big(c^m\big)/S^1\big)$$
$$= \big(H_{i(c^m)}\big(U_{c^m}^- \cup \big\{c^m\big\}, U_{c^m}^-\big) \otimes H_{q-i(c^m)}\big(N_{c^m}^- \cup \big\{c^m\big\}, N_{c^m}^-\big)\big)^{+\mathbf{Z}_m}.$$

(i) *When $\nu(c^m)=0$, there holds*

$$\overline{C}_q\big(E,c^m\big) = \begin{cases} \mathbf{Q}, & \text{if } i(c^m)-i(c)\in 2\mathbf{Z} \text{ and } q=i(c^m), \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *When $\nu(c^m)>0$, there holds*

$$\overline{C}_q\big(E,c^m\big) = H_{q-i(c^m)}\big(N_{c^m}^- \cup \big\{c^m\big\}, N_{c^m}^-\big)^{\epsilon(c^m)\mathbf{Z}_m},$$

where $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$.

We need the following

Definition 2.2. (Cf. [6,21,25].) Suppose c is a closed geodesic of multiplicity $m(c)=m$ satisfying (Iso). If N is a local characteristic manifold at c, $N^- = N\cap\Lambda(c)$ and $j\in\mathbf{Z}$, we define

$$k_j(c) \equiv \dim H_j\big(N^-\cup\{c\}, N^-\big),$$
$$k_j^{\pm1}(c) \equiv \dim H_j\big(N^-\cup\{c\}, N^-\big)^{\pm\mathbf{Z}_m}.$$

Clearly the integers $k_j(c)$ and $k_j^{\pm1}(c)$ equal to 0 when $j<0$ or $j>\nu(c)$, and can take only values 0 or 1 when $j=0$ or $j=\nu(c)$.

Proposition 2.3. (Cf. [25, Satz 6.13], [6,21].) *Let c be a prime closed geodesic satisfying* (Iso).

(i) *There holds $0\leqslant k_j^{\pm1}(c^m)\leqslant k_j(c^m)$ for all $m\in\mathbf{N}$ and $j\in\mathbf{Z}$.*
(ii) *For any $m\in\mathbf{N}$, there hold $k_0^{+1}(c^m)=k_0(c^m)$ and $k_0^{-1}(c^m)=0$.*
(iii) *In particular, if c^m is non-degenerate, i.e., $\nu(c^m)=0$, then $k_0^{+1}(c^m)=k_0(c^m)=1$ and $k_0^{-1}(c^m)=0$ hold.*

3. The structure of $H_*(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n; \mathbf{Q})$

In this section, we briefly describe the relative homological structure of the quotient space $\overline{\Lambda}\equiv\overline{\Lambda}S^n=\Lambda S^n/S^1$. Here we have $\overline{\Lambda}^0=\overline{\Lambda}^0S^n=\{\text{constant point curves in } S^n\}\cong S^n$.

Let (X,Y) be a space pair such that the Betti numbers $b_i=b_i(X,Y)=\dim H_i(X,Y;\mathbf{Q})$ are finite for all $i\in\mathbf{Z}$. As usual the *Poincaré series* of (X,Y) is defined by the formal power series $P(X,Y)=\sum_{i=0}^{\infty}b_it^i$. We need the following well-known results on Betti numbers and the Morse inequality.

Theorem 3.1. (Cf. Theorem 2.4 and Remark 2.5 of [24].)

(i) *When $n\in 2\mathbf{N}$, we have*

$$P(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)(t) = t^{n-1}\left(\frac{1}{1-t^2} + \frac{t^{2n-2}}{1-t^{2n-2}}\right)$$
$$= \left(t^{(n-1)} + t^{(n+1)} + t^{(n+3)} + \cdots\right) + \left(t^{3(n-1)} + t^{5(n-1)} + t^{7(n-1)} + \cdots\right),$$

which yields

$$b_q \equiv \dim H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) = \begin{cases} 2, & \text{if } q \in \mathcal{K} \equiv \{k(n-1) \mid 3 \leqslant k \in (2\mathbf{N}+1)\}, \\ 1, & \text{if } q \in \{(n-1)+2k \mid k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.1}$$

(ii) *When* $n \in (2\mathbf{N}+1)$, *we have*

$$P(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n)(t) = t^{n-1}\left(\frac{1}{1-t^2} + \frac{t^{n-1}}{1-t^{n-1}}\right)$$
$$= \left(t^{(n-1)} + t^{(n+1)} + t^{(n+3)} + \cdots\right) + \left(t^{2(n-1)} + t^{3(n-1)} + t^{4(n-1)} + \cdots\right),$$

which yields

$$b_q \equiv \dim H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0 S^n) = \begin{cases} 2, & \text{if } q \in \mathcal{K} \equiv \{k(n-1) \mid 2 \leqslant k \in \mathbf{N}\}, \\ 1, & \text{if } q \in \{(n-1)+2k \mid k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Theorem 3.2. (Cf. Theorem I.4.3 of [8], Theorem 6.1 of [25].) *Suppose that there exist only finitely many prime closed geodesics* $\{c_j\}_{1\leqslant j\leqslant k}$ *on a Finsler n-sphere* (S^n, F). *Set*

$$M_q = \sum_{1\leqslant j\leqslant k,\, m\geqslant 1} \dim \overline{C}_q(E, c_j^m), \quad \forall q \in \mathbf{Z}.$$

Then for every integer $q \geqslant 0$ *there holds*

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \geqslant b_q - b_{q-1} + \cdots + (-1)^q b_0, \tag{3.3}$$
$$M_q \geqslant b_q. \tag{3.4}$$

4. Classification of closed geodesics on bumpy Finsler manifolds

Let c be a closed geodesic on a Finsler manifold (M, F). Denote the linearized Poincaré map of c by P_c. By [17] in 2002 of Liu and Long (cf. Chapter 12 of [19]), the index iteration formulae in [18] work for Morse indices of iterated closed geodesics on Riemannian as well as Finsler manifolds. We call a closed geodesic c is *completely non-degenerate*, if c^m is non-degenerate for all $m \in \mathbf{N}$. When the Finsler metric F is bumpy, every closed geodesic c on (M, F) is completely non-degenerate. Thus by Theorems 8.1.4–8.1.7, 8.2.3 and 8.2.4, and 8.3.1 of [19], in the basic normal form decomposition of the symplectic matrix P_c (cf. Theorem 1.8.10 of [19]) there can exist only basic normal forms like $H(d)$ with $d \in \mathbf{R} \setminus \{0, \pm 1\}$, $R(\theta)$ and $N(\alpha, B)$ with θ/π and α/π being irrational (cf. notation below). Therefore according to the iteration formula of Morse indices, completely non-degenerate closed geodesics on a Finsler manifold (M, F) can be classified into the following 5 cases *NCG-1* to *NCG-5*.

To introduce this classification, we need some notations from [19]. Given any two real matrices of the square block form

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

the $\diamond$-sum of M_1 and M_2 is defined by the $2(i+j)\times 2(i+j)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

For convenience, we denote by $N(\alpha, B)^{\diamond r} \equiv N(\alpha_1, B_1) \diamond \cdots \diamond N(\alpha_r, B_r)$, where $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $B = (B_1, \ldots, B_r)$ for some $0 \leqslant r \leqslant [\frac{n-1}{2}]$. If $r = 0$ in the following, it means that no such a term $N(\alpha, B)^{\diamond r}$ appears. Here as in [19] we set

$$N(\alpha_i, B_i) = \begin{pmatrix} R(\alpha_i) & B_i \\ 0 & R(\alpha_i) \end{pmatrix},$$

$$R(\alpha_i) = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix}, \qquad B_i = \begin{pmatrix} b_{i1} & b_{i2} \\ b_{i3} & b_{i4} \end{pmatrix},$$

where $\alpha_i/\pi \in (0,2) \setminus (\mathbf{Q} \cup \{1\})$, $(b_{i1}, b_{i2}, b_{i3}, b_{i4}) \in \mathbf{R}^4$ for $1 \leqslant i \leqslant r$. We denote also $H(d) = \left(\begin{smallmatrix} d & 0 \\ 0 & 1/d \end{smallmatrix}\right)$ with $d \in \mathbf{R} \setminus \{0, \pm 1\}$.

The homotopy set $\Omega(M)$ of M in the symplectic group $\mathrm{Sp}(2n)$ was studied in [18] which is defined by

$$\Omega(M) = \bigl\{N \in \mathrm{Sp}(2n) \bigm| \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M)\, \forall \omega \in \Gamma\bigr\},$$

where $\sigma(M)$ denotes the spectrum of M, $\nu_\omega(M) \equiv \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I)$ for all $\omega \in \mathbf{U}$, and $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$. Let $\Omega^0(M)$ denote the path connected component of $\Omega(M)$ containing M (cf. [19, p. 38]).

By Theorems 8.2.3 to 8.2.4 of [19], the Morse indices of iterates of a completely non-degenerate closed geodesic c with $P_c = N(\alpha_i, B_i)$ satisfy the same formula

$$i(c) = 2p \quad \text{for some } p \in \mathbf{N}_0 \qquad \text{and} \qquad i\bigl(c^m\bigr) = 2mp, \quad \nu\bigl(c^m\bigr) = 0, \quad \forall m \geqslant 1.$$

Hence by Theorems 8.1.4–8.1.7 and 8.3.1 of [19], we have the following classification of completely non-degenerate closed geodesics c on a Finsler n-dimensional manifold, i.e., there exists a path $f_c \in C([0,1], \Omega^0(P_c))$ such that $f_c(0) = P_c$ and $f_c(1)$ have the following forms:

NCG-1. $f_c(1) = N(\alpha, B)^{\diamond r} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_{n-2r-1})$.

In this case, by Theorem 8.3.1 of [19], we have $i(c) = 2p + (n - 2r - 1)$ for some $p \in \mathbf{Z}$ such that $i(c) \geqslant 0$, and

$$i(c^m) = 2mp + 2\sum_{i=1}^{n-2r-1}\left[\frac{m\theta_i}{2\pi}\right] + (n - 2r - 1), \qquad \nu(c^m) = 0, \quad \forall m \geqslant 1. \tag{4.1}$$

NCG-2. $f_c(1) = N(\alpha, B)^{\diamond r} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_k) \diamond H(d_{k+1}) \diamond \cdots \diamond H(d_{n-2r-1})$ *with* $k \in 2\mathbf{N}$ *and* $2 \leqslant k \leqslant n - 2r - 2$.

In this case, by Theorem 8.3.1 of [19], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = m(p - k) + 2\sum_{i=1}^{k}\left[\frac{m\theta_i}{2\pi}\right] + k, \qquad \nu(c^m) = 0, \quad \forall m \geqslant 1. \tag{4.2}$$

NCG-3. $f_c(1) = N(\alpha, B)^{\diamond r} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_k) \diamond H(d_{k+1}) \diamond \cdots \diamond H(d_{n-2r-1})$ *with* $k \in (2\mathbf{N} - 1)$ *and* $3 \leqslant k \leqslant n - 2r - 2$.

In this case, by Theorem 8.3.1 of [19], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = m(p - k) + 2\sum_{i=1}^{k}\left[\frac{m\theta_i}{2\pi}\right] + k, \qquad \nu(c^m) = 0, \quad \forall m \geqslant 1. \tag{4.3}$$

NCG-4. $f_c(1) = N(\alpha, B)^{\diamond r} \diamond R(\theta_1) \diamond H(d_2) \diamond \cdots \diamond H(d_{n-2r-1})$.

In this case, by Theorem 8.3.1 of [19], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = m(p - 1) + 2\left[\frac{m\theta_1}{2\pi}\right] + 1, \qquad \nu(c^m) = 0, \quad \forall m \geqslant 1. \tag{4.4}$$

NCG-5. $f_c(1) = N(\alpha, B)^{\diamond r} \diamond H(d_1) \diamond \cdots \diamond H(d_{n-1})$.

In this case, by Theorem 8.3.1 of [19], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = mp, \qquad \nu(c^m) = 0, \quad \forall m \geqslant 1. \tag{4.5}$$

5. A mean index identity

We need a notation from [21].

Definition 5.1. Let c be a completely non-degenerate prime closed geodesic on (S^n, F). For each $m \in \mathbf{N}$, the critical type numbers of c^m is defined by

$$K(c^m) \equiv \left(k_0^\epsilon(c^m), k_1^\epsilon(c^m), \ldots, k_n^\epsilon(c^m)\right) = \left(k_0^\epsilon(c^m), 0, \ldots, 0\right), \tag{5.1}$$

where $\epsilon = \epsilon(c^m) = (-1)^{i(c^m)-i(c)}$. Note that only $k_0^\epsilon(c^m)$ may be non-zero for $m \geqslant 1$ by Definition 2.2 and Proposition 2.3. We call a completely non-degenerate prime closed geodesic c homologically invisible if $k_0^\epsilon(c^m) = 0$ for all $m \in \mathbf{N}$, or homologically visible otherwise.

Lemma 5.2. *Let c be a completely non-degenerate prime closed geodesic on a Finsler n-sphere (S^n, F). Then there exist a minimal integer $N \in \mathbf{N}$ such that $K(c^m) = K(c^{m+N})$ for all $m \in \mathbf{N}$. According to the classification in Section* 4, *we have*

$$N = 1, \quad \text{if } c \text{ belongs to NCG-1};$$

$$N = \begin{cases} 1, & \text{if } p \text{ is even}, \\ 2, & \text{if } p \text{ is odd}, \end{cases} \quad \text{if } c \text{ belongs to NCG-2 or NCG-5};$$

$$N = \begin{cases} 2, & \text{if } p \text{ is even}, \\ 1, & \text{if } p \text{ is odd}, \end{cases} \quad \text{if } c \text{ belongs to NCG-3 or NCG-4}.$$

Proof. In fact, N depends only on the parity of $i(c^m) - i(c)$ for any $m \in \mathbf{N}$ by Proposition 2.3. More precisely,

$$N = \begin{cases} 1, & \text{if } i(c^m) - i(c) \text{ is even for any } m \in \mathbf{N}, \\ 2, & \text{otherwise}. \end{cases}$$

By the classification of Section 4, we have the following details. In *NCG-1*, $i(c^m) - i(c)$ is even. In *NCG-2* and *NCG-3*, $i(c^m) - i(c) = (m-1)(p-k) \bmod 2$. In *NCG-4*, $i(c^m) - i(c) = (m-1)(p-1) \bmod 2$. In *NCG-5*, $i(c^m) - i(c) = (m-1)p$. Therefore Lemma 5.2 follows. □

Suppose that there exist only finitely many completely non-degenerate prime closed geodesics $\{c_j\}_{1\leqslant j\leqslant k}$ for $1 \leqslant j \leqslant k$ on a bumpy Finsler n-sphere (S^n, F). The *Morse series* $M(t)$ of the energy functional E on the space $(\Lambda S^n/S^1, \Lambda^0 S^n/S^1)$ is defined by

$$M(t) = \sum_{\substack{q\geqslant 0,\, m\neq 0 \\ 1\leqslant j\leqslant k}} \dim \overline{C}_q\big(E, c_j^m\big) t^q.$$

Then it yields a formal power series $Q(t) = \sum_{i=0}^{\infty} q_i t^i$ with non-negative integer coefficients q_i such that

$$M(t) = P\big(\Lambda S^n/S^1, \Lambda^0 S^n/S^1\big)(t) + (1+t)Q(t). \tag{5.2}$$

For a formal power series $R(t) = \sum_{i=0}^{\infty} r_i t^i$, we denote by $R^n(t) = \sum_{i=0}^{n} r_i t^i$ for $n \in \mathbf{N}$ the corresponding truncated polynomials. Using this notation, (5.2) becomes

$$(-1)^m q_m = M^m(-1) - P^m(-1) \quad \forall m \in \mathbf{N}. \tag{5.3}$$

By Satz 7.8 of [25] we have specially for spheres:

$$\lim_{m\to\infty} \frac{1}{m} P^m\big(\Lambda S^n/S^1, \Lambda^0 S^n/S^1\big)(-1) = \begin{cases} -\frac{n}{2(n-1)}, & \text{if } n \text{ is even}, \\ \frac{n+1}{2(n-1)}, & \text{if } n \text{ is odd}. \end{cases} \tag{5.4}$$

A general version of the following mean index identity was proved in Theorem 3 in [24] and [25] of Rademacher. Our following theorem gives more precise coefficients in the identity than those in [24] and [25]. This more precise information is crucial in the proof of our main Theorem 1.2 later.

Theorem 5.3. *Suppose that there exist only finitely many homologically visible prime closed geodesics $\{c_j\}_{1\leqslant j\leqslant k}$ on a bumpy Finsler n-sphere (S^n, F) with $\hat{i}(c_j) > 0$. Then the following identity holds*

$$\sum_{1\leqslant j\leqslant k,\,1\leqslant m\leqslant N_j} (-1)^{i(c_j^m)} k_0^{\epsilon}\big(c_j^m\big)\frac{1}{N_j\hat{i}(c_j)} = \begin{cases} -\frac{n}{2(n-1)}, & \text{if } n \text{ is even},\\ \frac{n+1}{2(n-1)}, & \text{if } n \text{ is odd},\end{cases} \tag{5.5}$$

where $N_j = N(c_j) \in \mathbf{N}$ is the number defined in Lemma 5.2 for c_j, $k_0^{\epsilon}(c_j^m)$s are the critical type numbers of c_j^m, $\epsilon \equiv \epsilon(c_j^m) = (-1)^{i(c_j^m)-i(c_j)}$.

Proof. Because $\dim \overline{C}_q(E, c_j^m)$ can be non-zero only for $q = i(c_j^m)$ by Proposition 2.1, the formal Poincaré series $M(t)$ becomes

$$M(t) = \sum_{1\leqslant j\leqslant k,\, m\geqslant 1} k_0^{\epsilon}\big(c_j^m\big)t^{i(c_j^m)} = \sum_{1\leqslant j\leqslant k,\,1\leqslant m\leqslant N_j,\, s\geqslant 0} k_0^{\epsilon}\big(c_j^m\big)t^{i(c_j^{sN_j+m})}, \tag{5.6}$$

where the last equality follows from Lemma 5.2. Write $M(t) = \sum_{h=0}^{\infty} w_h t^h$. Then we have

$$w_h = \sum_{1\leqslant j\leqslant k,\,1\leqslant m\leqslant N_j} k_0^{\epsilon}\big(c_j^m\big)^{\#}\big\{s\in\mathbf{N}_0 \bigm| i\big(c_j^{sN_j+m}\big) = h\big\}, \tag{5.7}$$

where $^{\#}A$ denotes the total number of elements in a set A.

Claim 1. $\{w_h\}_{h\geqslant 0}$ *is bounded.*

In fact, we have

$$\begin{aligned}
&^{\#}\big\{s\in\mathbf{N}_0 \bigm| i\big(c_j^{sN_j+m}\big) = h\big\}\\
&\quad = {}^{\#}\big\{s\in\mathbf{N}_0 \bigm| i\big(c_j^{sN_j+m}\big) = h,\ \big|i\big(c_j^{sN_j+m}\big) - (sN_j+m)\hat{i}(c_j)\big| \leqslant n-1\big\}\\
&\quad \leqslant {}^{\#}\big\{s\in\mathbf{N}_0 \bigm| \big|h - (sN_j+m)\hat{i}(c_j)\big| \leqslant n-1\big\}\\
&\quad = {}^{\#}\left\{s\in\mathbf{N}_0 \Bigm| \frac{h-n+1-m\hat{i}(c_j)}{N_j\hat{i}(c_j)} \leqslant s \leqslant \frac{h+n-1-m\hat{i}(c_j)}{N_j\hat{i}(c_j)}\right\}\\
&\quad \leqslant \frac{2(n-1)}{N_j\hat{i}(c_j)} + 1,
\end{aligned}$$

where the first equality follows from the fact $|i(c^m) - m\hat{i}(c)| \leqslant n-1$ (cf. Theorem 1.4 on p. 69 of [24]). Hence Claim 1 holds.

Next we estimate $M^n(-1)$. By (5.7) we have

$$M^r(-1) = \sum_{h=0}^{r} w_h(-1)^h$$
$$= \sum_{1\leqslant j\leqslant k,\, 1\leqslant m\leqslant N_j} (-1)^{i(c_j^m)} k_0^{\epsilon}\big(c_j^m\big)^{\#}\big\{s\in \mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant r\big\}. \tag{5.8}$$

Claim 2. *There is a real constant $C>0$ independent of r, but depending on c_j for $1\leqslant j\leqslant k$ such that*

$$\left| M^r(-1) - \sum_{1\leqslant j\leqslant k,\, 1\leqslant m\leqslant N_j} (-1)^{i(c_j^m)} k_0^{\epsilon}\big(c_j^m\big)\frac{r}{N_j\hat{i}(c_j)}\right| \leqslant C. \tag{5.9}$$

In fact, we have

$$\begin{aligned}
&{}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant r\big\}\\
&\quad = {}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant r,\ \big|i\big(c_j^{sN_j+m}\big)-(sN_j+m)\hat{i}(c_j)\big|\leqslant n-1\big\}\\
&\quad \leqslant {}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; 0\leqslant (sN_j+m)\hat{i}(c_j)\leqslant r+n-1\big\}\\
&\quad = {}^{\#}\left\{s\in\mathbf{N}_0 \;\middle|\; 0\leqslant s\leqslant \frac{r+n-1-m\hat{i}(c_j)}{N_j\hat{i}(c_j)}\right\}\\
&\quad \leqslant \frac{r+n-1}{N_j\hat{i}(c_j)},
\end{aligned} \tag{5.10}$$

where the last inequality uses $\frac{1}{2}\leqslant \frac{m}{N_j}\leqslant 2$ by the definition of N_j and $1\leqslant m\leqslant N_j$.

On the other hand, we have

$$\begin{aligned}
&{}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant r\big\}\\
&\quad = {}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant r,\ \big|i\big(c_j^{sN_j+m}\big)-(sN_j+m)\hat{i}(c_j)\big|\leqslant n-1\big\}\\
&\quad \geqslant {}^{\#}\big\{s\in\mathbf{N}_0 \;\big|\; i\big(c_j^{sN_j+m}\big)\leqslant (sN_j+m)\hat{i}(c_j)+(n-1)\leqslant r\big\}\\
&\quad \geqslant {}^{\#}\left\{s\in\mathbf{N}_0 \;\middle|\; 0\leqslant s\leqslant \frac{r-n+1-m\hat{i}(c_j)}{N_j\hat{i}(c_j)}\right\}\\
&\quad \geqslant \frac{r-n+1}{N_j\hat{i}(c_j)}-2,
\end{aligned} \tag{5.11}$$

where the last inequality uses $\frac{1}{2}\leqslant \frac{m}{N_j}\leqslant 2$ by the definition of N_j and $1\leqslant m\leqslant N_j$.

By (5.10) and (5.11), we obtain (5.9).

Since the sequence $\{w_h\}$ is bounded and $w_r = b_r + q_r + q_{r-1}$, the sequence $\{q_h\}_{h\geqslant 0}$ of $Q(t)$ is bounded. Hence, by (5.3) we obtain

$$\lim_{r\to\infty}\frac{1}{r}M^r(-1) = \lim_{r\to\infty}\frac{1}{r}P^r(-1) = \begin{cases} -\frac{n}{2(n-1)}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2(n-1)}, & \text{if } n \text{ is odd.} \end{cases} \tag{5.12}$$

Hence (5.5) holds. □

Remark 5.4. Bangert and Long in [6] as well as Long and Wang in [21] established such an mean index identity with exact coefficients on Finsler 2-spheres. For readers convenience, following ideas in [6] and [21] we give a complete proof for (S^n, F) here.

6. Proof of Theorem 1.2

Assuming the contrary, we prove Theorem 1.2 by contradiction. That is, assume the following condition in this section:

(F) *There exists only one prime closed geodesic c on the bumpy Finsler $S^n = (S^n, F)$.*

Lemma 6.1. *Under the assumption* (F)*, the mean index of the closed geodesic c must satisfy $\hat{i}(c) > 0$.*

Proof. If $\hat{i}(c) = 0$, then we have $i(c^m) = 0$ for all $m \geqslant 1$ (cf. [17, Corollary 4.2]). By Theorem 3.1, we have $b_{n-1} = 1$. By Proposition 2.1, we have

$$\overline{C}_0(E, c^m) = \mathbf{Q}, \qquad \overline{C}_q(E, c^m) = 0 \quad \text{for } q \in \mathbf{N}.$$

By Theorem 3.2, we have $0 = M_{n-1} \geqslant b_{n-1} = 1$ with $n \geqslant 2$, which implies $\hat{i}(c) > 0$. □

Lemma 6.2. *Under the assumption* (F)*, the index of the closed geodesic c must satisfy $i(c) \leqslant n-1$.*

Proof. By contradiction, assume $i(c) > n-1$. By Corollary 4.2 of [17] (cf. (i) of Theorem 12.1.1 of [19]), we have

$$i(c^m) \geqslant i(c), \quad \forall m \in \mathbf{N}. \tag{6.1}$$

Hence $i(c^m) > n-1$ for all $m \geqslant 1$. By Proposition 2.1, we have

$$\overline{C}_q(E, c^m) = 0 \quad \text{for } q \in [0, n-1] \cap \mathbf{N}_0.$$

Hence $\overline{C}_{n-1}(E, c^m) = 0$ for all $m \in \mathbf{N}$, which implies $M_{n-1} = 0$. By Theorem 3.2, we have $0 = M_{n-1} \geqslant b_{n-1} = 1$, which is a contradiction. So we have $i(c) \leqslant n-1$. □

Lemma 6.3. *Under the assumption* (F)*, the index of the closed geodesic c must satisfy*

$$i(c) \geqslant n-1,$$

if one of the following conditions is satisfied:

(i) *when n is even,* $i(c)$ *is odd and the Morse-type numbers* $M_{2k} = 0$ *for all* $k \in \mathbf{N}$;
(ii) *when n is odd,* $i(c)$ *is even and the Morse-type numbers* $M_{2k-1} = 0$ *for all* $k \in \mathbf{N}$.

Proof. Suppose (i) is satisfied. Assume $i(c) < n - 1$. Then $i(c) = 2k_0 - 1 \leqslant n - 3$ for some $k_0 \in \mathbf{N}$. So the Morse-type number $M_{2k_0-1} \geqslant 1$ by Propositions 2.1 and 2.3. By (6.1) and $i(c) = 2k_0 - 1$, we obtain $M_{2k-1} = 0$ for $0 \leqslant k \leqslant k_0 - 1$. But by (i) of Theorem 3.1, $b_k = 0$ for any $k < n - 1$. Therefore, by Theorem 3.2 and the condition (i), we have

$$-1 \geqslant -M_{2k_0-1} = M_{2k_0} - M_{2k_0-1} + \cdots - M_0 \geqslant b_{2k_0} - b_{2k_0-1} + \cdots - b_0 = 0, \tag{6.2}$$

which is a contradiction.

Suppose (ii) is satisfied. Assume $i(c) < n - 1$. Then $i(c) = 2k_0 \leqslant n - 3$ for some $k_0 \in \mathbf{N}$. So the Morse-type number $M_{2k_0} \geqslant 1$ by Propositions 2.1 and 2.3, and $b_k = 0$ for any $k < n - 1$ by (i) of Theorem 3.1. Therefore, similarly to (6.2), by Theorem 3.2 we have

$$-1 \geqslant -M_{2k_0} = M_{2k_0+1} - M_{2k_0} + \cdots - M_0 \geqslant b_{2k_0+1} - b_{2k_0} + \cdots - b_0 = 0, \tag{6.3}$$

which is a contradiction. Lemma 6.3 is proved. □

As an immediate consequence of Lemmas 6.2 and 6.3, we have the following corollary.

Corollary 6.4. *Under the conditions* (F), *and* (i) *or* (ii) *of Lemma* 6.3, *the index of the closed geodesic c must satisfy* $i(c) = n - 1$.

By Theorem 3.1, the first appearance of the Betti number b_q which takes the value 2 is when

$$q = \begin{cases} 3(n-1), & \text{if } n \text{ is even}, \\ 2(n-1), & \text{if } n \text{ is odd}. \end{cases}$$

To study Case 1 in Step 1 and Case 1 in Step 2 in our proof of Theorem 1.2 below, a basic idea is that we want to find a contradiction in both cases by Morse inequality before the Betti number b_q reaching the value 2, i.e., when

$$q \leqslant \begin{cases} 3(n-1) - 2, & \text{if } n \text{ is even}, \\ 2(n-1) - 2, & \text{if } n \text{ is odd}. \end{cases}$$

Hence we construct the following sets

$$\Theta(n) = \begin{cases} \{j \in 2\mathbf{N} - 1 \mid n - 1 \leqslant j \leqslant 3n - 5\}, & \text{when } n \text{ is even}, \\ \{j \in 2\mathbf{N} \mid n - 1 \leqslant j \leqslant 2n - 4\}, & \text{when } n \text{ is odd}. \end{cases} \tag{6.4}$$

Lemma 6.5. *Assume the conditions* (F), *and* (i) *or* (ii) *of Lemma* 6.3 *hold. Suppose* $\frac{i(c^m) - i(c)}{2} \in \{0, 1, \ldots, m - 1\}$ *for all m. Then for every* $n - 1 + 2k \in \Theta(n)$, *there exists a unique iteration* c^{k+1} *of c such that*

$$i(c^{k+1}) = n - 1 + 2k. \tag{6.5}$$

Proof. By Corollary 6.4, $i(c) = n-1$. So (6.5) holds when $k=0$. Assume that there exists another iteration c^{m_0} such that $i(c^{m_0}) = i(c) = n-1$, by Proposition 2.1 and $i(c^m) - i(c) \in 2\mathbf{N}_0$ for all m, it yields $M_{n-1} \geqslant 2$ and $M_r = 0$ for any $r < n-1$ by (6.1). On the other hand, by Theorem 3.1, $b_{n-1} = 1$ and $b_n = b_r = 0$ for any $r < n-1$. Noting that $M_n = 0$ by the condition (i) or (ii), by Theorem 3.2 we have

$$-2 \geqslant M_n - M_{n-1} + \cdots + (-1)^n M_0 \geqslant b_n - b_{n-1} + \cdots + (-1)^n b_0 = -1,$$

which is a contradiction. Thus Lemma 6.5 holds for $k=0$.

By induction, we assume that Lemma 6.5 holds for all $k \leqslant \hat{k}$, where $1 \leqslant \hat{k} < \max \Theta(n)$, i.e., there exists a unique iteration c^{k+1} such that (6.5) holds for all $k \leqslant \hat{k}$. Hence we have

$$M_q = 1 \quad \text{for } q \in \hat{\Theta}(n) \equiv \{ n-1+2t \mid t \in [0,\hat{k}] \cap \mathbf{N}_0 \} \subset \Theta(n), \tag{6.6}$$

by Proposition 2.1 and $i(c^m) - i(c) \in 2\mathbf{N}_0$ for all m.

Firstly, we prove that $i(c^{\hat{k}+2}) = n-1+2(\hat{k}+1)$. By the condition $\frac{i(c^m)-i(c)}{2} \in \{0, 1, \ldots, m-1\}$ for all m, it yields $i(c^{\hat{k}+2}) = n-1+2t$, $t \in \{0, 1, \ldots, \hat{k}+1\}$. If $t \in \{0, 1, \ldots, \hat{k}\}$, then there must exist some $s \in [0, \hat{k}] \cap \mathbf{N}_0$ such that $i(c^{\hat{k}+2}) = i(c^{s+1}) = n-1+2s$, which contradicts to the uniqueness of c^{s+1}. So the only possibility is $i(c^{\hat{k}+2}) = n-1+2(\hat{k}+1)$.

On the other hand, assume that there exists another iteration c^{m_0} such that $i(c^{m_0}) = i(c^{\hat{k}+2}) = n-1+2(\hat{k}+1) \equiv \kappa \in \Theta(n)$. Then it yields $M_\kappa \geqslant 2$ by Proposition 2.1 and $i(c^m) - i(c) \in 2\mathbf{N}_0$ for all m. Note that for $l \in [0, \kappa-1] \setminus \hat{\Theta}(n)$, there holds $M_{\kappa+1} = M_l = 0$ by the condition (i) or (ii). Therefore among all the M_l's with $l < \kappa$, half of them are zero, and half of them are 1. By Theorem 3.1, $b_{\kappa+1} = b_l = 0$ for the same l mentioned above and $b_\kappa = b_q = 1$ for q in (6.6). Note that $\kappa - n \in (2\mathbf{Z}+1)$ by the definition of κ. Therefore, by Theorem 3.2 we have

$$\begin{aligned} -\left(2 + \frac{\kappa-n+1}{2}\right) &\geqslant M_{\kappa+1} - M_\kappa + M_{\kappa-1} + \cdots - M_{n-1} \\ &\geqslant b_{\kappa+1} - b_\kappa - b_{\kappa-1} + \cdots - b_{n-1} \\ &= -\left(1 + \frac{\kappa-n+1}{2}\right), \end{aligned}$$

which is a contradiction. Therefore Lemma 6.5 holds for $k = \hat{k}+1$. This completes the proof. □

Now we can give

Proof of Theorem 1.2. We carry out the proof in two steps under the assumption (F) on (S^n, F).

Step 1. When n is even, we claim that there must be another prime closed geodesic.

We study the problem in five cases according to our classification in Section 4.

Case 1. c belongs to NCG-1.

In this case, $i(c^m)$ is odd and $i(c^m) - i(c)$ is even for any $m \in \mathbf{N}$. Hence $\epsilon = \epsilon(c^m) = 1$ and $k_0^\epsilon(c^m) = 1$ by Proposition 2.3. By Lemma 5.2 and Theorem 5.3, we have $-\frac{1}{\hat{i}(c)} = -\frac{n}{2(n-1)}$, i.e.,

$$2p + \sum_{i=1}^{n-2r-1} \frac{\theta_i}{\pi} = \hat{i}(c) = \frac{2(n-1)}{n} < 2. \tag{6.7}$$

On the other hand, by Proposition 2.1 and the oddness of $i(c^m)$, we have the Morse-type numbers

$$M_k = 0 \quad \text{for all } k \in 2\mathbf{N}. \tag{6.8}$$

Hence, by Corollary 6.4, $2p + n - 2r - 1 = i(c) = n - 1$, which yields $p = r \in \mathbf{N}_0$. Together with (6.7), it yields $r = p = 0$. So in this case, we have

$$i(c) = n - 1, \qquad i\left(c^m\right) = 2\sum_{i=1}^{n-1}\left[\frac{m\theta_i}{2\pi}\right] + n - 1 \quad \text{and} \quad \hat{i}(c) = \sum_{i=1}^{n-1}\frac{\theta_i}{\pi} = \frac{2(n-1)}{n} < 2. \tag{6.9}$$

Note that, by (6.9), for any $m \in \mathbf{N}$, there holds

$$\sum_{i=1}^{n-1}\frac{m\theta_i}{2\pi} = \frac{m(n-1)}{n} < m. \tag{6.10}$$

Hence

$$\sum_{i=1}^{n-1}\left[\frac{m\theta_i}{2\pi}\right] \in \{0, 1, \ldots, m-1\}. \tag{6.11}$$

Claim 1. *For any $m \in [1, n-1] \cap \mathbf{N}$, there holds $i(c^m) = 2(m-1) + n - 1$.*

In fact, we have $i(c) = n - 1$ by (6.9). By induction, assume that $i(c^k) = 2(k-1) + n - 1$ for some $k \in [1, n-2] \cap \mathbf{N}$. By (6.9) and (6.11), for $k + 1$ we have

$$i\left(c^{k+1}\right) = 2\sum_{i=1}^{n-1}\left[\frac{(k+1)\theta_i}{2\pi}\right] + n - 1 \in \left\{n - 1 + 2s \mid s \in [0, k] \cap \mathbf{N}_0\right\}. \tag{6.12}$$

Note that the conditions of Lemma 6.5 are satisfied by (6.8), (6.9) and (6.11). If $i(c^{k+1}) \leqslant n - 1 + 2(k-1)$, then $i(c^{k+1}) = i(c^r)$ for some $r \in [1, k] \cap \mathbf{N}$, which yields a contradiction to Lemma 6.5. So the only possibility is $i(c^{k+1}) = 2k + n - 1$. Claim 1 is proved.

Next we consider $i(c^n)$. By (6.10), we have

$$\sum_{i=1}^{n-1}\frac{n\theta_i}{2\pi} = \frac{n(n-1)}{n} = n - 1. \tag{6.13}$$

Noting that $\{\frac{n\theta_i}{2\pi}\}$ is an irrational number sequence, by (6.13) it yields

$$\sum_{i=1}^{n-1}\left[\frac{n\theta_i}{2\pi}\right] \in \{0, 1, \ldots, n-2\}. \tag{6.14}$$

Hence

$$i\left(c^n\right)=2\sum_{i=1}^{n-1}\left[\frac{n\theta_i}{2\pi}\right]+n-1\in\left\{n-1+2s \mid s\in[0,n-2]\cap\mathbf{N}_0\right\}. \tag{6.15}$$

Therefore, by Claim 1 and (6.15), there must exists some $r\in[1,n-1]\cap\mathbf{N}$ such that $i(c^n)=i(c^r)$, which is also a contradiction to Lemma 6.5.

Case 2. c belongs to NCG-2.

Subcase 2.1. If $i(c)=p$ is even, we have $N=1$ by Lemma 5.2 and $i(c^m)$ is even for any $m\in\mathbf{N}$. Hence $\epsilon=1$ and $k_0^{\epsilon}(c^m)=1$ by Proposition 2.3. Thus, by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)}=-\frac{n}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Subcase 2.2. If $i(c)=p$ is odd, $N=2$ by Lemma 5.2 and

$$i\left(c^m\right) \text{ is } \begin{cases}\text{even}, & \text{if } m \text{ is even},\\ \text{odd}, & \text{if } m \text{ is odd}.\end{cases} \tag{6.16}$$

So by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)}=\frac{n}{n-1}$, which implies

$$\hat{i}(c)=p-k+\sum_{i=1}^{k}\frac{\theta_i}{\pi}=\frac{n-1}{n}<1. \tag{6.17}$$

On the other hand, by (6.16) and Proposition 2.1, we have the Morse-type numbers $M_k=0$ for any $k\in 2\mathbf{N}$. So by Corollary 6.4, we have $i(c)=p=n-1$. Hence by (6.17), it yields $n-2<k$. This yields a contradiction to $k\leqslant n-2r-2$ in the definition of *NCG-2*.

Case 3. c belongs to NCG-3.

Subcase 3.1. If $i(c)=p$ is even, we obtain $N=2$ by Lemma 5.2 and $i(c^2)$ is odd. Hence $\epsilon=\epsilon(c^2)=-1$ and $k_0^{\epsilon}(c^2)=0$ So by Theorem 5.3, it yields $\frac{1}{2\hat{i}(c)}=-\frac{n}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Subcase 3.2. If $i(c)=p$ is odd, we obtain $N=1$ by Lemma 5.2 and $i(c^m)$ is odd for any m. Hence $\epsilon=1$ and $k_0^{\epsilon}(c^m)=1$ by Proposition 2.3. So by Theorem 5.3, it yields $-\frac{1}{\hat{i}(c)}=-\frac{n}{2(n-1)}$, which implies

$$\hat{i}(c)=p-k+\sum_{i=1}^{k}\frac{\theta_i}{\pi}=\frac{2(n-1)}{n}<2. \tag{6.18}$$

Noting that $p-k$ is even, this yields $p\leqslant k$. Because the Morse-type numbers $M_k=0$ holds for any $k\in 2\mathbf{N}$ by Proposition 2.1, by Corollary 6.4 we have $i(c)=p=n-1$. Hence $n-1=p\leqslant k$. This yields a contradiction to $k\leqslant n-2r-2$ in the definition of *NCG-3*.

Case 4. c belongs to NCG-4.

Subcase 4.1. If $i(c)=p$ is even, $N=2$ by Lemma 5.2 and $i(c^2)$ is odd. Hence $\epsilon=\epsilon(c^2)=-1$ and $k_0^{\epsilon}(c^2)=0$. So by Theorem 5.3, it yields $\frac{1}{2\hat{i}(c)}=-\frac{n}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Subcase 4.2. If $i(c) = p$ is odd, $N = 1$ by Lemma 5.2 and $i(c^m)$ is odd for all m. Hence $\epsilon = 1$ and $k_0^\epsilon(c^m) = 1$ by Proposition 2.3. Thus by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)} = \frac{n}{2(n-1)}$. But in this case $\hat{i}(c) = (p-1) + \frac{\theta_1}{\pi}$ is an irrational number. This leads to a contradiction.

Case 5. c belongs to NCG-5.

Subcase 5.1. If $i(c) = p$ is even, $N = 1$ by Lemma 5.2 and $i(c^m)$ is even for all m. Hence $\epsilon = 1$ and $k_0^\epsilon(c^m) = 1$ by Proposition 2.3. So by Theorem 5.3, there holds $\frac{1}{\hat{i}(c)} = -\frac{n}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Subcase 5.2. If $i(c) = p$ is odd, we obtain $N = 2$ by Lemma 5.2 and $i(c^2)$ is even. Hence $\epsilon = \epsilon(c^2) = -1$ and $k_0^\epsilon(c^2) = 0$. Thus by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)} = \frac{n}{n-1}$. Hence $\hat{i}(c) = p = \frac{n-1}{n}$, which yields a contradiction because $p \in \mathbf{N}_0$.

Therefore when n is even, we have proved that there must exist another prime closed geodesic on the bumpy Finsler n-sphere (S^n, F).

Step 2. When n is odd, we claim that there must be another prime closed geodesic.

We continue our study in five cases according to the classification in Section 4.

Case 1. c belongs to NCG-1.

In this case, $i(c^m)$ is even for all m. Hence $\epsilon = 1$ and $k_0^\epsilon(c^m) = 1$ by Proposition 2.3. By Lemma 5.2 and Theorem 5.3, we have $\frac{1}{\hat{i}(c)} = \frac{n+1}{2(n-1)}$, i.e.,

$$2p + \sum_{i=1}^{n-2r-1} \frac{\theta_i}{\pi} = \hat{i}(c) = \frac{2(n-1)}{n+1} < 2. \tag{6.19}$$

On the other hand, by Proposition 2.1 and the evenness of $i(c^m)$, we have the Morse-type numbers

$$M_k = 0 \quad \forall k \in 2\mathbf{N} - 1. \tag{6.20}$$

Thus by Corollary 6.4, we have $2p + n - 2r - 1 = i(c) = n - 1$, which implies $p = r \in \mathbf{N}_0$. Together with (6.19), it yields $r = p = 0$. Hence in this case, we have

$$i(c) = n-1, \qquad i\left(c^m\right) = 2\sum_{i=1}^{n-1}\left[\frac{m\theta_i}{2\pi}\right] + n - 1 \quad \text{and} \quad \hat{i}(c) = \sum_{i=1}^{n-1} \frac{\theta_i}{\pi} = \frac{2(n-1)}{n+1} < 2. \tag{6.21}$$

Note that, by (6.21), for any $m \in \mathbf{N}$ there holds

$$\sum_{i=1}^{n-1} \frac{m\theta_i}{2\pi} = \frac{m(n-1)}{n+1} < m. \tag{6.22}$$

Hence

$$\sum_{i=1}^{n-1}\left[\frac{m\theta_i}{2\pi}\right] \in \{0, 1, \ldots, m-1\}. \tag{6.23}$$

Let

$$m_1 = \frac{n-1}{2}, \qquad m_2 = \frac{n+1}{2}. \tag{6.24}$$

Claim 1. *For any $m \in [1, m_1] \cap \mathbf{N}$, there holds $i(c^m) = 2(m-1) + n - 1$.*

In fact, we have $i(c) = n - 1$. By induction, assume that $i(c^k) = 2(k-1) + n - 1$ for some $k \in [1, m_1 - 1] \cap \mathbf{N}$. By (6.21) and (6.23), for $k+1$ we have

$$i\big(c^{k+1}\big) = 2\sum_{i=1}^{n-1}\left[\frac{(k+1)\theta_i}{2\pi}\right] + n - 1 \in \big\{n - 1 + 2s \bigm| s \in [0, k] \cap \mathbf{N}_0\big\}. \tag{6.25}$$

Note that the conditions of Lemma 6.5 are satisfied by (6.20), (6.21) and (6.23). If $i(c^{k+1}) \leqslant n - 1 + 2(k-1)$, then $i(c^{k+1}) = i(c^r)$ holds for some $r \in [1, k] \cap \mathbf{N}$, which yields a contradiction to Lemma 6.5. Therefore the only possibility is $i(c^{k+1}) = 2k + n - 1$, which proves Claim 1.

Next we consider $i(c^{m_2})$. By (6.22) and (6.24), we have

$$\sum_{i=1}^{n-1}\frac{m_2\theta_i}{2\pi} = \frac{(n+1)(n-1)}{2(n+1)} = \frac{n-1}{2}. \tag{6.26}$$

Noting that $\{\frac{m_2\theta_i}{2\pi}\}$ is an irrational number sequence, by (6.26) it yields

$$\sum_{i=1}^{n-1}\left[\frac{m_2\theta_i}{2\pi}\right] \in \left\{0, 1, \ldots, \frac{n-1}{2} - 1\right\}. \tag{6.27}$$

Hence

$$i\big(c^{m_2}\big) = 2\sum_{i=1}^{n-1}\left[\frac{m_2\theta_i}{2\pi}\right] + n - 1 \in \left\{n - 1 + 2s \,\middle|\, s \in \left[0, \frac{n-1}{2} - 1\right] \cap \mathbf{N}_0\right\}. \tag{6.28}$$

Therefore, by Claim 1 and (6.28), there must exist some $r \in [1, m_1] \cap \mathbf{N}$ such that $i(c^{m_2}) = i(c^r)$, which is also a contradiction to Lemma 6.5.

Case 2. c belongs to NCG-2.

Subcase 2.1. If $i(c) = p$ is even, $N = 1$ by Lemma 5.2 and $i(c^m)$ is even for all m. Hence $\epsilon = 1$ and $k_0^{\epsilon}(c^m) = 1$ by Proposition 2.3. So by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)} = \frac{n+1}{2(n-1)}$, which implies

$$\hat{i}(c) = p - k + \sum_{i=1}^{k}\frac{\theta_i}{\pi} = \frac{2(n-1)}{n+1} < 2. \tag{6.29}$$

Noting that $p - k$ is even, this yields $p \leqslant k$. Because the Morse-type numbers $M_k = 0$ for all $k \in 2\mathbf{N} - 1$, by Corollary 6.4 we have $i(c) = p = n - 1$. Hence $n - 1 = p \leqslant k$. This contradicts to $k \leqslant n - 2r - 2$ by the definition of *NCG-2*.

Subcase 2.2. If $i(c) = p$ is odd, we have $N = 2$ by Lemma 5.2 and $i(c^2)$ is even. Hence $\epsilon = \epsilon(c^2) = -1$ and $k_0^\epsilon(c^2) = 0$. So by Theorem 5.3, it yields $-\frac{1}{\hat{i}(c)} = \frac{n+1}{n-1}$, which yields a contradiction.

Case 3. c belongs to NCG-3.

Subcase 3.1. If $i(c) = p$ is even, $N = 2$ by Lemma 5.2 and

$$i(c^m) \text{ is } \begin{cases} \text{even}, & \text{if } m \text{ is odd}, \\ \text{odd}, & \text{if } m \text{ is even}. \end{cases} \tag{6.30}$$

Therefore by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)} = \frac{n+1}{n-1}$, which implies

$$\hat{i}(c) = p - k + \sum_{i=1}^{k} \frac{\theta_i}{\pi} = \frac{n-1}{n+1} < 1. \tag{6.31}$$

By (6.30) and Proposition 2.1, we have the Morse-type numbers $M_k = 0$ for all $k \in 2\mathbf{N} - 1$. Thus by Corollary 6.4, we have $i(c) = p = n - 1$. Hence, by (6.31), it yields $n - 2 < k$. This is a contradiction to $k \leqslant n - 2r - 2$ by the definition of *NCG-3*.

Subcase 3.2. If $i(c) = p$ is odd, $N = 1$ by Lemma 5.2. So, by Theorem 5.3, it yields $-\frac{1}{\hat{i}(c)} = \frac{n+1}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Case 4. c belongs to NCG-4.

Subcase 4.1. If $i(c) = p$ is even, $N = 2$ by Lemma 5.2 and $i(c^2)$ is odd. Hence $\epsilon = \epsilon(c^2) = -1$ and $k_0^\epsilon(c^2) = 0$. So by Theorem 5.3, it yields $\frac{1}{\hat{i}(c)} = \frac{n+1}{n-1}$. But in this case $\hat{i}(c) = (p-1) + \frac{\theta_1}{\pi}$ is an irrational number. This leads to a contradiction.

Subcase 4.2. If $i(c) = p$ is odd, we have $N = 1$ by Lemma 5.2. Thus by Theorem 5.3, we have $-\frac{1}{\hat{i}(c)} = \frac{n+1}{2(n-1)}$, which yields a contradiction by Lemma 6.1.

Case 5. c belongs to NCG-5.

Subcase 5.1. If $i(c) = p$ is even, $N = 1$ by Lemma 5.2. So by Theorem 5.3, it yields $\frac{1}{p} = \frac{1}{\hat{i}(c)} = \frac{n+1}{2(n-1)}$. Hence we have

$$1 > \frac{n-1}{n+1} = \frac{p}{2} \geqslant 1, \tag{6.32}$$

which is a contradiction.

Subcase 5.2. If $i(c) = p$ is odd, $N = 2$ by Lemma 5.2 and $i(c^2)$ is even. Hence $\epsilon = \epsilon(c^2) = -1$ and $k_0^\epsilon(c^2) = 0$. Thus by Theorem 5.3, there holds $-\frac{1}{\hat{i}(c)} = \frac{n+1}{n-1}$, which yields a contradiction by Lemma 6.1.

So when n is odd, there must exist another prime closed geodesic on the bumpy Finsler n-sphere (S^n, F).

The Steps 1 and 2 complete the proof of Theorem 1.2. □

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