Well-graded families of relations

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Abstract

Any semiorder on a finite set can be reached from any other semiorder on the same set by elementary steps consisting either in the addition or in the removal of a single ordered pair, in such a way that only semiorders are generated at every step, and also that the number of steps equals the distance between the two semiorders. Similar results are also established for other families of relations (partial orders, biorders, interval orders). These combinatorial results are used in another paper to develop a stochastic theory describing the emergence and the evolution of preference relations (Falmagne and Doignon, [7]).

1. Introduction

An interval order on a finite set X is a relation R on X for which there exist two mappings f: X → ℝ and t: X → ℝ+ such that for all x, y in X:

\[ x R y \iff f(x) > f(y) + t(y). \]

Semiorders are obtained when t takes a constant value. These concepts were introduced by Luce [11] (for semiorders; see also Scott and Suppes [16]) and Fishburn [8] (for interval orders). For background, the reader is referred to [9, 14, 15, 17, 18]. Intrinsic characterizations will be recalled below.

Any semiorder can be transformed into some other semiorder on the same finite set by adding or removing some ordered pair. Moreover, for any two semiorders R and R' on the same set, there is a sequence of exactly n such elementary steps transforming R into R', where n = d(R, R') = |RA R'| is measured by the standard distance (for sets)
between $R$ and $R'$. The purpose of this paper is to establish this result, together with similar ones for other families of relations (partial orders, biorders, interval orders). The notation and terminology of Roberts [14] will be adhered to, with a few explicit departures.

**Definition 1.** A collection $\mathcal{F}$ of subsets of a finite set $E$ is well graded when, for any two members $R$ and $S$ of $\mathcal{F}$ at distance $k$, there always exist sets $R = F_0, F_1, \ldots, F_k = S$ in $\mathcal{F}$ such that $d(F_{i-1}, F_i) = 1$, for $i = 1, 2, \ldots, k$.

Definition 1 applies to specific families of relations regarded as sets of pairs. Our main result asserts that the family of all semiorders on a finite set is well graded.

Motivation for investigating the concept of wellgradedness comes from a stochastic theory developed by Falmagne and Doignon [7] (Bogart [1] gives a hint of a related development for a special case making all the partial orders on a set asymptotically equiprobable). The theory purports to explain the emergence and the evolution of preference relations through the random occurrence of quantum tokens of information. A preference relation is regarded as the current state of a subject. The occurrence of a token may modify the current relation by the addition or the removal of a single pair. In an exemplary special case of this theory, the process is a random walk on the class of all semiorders on a set, and the asymptotic probability of any semiorder can be computed. The wellgradedness property of the family of all semiorders on a finite set is critical for establishing these results. The wellgradedness of a family of sets has also been investigated in the context of knowledge spaces, which are combinatorial structures playing a role in the design of efficient algorithms for the assessment of knowledge (see e.g. [6]).

Two byproducts of our investigations are related to Pirlot [13]. First, a simpler, purely combinatorial proof is provided for his result stating that the ‘noses’ and ‘hollows’ of a reduced semiorder determine this semiorder. Second, a straightforward characterization of noses and hollows answers a question raised in the same paper.

While we only consider a few classes of preference relations here, many other families of combinatorial structures could be tested for wellgradedness. The reader interested in this enterprise would benefit from using Proposition 3 below.

2. Background

In this paper, we take $X$ and $Y$ to be two finite, nonempty sets, with $Y$ not necessarily disjoint or distinct from $X$. We abbreviate the pair $(x, y) \in X \times Y$ as $xy$. For any relation $R$ from $X$ to $Y$, that is $R \subseteq X \times Y$, we denote by $\bar{R} = (X \times Y) \setminus R$ the complement of $R$ (w.r.t. $X \times Y$). More generally, the complement of a subset $R$ w.r.t. a set $E$ is $\bar{R} = E \setminus R$. The product (or composition) of relations $R_1, R_2, \ldots, R_k$ is denoted by $R_1 R_2 \cdots R_k$. We will always designate by $I$ the identity relation on the set $X$, that is
Given a relation $T$ on $X$ and a positive integer $n$, we write $T^n$ for the $n$th power of the relation $T$. By convention, we define $T^0 = I$.

Consider the following axioms for a relation $R$ from $X$ to $Y$:

(I) $\forall x \in X: \not xRx$ (irreflexivity);
(B) $\forall x, x' \in X, \forall y, y' \in Y: (xRy \text{ and } x'Ry') \Rightarrow (xRy' \text{ or } x'Ry)$;
(S) $\forall x, y, z, t \in X: (xRy \text{ and } yRz) \Rightarrow (xRt \text{ or } tRz)$.

Suppose first that $X = Y$. The relation $R$ is an interval order on $X$ iff it satisfies Axioms (I) and (B). It is a semiorder iff it satisfies Axioms (I), (B) and (S). Any interval order, thus any semiorder, $R$ on $X$ also satisfies

(T) $\forall x, y, z \in X: (xRy \text{ and } yRz) \Rightarrow xRz$ (transitivity).

In other words, all interval orders are (strict) partial orders, i.e. irreflexive and transitive relations on a set.

The following generalization of interval orders is also of interest. A biorder from $X$ to $Y$ (where $Y$ is not necessarily identical to $X$) is any relation $R \subseteq X \times Y$ satisfying Condition (B) (for background, see e.g. [3], and the references included in that paper; biorders are also called Ferrers relations by other authors). An interval order is nothing else than an irreflexive biorder from a set to itself. There is a (somewhat overlooked) connection between biorders and ‘difference graphs’ in the sense of Hammer et al. [10]. A bipartite graph $G = (V, E)$, with vertex set $V$ partitioned into two stable sets $X$ and $Y$, is a difference graph iff the relation obtained by orienting all edges from $X$ to $Y$ is a biorder.

A compact reformulation of Conditions (I), (T), (B) and (S) in terms of products of relations will be most useful:

(I) $R \cap I = \emptyset$;
(T) $RR \subseteq R$;
(B) $R\bar{R}^{-1}R \subseteq R$;
(S) $RR\bar{R}^{-1} \subseteq R$.

3. Wellgradedness and the fringes

Let $\mathcal{F}$ be a family of subsets of a finite set $E$. A crucial tool in the study of a well-graded family $\mathcal{F}$ lies in the concept of the ‘fringes’ of a set $R$ in $\mathcal{F}$.

**Definition 2.** Let $R$ be any set in $\mathcal{F}$. The inner fringe $R^\delta$ of $R$ (w.r.t. $\mathcal{F}$) consists of all elements $e$ in $R$ such that $R \setminus \{e\}$ is another set in $\mathcal{F}$. Similarly, the outer fringe $R^\triangledown$ of $R$ (w.r.t. $\mathcal{F}$) is formed by all elements $e$ in $\bar{R}$ such that $R \cup \{e\}$ is another set in $\mathcal{F}$.

In particular, the inner (resp. outer) fringe of any semiorder $R$ on a set $X$ is the collection of all pairs $xy \in R$ (resp. $xy \in \bar{R}$) such that $R \setminus \{xy\}$ (resp. $R \cup \{xy\}$) is a semiorder on $X$. 
Proposition 3. The three following conditions on the family $\mathcal{F}$ of subsets are logically equivalent:

1. $\mathcal{F}$ is well graded;
2. any two sets $R$ and $S$ in $\mathcal{F}$ which satisfy $R^c \subseteq S$ and $R^c \subseteq S$ must be equal;
3. any two sets $R$ and $S$ in $\mathcal{F}$ which satisfy $R^c \subseteq S$, $R^c \subseteq S$, $R^c \subseteq R$, $S^c \subseteq R$ must be equal.

Proof. (1) $\Rightarrow$ (2): Take any two sets $R$ and $S$ in $\mathcal{F}$ satisfying $R^c \subseteq S$ and $R^c \subseteq S$. If the distance between $R$ and $S$ is $k > 0$, there are sets $R = F_0, F_1, \ldots, F_k = S$ in $\mathcal{F}$ such that $d(F_{i-1}, F_i) = 1$, for $i = 1, 2, \ldots, k$. Then $R \setminus F_1 = \{e\}$ for some $e \in R^c \setminus S$ or $F_1 \setminus R = \{e'\}$ for some $e' \in R^c \cap S$, in both cases a contradiction.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Let $R$ and $S$ be two sets in $\mathcal{F}$ at distance $k > 0$. Since $R \neq S$, Condition (3) implies that for some $e$ in $E$:

$$e \in (R^c \setminus S) \cup (R^c \cap S) \cup (S^c \setminus R) \cup (S^c \cap R).$$

If $e \in R^c \setminus S$, we set $F_1 = R \setminus \{e\}$. In the other three cases, we set $F_1 = R \cup \{e\}$ or $F_{k-1} = S \setminus \{e\}$ or $F_{k-1} = S^c \cup \{e\}$. We obtain either $d(R, F_1) = 1$ and $d(F_1, S) = k - 1$ (in the first two cases), or $d(S, F_{k-1}) = 1$ and $d(F_{k-1}, R) = k - 1$ (in the last two cases). The result follows by induction. \[\square\]

As a consequence, any set in a well-graded family $\mathcal{F}$ is defined by its fringes in the sense that

$$\forall R, S \in \mathcal{F}: (R^c = S^c \text{ and } S^c \neq R^c) \iff R = S.$$  

However, the above condition does not imply wellgradedness (a counterexample is easily obtained). On the other hand, if a family $\mathcal{L}$ of subsets of $E$ is closed under intersection (in particular $E \in \mathcal{L}$), then it is well graded iff any set in $\mathcal{L}$ is defined by its inner fringe only, in the following sense:

$$\forall R, S \in \mathcal{F}: R^c = S^c \iff R = S$$

(for a justification, see after the next proposition).

Proposition 4. Let $\mathcal{L}$ be a family of subsets of a finite set $E$ which is closed under intersection. Then $\mathcal{L}$ is well graded iff the following condition is fulfilled:

$$(\ast) \text{ for any two sets } K, L \text{ in } \mathcal{M} \text{ with } K \subseteq L \text{ and } |L \setminus K| = m, \text{ there exist sets } K = L_0, L_1, \ldots, L_m = L \text{ in } \mathcal{M} \text{ such that } L_{j-1} \subseteq L_j \text{ and } |L_j \setminus L_{j-1}| = 1 \text{ for } j = 1, 2, \ldots, m.$$  

Proof. That Condition $(\ast)$ is necessary for wellgradedness is easily seen by taking $R = K$ and $S = L$ in the definition of wellgradedness. Conversely, to build a sequence from $R$ to $S$ in $\mathcal{L}$ as in Definition 1, one can concatenate a sequence from $R$ to $R \cap S$ with another sequence from $R \cap S$ to $S$. \[\square\]
In Edelman and Jamison [5] terminology, the family \( \mathcal{L} \) is a convex geometry iff it satisfies Condition (*) and contains the empty set \( \emptyset \). (In case \( \mathcal{L} \) satisfies Condition (*), but \( \emptyset \notin \mathcal{L} \), notice that the family \( \{ L \setminus Z \, | \, L \in \mathcal{L} \} \) is a convex geometry where \( Z = \bigcap \mathcal{L} \).) Their Theorem 2.1 implies that the family \( \mathcal{L} \) is well graded iff each set \( R \) in \( \mathcal{L} \) satisfies \( R = \bigcap \{ L \in \mathcal{L} \, | \, R^f \subseteq L \} \). We immediately exercise these concepts in the case of the family of all partial orders on some finite set \( E \), omitting proofs.

**Proposition 5.** The inner and outer fringes of a partial order \( P \) are described by the two formulas \( P^f = P \setminus PP \) and \( P^c = P \setminus (I \cup PP^{-1} \cup P^{-1}P) \).

Thus, the inner fringe of a partial order is its Hasse diagram while the outer fringe consists of the ‘non-forced pairs’ in the sense of Trotter [18]. As an application of Proposition 4, we get:

**Proposition 6.** The family \( \mathcal{F} \) of all partial orders on a finite set \( X \) is well graded.

A similar result was obtained by Bogart [1] (see also [12]). Note that neither the class of linear or simple orders (when the ground set \( X \) has at least two elements) nor the class of weak orders (when \( X \) has at least three elements) are well graded. For linear orders, an extension of the wellgradedness concept is obtained by considering elementary steps of size two; see e.g. Ovchinnikov [12]. Doignon and Falmagne [4] give an application and a list of several, related references.

We now turn to new examples of well graded families of relations. Proofs will be a bit more intricate, since removing e.g. from an interval order some pair in its Hasse diagram does not necessarily leave an interval order.

### 4. The wellgradedness of biorders

**Proposition 7.** The inner and outer fringes of a biorder \( R \) from the set \( X \) to the set \( Y \) are respectively equal to \( R^f = R \setminus RR^{-1}R \) and \( R^c = R \setminus RR^{-1}R \).

The proof is left to the reader. To establish that the family of all biorders between two finite sets \( X \) and \( Y \) is well graded, we only need to prove Condition (2) in Proposition 3. Some auxiliary results will be useful.

We recall that for any relation \( R \), the products \( RR^{-1} \) and \( R^{-1}R \) are irreflexive. Moreover, if \( R \) is a biorder, then for any positive integer \( n \), the \( n \)-th power \( (RR^{-1})^n \) of the product \( RR^{-1} \) is also irreflexive. We shall use the following fact:

**Proposition 8.** Let \( R \) be a biorder from a set \( X \) to a set \( Y \). Then we have necessarily

\[
R = \bigcup_{k=0}^{\infty} (RR^{-1})^k R = \bigcup_{k=0}^{\infty} (R^f(R^c)^{-1})^k R^f.
\]
Proof. We show $R \subseteq \bigcup_{k=0}^{\infty} (R \bar{R}^{-1})^k R \subseteq \bigcup_{k=0}^{\infty} (R^\| (R^\|)^{-1})^k R^\| \subseteq R$.

The first inclusion is obvious: take $k = 0$. To establish the second inclusion, suppose that $xy \in \bigcup_{k=0}^{\infty} (R \bar{R}^{-1})^k R$. Thus, $xy \in (R \bar{R}^{-1})^k R$ for some $k \geq 0$. Because $(R \bar{R}^{-1})^n$ is irreflexive for any positive integer $n$ and $X$ is finite, we can assume without loss of generality that $k$ is maximal. This implies that each of the $k + 1$ factors $R$ in the formula $(R \bar{R}^{-1})^k R$ can be replaced with $R J$ (while keeping $xy$ in the product). Indeed, if this were not the case, such a factor $R$ could be replaced with $R \bar{R}^{-1} R$, and we would find $xy \in (R \bar{R}^{-1})^{k+1} R$, contradicting the maximality of $k$. The fact that each of the $k$ factors $\bar{R}^{-1}$ in the formula $(R \bar{R}^{-1})^k R$ can be replaced with $(R^\|)^{-1}$ is proved by similar arguments. We conclude that the second inclusion holds.

The third inclusion results from $R^\| \subseteq R$ and $R^\| (R^\|)^{-1} R \subseteq R$, which is implied by the biorder inclusion $R \bar{R}^{-1} R \subseteq R$. □

Proposition 9. Let $R$ and $S$ be two biorders from $X$ to $Y$. Then

$$(R^\| \subseteq S \text{ and } R^\| \subseteq S) \Rightarrow R = S.$$

Proof. We have

$$R = \bigcup_{k=0}^{\infty} (R^\| (R^\|)^{-1})^k R^\| \quad (\text{by Proposition } 8)$$

$$\subseteq \bigcup_{k=0}^{\infty} (SS^{-1})^k S \quad (\text{by hypothesis, } R^\| \subseteq S \text{ and } R^\| \subseteq \bar{S})$$

$$= S \quad (\text{by Proposition } 8).$$

To prove the converse inclusion, notice that $\bar{R}$ and $\bar{S}$ are themselves biorders. Moreover, $(\bar{R})^\| = R^\| \text{ and } (\bar{S})^\| = (\bar{S})$. This means that our hypothesis translates as $(\bar{R})^\| \subseteq \bar{S}$ and $(\bar{R})^\| \subseteq (\bar{S})$. The argument used above gives thus $\bar{R} \subseteq \bar{S}$, that is $S \subseteq R$. □

Corollary 10. The family of all interval orders on a finite set is well graded.

5. The fringes of a semiorder and wellgradedness

The case of the semiorders on a finite set $X$ is settled according to almost the same scheme as that of the biorders.

Proposition 11. The inner and outer fringes of a semiorder $R$ are respectively equal to

$$R^\| = R \setminus (R \bar{R}^{-1} R \cup \bar{R}^{-1} RR \cup RR \bar{R}^{-1})$$
A semiorder $R$ on $X$ is reduced when for all $x, y \in X$,

$$\forall z \in X : xRz \Leftrightarrow yRz \text{ and } zRx \Leftrightarrow zRy \Rightarrow x = y.$$ 

Using terminology due to Pirlot [13], it can be checked that the 'noses' of a reduced semiorder are exactly the pairs in $R^e$, while the 'hollows' of $R$ that are not in $I$ constitute $R^c$ (a pair $xx$ is a hollow when for $y \in X \setminus \{x\}$, there holds $xRy$ or $yRx$). Proposition 11 thus offers a simple solution to a problem raised in the conclusions section of Pirlot's paper: characterizing the noses and the hollows directly in terms of the reduced semiorder. We now provide a simplified, combinatorial approach to the main result of Pirlot [13] stating that a reduced semiorder is determined by its noses and hollows (we avoid the reference to so-called 'minimal representations'). To precisely state this result, let us introduce some new notations.

For two relations $R$ and $S$ on the same set, we define another relation $R \succ S$ as the union of all products of factors each of which is $R$ or $S$ with strictly more factors $R$ than $S$. Similarly, $R \succ = S$ is the union of all products of factors each of which is $R$ or $S$ with at least as many factors $R$ as $S$.

For instance, $R \succ R^{-1}$ is the union of

- $R$, $RR$, $RRR^{-1}$, $RR^{-1}R$, $R^{-1}RR$, $RRR$, $RRR^{-1}$, $RR^{-1}R$, $RR^{-1}RR$, $R^{-1}RRR$, $RRR^{-1}R^{-1}$, etc.

**Proposition 12.** Let $R$ be a semiorder on the finite set $X$. Then

$$R = R \succ R^{-1} = R^e \succ (R^e)^{-1}.$$ 

For completeness, we first establish the following well-known result (see e.g. [15, 2]). From now on, the term 'product' will always mean 'product of factors each of which is $R$ or $R^{-1}$', and we will refer either to the list of these factors or to the set of pairs forming the product. For the proof of the following lemma, notice that Axiom (B) is equivalent to

$$(B') \quad R^{-1}RR^{-1} \subseteq R^{-1}.$$
Similarly, Axiom (S) is equivalent to each of the three forms
\[(S') \quad \bar{R}^{-1}RR \subseteq R;\]
\[(S'') \quad \bar{R}^{-1}\bar{R}^{-1}R \subseteq \bar{R}^{-1};\]
\[(S'''') \quad R\bar{R}^{-1}\bar{R}^{-1} \subseteq \bar{R}^{-1}.\]

**Lemma 13.** The relation \(R \times \bar{R}^{-1}\) is irreflexive.

**Proof of Lemma 13.** It suffices to show that \(C \subseteq \bar{R}^{-1}\) for any product \(C\) having at least as many factors \(R\) as factors \(\bar{R}^{-1}\). This certainly holds if \(C\) consists of just one factor (\(R\) is irreflexive), or two factors (since \(\bar{R}^{-1}R\) and \(R\bar{R}^{-1}\) are irreflexive and \(RR \subseteq R\)). Suppose that \(C\) has \(k\) factors, with \(k \geq 3\). If \(C\) has all of its factors equal to \(R\), then it is again irreflexive (by the transitivity and the irreflexivity of \(R\)). In the remaining cases, the expression of \(C\) must necessarily contain one of the six products:

\[
\begin{align*}
(1) \quad & RR\bar{R}^{-1} \\
(2) \quad & R\bar{R}^{-1}R \\
(3) \quad & \bar{R}^{-1}RR \\
(4) \quad & R\bar{R}^{-1}\bar{R}^{-1} \\
(5) \quad & \bar{R}^{-1}R\bar{R}^{-1} \\
(6) \quad & \bar{R}^{-1}\bar{R}^{-1}R.
\end{align*}
\]

Using Axioms (B) and (S) or their equivalent forms (B') and (S'), (S''), (S'''), the first three products can be replaced by \(R\), and the last three by \(\bar{R}^{-1}\), so that the resulting product \(C'\) contains \(C\), has also at least as many factors \(R\) as factors \(\bar{R}^{-1}\), and has \(k - 2\) factors. The result follows by induction. \(\square\)

**Proof of Proposition 12.** We show
\[
R \subseteq R \Rightarrow \bar{R}^{-1} \subseteq R^\prime \Rightarrow (R^\prime)^{-1} \subseteq R.
\]

The first inclusion is obvious.

Let now \(xy \in R \Rightarrow \bar{R}^{-1}\). Associate to each product \(C\) the difference \(\mu(C)\) between the number of factors \(R\) and the number of factors \(\bar{R}^{-1}\). Notice that if a product \(C\) has a subsequence \(D\) of consecutive factors whose product meets \(I\) (i.e. this product contains a pair \(xx\)), then removing these factors from \(C\) leaves a product \(C'\) with \(\mu(C') > \mu(C)\). This follows from Lemma 13: deleting the factors forming \(D\) increases the value of \(\mu\).

Consider now the collection \(\mathcal{S}\) of all sequences \(x = x_0, x_1, \ldots, x_k = y\) such that \(x_{i-1}(R \cup \bar{R}^{-1})x_i\) for \(i = 1, 2, \ldots, k\). By our assumption, there exists such a sequence. Denote by \(C\) the product corresponding to the sequence written above, i.e. \(C\) is the product of \(k\) factors with the \(i\)th factor being \(R\) when \(x_{i-1}Rx_i\), and \(\bar{R}^{-1}\) otherwise. If \(x_i = x_j\) for some \(i, j\) with \(0 \leq i < j \leq k\), then removing \(x_{i+1}, x_{i+2}, \ldots, x_j\) leaves another sequence in \(\mathcal{S}\) whose corresponding product \(C'\) satisfies \(\mu(C') > \mu(C)\) (by the above paragraph). As the sequences in \(\mathcal{S}\) having no repeated element are in finite number, we infer the existence of some sequence in \(\mathcal{S}\) that maximizes \(\mu(D)\), where \(D\) is the corresponding product. Then among all sequences maximizing \(\mu\), select one whose corresponding product \(M\) has maximum number of factors \(R\). By the assumption \(xy \in R \Rightarrow \bar{R}^{-1}\), the product \(M\) has strictly more factors \(R\) than \(\bar{R}^{-1}\). We then show that any \(R\) in the expression of \(M\) can be replaced with \(R^\prime\). If this were not true for
some factor $R$, we could by Proposition 11 replace this factor by $R\bar{R}^{-1}R$, $\bar{R}^{-1}RR$ or $RR\bar{R}^{-1}$. In each of these three cases, we would obtain a product $N$ still containing $xy$, satisfying $\mu(N) = \mu(M)$, and having more factors $R$ than $M$ has, a contradiction. In a similar manner, it can be proved that each factor $\bar{R}^{-1}$ in the expression of $M$ can be replaced with $(R^c)^{-1}$. Thus $R \gg \bar{R}^{-1} \subseteq R^e \gg (R^c)^{-1}$.

Finally, if $xy \in (R^e \gg (R^c)^{-1}) \setminus R$, we have $yx \in \bar{R}^{-1}$. As $R^e \subseteq R$ and $(R^c)^{-1} \subseteq \bar{R}^{-1}$, it follows that the pair $xx$ belongs to $R \times \bar{R}^{-1}$, in contradiction with Lemma 13.

Lemma 14. If $R$ and $S$ are semiorders on $X$ with $R^e \subseteq S$, $R^c \subseteq S^c$, $S^e \subseteq R$ and $S^c \subseteq \bar{R}$, then $R = S$.

Proof. Use Proposition 12.

Proposition 15. The family of all semiorders on $X$ is well graded.

Proof. Use Condition (3) from Proposition 3 together with Lemma 14.

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