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On Variational Principles for Linear Initial Value Problems

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Euler–Lagrange and Euler–Hamilton variational principles are presented for a class of linear initial value problems.

1. INTRODUCTION

Initial value problems arise in diffusion, classical mechanics, and other areas of mathematical physics and it is of some interest to provide a variational formulation of them. Early attempts, represented by Hamilton's principle and other action principles, are inadequate because they actually refer to *boundary* value problems, in which comparison curves must agree with the critical curve at *both* ends of the time interval. The unsatisfactory nature of these results arises basically because the function space is that of standard calculus of variations with inner product

$$\langle \phi, \psi \rangle = \int_0^T \phi(t)\, \psi(t)\, dt. \tag{1.1}$$

For such spaces, boundary terms at $t = 0$ and $t = T$ enter naturally, and furthermore the operator $\partial/\partial t$ characteristic of diffusion is not symmetric, so that terms such as $\partial\psi/\partial t$ are not potential.

One way around these difficulties has been explored recently by Tonti [1] using the idea of a convolution inner product

$$\langle \phi, \psi \rangle_c = \int_0^T \phi(t)\, \psi(T - t)\, dt. \tag{1.2}$$

With this bilinear form the operator $\partial/\partial t$ is symmetric and it is possible to give a variational formulation of initial value problems. Tonti's results [1] refer to the Euler–Lagrange formulation of certain problems and they involve essential conditions or constraints on the admissible functions. In this paper we show how these constraints can be removed so that the initial conditions arise as *natural* conditions from the variational theory. In addition, we provide a canonical Euler–Hamilton formulation of the results.

2. A Class of Initial Value Problems

We consider a class of initial value problems described by

$$A\ddot{\mathbf{q}} + B\dot{\mathbf{q}} + C\mathbf{q} = \mathbf{f}, \qquad t > 0 \tag{2.1}$$

with

$$\mathbf{q}(0) = \mathbf{a}, \tag{2.2}$$

and

$$\dot{\mathbf{q}}(0) = \mathbf{b}. \tag{2.3}$$

Here the vectors have n components, $\mathbf{f}$, $\mathbf{a}$, and $\mathbf{b}$ being given, and A, B, and C denote given symmetric $n \times n$ matrices. The solution $\mathbf{q}^*$ of (2.1) to (2.3) will be regarded as an element in the convolution space with inner product

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_c = \int_0^T \sum_{i=1}^n \phi_i(t)\, \psi_i(T - t)\, dt. \tag{2.4}$$

It is also convenient to define a reduced inner product

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_0 = \sum_{i=1}^n \phi_i(T)\, \psi_i(0). \tag{2.5}$$

With these definitions we find that

$$\langle \boldsymbol{\phi}, \dot{\boldsymbol{\psi}} \rangle_c = \langle \dot{\boldsymbol{\phi}}, \boldsymbol{\psi} \rangle_c - \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_0 + \langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle_0\,, \tag{2.6}$$

and so the operator

$$L = d/dt \tag{2.7}$$

has formal adjoint

$$L^* = d/dt \tag{2.8}$$

in the convolution space.

3. Variational Principle

To obtain a variational formulation of the initial value problem in (2.1)–(2.3), we seek a potential $J(\mathbf{q})$ such that $J(\mathbf{q})$ is stationary at $\mathbf{q}^*$, that is

$$\delta J(\mathbf{q}^*) = 0. \tag{3.1}$$

A suitable potential $J(\mathbf{q})$ will contain two parts, one corresponding to the differential equation (2.1) and the other corresponding to the initial conditions

(2.2) and (2.3). Taking these two aspects into account we find the following result.

THEOREM 3.1. *The functional*

$$J(\mathbf{q}) = \tfrac{1}{2}\langle \dot{\mathbf{q}}, A\dot{\mathbf{q}}\rangle_c + \tfrac{1}{2}\langle \mathbf{q}, B\dot{\mathbf{q}}\rangle_c + \tfrac{1}{2}\langle \mathbf{q}, C\mathbf{q}\rangle_c - \langle \mathbf{f}, \mathbf{q}\rangle_c$$
$$+ \langle \dot{\mathbf{q}}, A(\mathbf{q} - \mathbf{a})\rangle_0 - \langle \mathbf{q}, A\mathbf{b}\rangle_0 + \tfrac{1}{2}\langle \mathbf{q}, B\mathbf{q}\rangle_0 - \langle \mathbf{q}, B\mathbf{a}\rangle_0 \quad (3.2)$$

is stationary at the solution $\mathbf{q}^*$ *of* (2.1)–(2.3).

Proof. The first variation of the functional (3.2) is

$$\delta J = \left[\frac{d}{d\epsilon} J(\mathbf{q} + \epsilon\boldsymbol{\xi})\right]_{\epsilon\to 0}$$
$$= \langle \dot{\boldsymbol{\xi}}, A\dot{\mathbf{q}}\rangle_c + \tfrac{1}{2}\langle \mathbf{q}, B\dot{\boldsymbol{\xi}}\rangle_c + \tfrac{1}{2}\langle \boldsymbol{\xi}, B\dot{\mathbf{q}}\rangle_c + \langle \boldsymbol{\xi}, C\mathbf{q}\rangle_c - \langle \boldsymbol{\xi}, \mathbf{f}\rangle_c$$
$$+ \langle \dot{\boldsymbol{\xi}}, A(\mathbf{q} - \mathbf{a})\rangle_0 + \langle \dot{\mathbf{q}}, A\boldsymbol{\xi}\rangle_0 - \langle \boldsymbol{\xi}, A\mathbf{b}\rangle_0$$
$$+ \tfrac{1}{2}\langle \boldsymbol{\xi}, B\mathbf{q}\rangle_0 + \tfrac{1}{2}\langle \mathbf{q}, B\boldsymbol{\xi}\rangle_0 - \langle \boldsymbol{\xi}, B\mathbf{a}\rangle_0, \qquad \text{at} \quad \mathbf{q} = \mathbf{q}^*.$$

Integrating the first and second terms by parts using (2.6), we find that

$$\delta J = \langle \boldsymbol{\xi}, A\ddot{\mathbf{q}} + B\dot{\mathbf{q}} + C\mathbf{q} - \mathbf{f}\rangle_c + \langle \dot{\boldsymbol{\xi}}, A(\mathbf{q} - \mathbf{a})\rangle_0$$
$$+ \langle \boldsymbol{\xi}, A(\dot{\mathbf{q}} - \mathbf{b})\rangle_0 + \langle \boldsymbol{\xi}, B(\mathbf{q} - \mathbf{a})\rangle_0, \qquad \text{at} \quad \mathbf{q} = \mathbf{q}^*. \quad (3.3)$$

Hence, for arbitrary variations $\boldsymbol{\xi}$, $\delta J = 0$ implies that $\mathbf{q}^*$ satisfies equations (2.1)–(2.3). This proves the theorem.

We note that in this variational principle both of the initial conditions are natural conditions, and so *no* essential conditions need be imposed on the admissible functions $\mathbf{q}$.

4. CANONICAL FORMULATION

The problem in (2.1)–(2.3) will now be formulated in terms of canonical equations. In general these equations take the form (see [2])

$$L\mathbf{q} = \partial W/\partial \mathbf{u}, \quad (4.1)$$

$$L^*\mathbf{u} = \partial W/\partial \mathbf{q}, \quad (4.2)$$

where $\mathbf{u}$ and $\mathbf{q}$ are canonical variables, L^* is the formal adjoint of L, and $W = W(\mathbf{u}, \mathbf{q})$ is the Hamilton functional.

For L and L^* we choose

$$L = d/dt, \qquad L^* = d/dt, \tag{4.3}$$

(see Eq. (2.6)), and for $\mathbf{u}$ we take

$$\mathbf{u} = A\dot{\mathbf{q}} + \tfrac{1}{2}B\mathbf{q}. \tag{4.4}$$

Assuming that A^{-1} exists, we can rewrite (4.4) as

$$\dot{\mathbf{q}} = A^{-1}\mathbf{u} - \tfrac{1}{2}A^{-1}B\mathbf{q}, \tag{4.5}$$

and then (2.1) may be written as

$$\dot{\mathbf{u}} = \mathbf{f} - C\mathbf{q} + \tfrac{1}{4}BA^{-1}B\mathbf{q} - \tfrac{1}{2}BA^{-1}\mathbf{u}. \tag{4.6}$$

These are canonical equations of the form

$$\dot{\mathbf{q}} = \partial W/\partial\mathbf{u}, \tag{4.7}$$

$$\dot{\mathbf{u}} = \partial W/\partial\mathbf{q}, \tag{4.8}$$

where a suitable Hamiltonian W is

$$W(\mathbf{u}, \mathbf{q}) = \tfrac{1}{2}\langle\mathbf{u}, A^{-1}\mathbf{u}\rangle_c - \langle\mathbf{q}, \tfrac{1}{2}BA^{-1}\mathbf{u}\rangle_c - \tfrac{1}{2}\langle\mathbf{q}, C\mathbf{q}\rangle_c + \tfrac{1}{2}\langle\mathbf{q}, \tfrac{1}{4}BA^{-1}B\mathbf{q}\rangle_c + \langle\mathbf{f}, \mathbf{q}\rangle_c. \tag{4.9}$$

In terms of the canonical variables, the initial conditions (2.2) and (2.3) become

$$\mathbf{q}(0) = \mathbf{a}, \tag{4.10}$$

$$\mathbf{u}(0) = A\mathbf{b} + \tfrac{1}{2}B\mathbf{a}. \tag{4.11}$$

It is readily checked that the second derivatives of W in (4.9) have the required symmetry properties, namely,

$$\left(\frac{\partial^2 W}{\partial\mathbf{u}^2}\right)^* = \frac{\partial^2 W}{\partial\mathbf{u}^2}, \quad \left(\frac{\partial^2 W}{\partial\mathbf{u}\,\partial\mathbf{q}}\right)^* = \frac{\partial^2 W}{\partial\mathbf{q}\,\partial\mathbf{u}}, \quad \left(\frac{\partial^2 W}{\partial\mathbf{q}^2}\right)^* = \frac{\partial^2 W}{\partial\mathbf{q}^2}. \tag{4.12}$$

Equations (4.7) to (4.11) provide a canonical formulation of the problem in Section 2. We now obtain the associated variational principles.

Introduce the action functional

$$I(\mathbf{u}, \mathbf{q}) = \langle\mathbf{u}, \dot{\mathbf{q}}\rangle_c - W(\mathbf{u}, \mathbf{q}) + \langle\mathbf{u}, \mathbf{q} - \mathbf{a}\rangle_0 - \langle\mathbf{q}, A\mathbf{b} + \tfrac{1}{2}B\mathbf{a}\rangle_0, \tag{4.13}$$

$$= \langle\dot{\mathbf{u}}, \mathbf{q}\rangle_c - W(\mathbf{u}, \mathbf{q}) - \langle\mathbf{u}, \mathbf{a}\rangle_0 + \langle\mathbf{q}, \mathbf{u} - (A\mathbf{b} + \tfrac{1}{2}B\mathbf{a})\rangle_0. \tag{4.14}$$

From these expressions we find that

$$\frac{\partial I}{\partial \mathbf{u}} = \left(\dot{\mathbf{q}} - \frac{\partial W}{\partial \mathbf{u}}\right)_{t>0} + \{\mathbf{q}(0) - \mathbf{a}\}, \tag{4.15}$$

and

$$\frac{\partial I}{\partial \mathbf{q}} = \left(\dot{\mathbf{u}} - \frac{\partial W}{\partial \mathbf{q}}\right)_{t>0} + \{\mathbf{u}(0) - (A\mathbf{b} + \tfrac{1}{2}B\mathbf{a})\}. \tag{4.16}$$

The action $I(\mathbf{u}, \mathbf{q})$ is stationary where $\partial I/\partial \mathbf{u} = 0$, $\partial I/\partial \mathbf{q} = 0$, and so we have the variational principle

THEOREM 4.1. *The functional* $I(\mathbf{u}, \mathbf{q})$ *in* (4.13), (4.14), *is stationary at the solution* $(\mathbf{u}^*, \mathbf{q}^*)$ *of the initial value problem* (4.7) *to* (4.11). *In this theorem the conditions* (4.10) *and* (4.11) *are natural conditions.*

The functional $I(\mathbf{u}, \mathbf{q})$ can be used to obtain two additional variational principles.

Using (4.13) we define the functional $K(\mathbf{q})$ by

$$K(\mathbf{q}) = I(\mathbf{u}, \mathbf{q}), \tag{4.17}$$

where $\mathbf{u}$ is such that the first canonical equation (4.7) holds identically, that is

$$\mathbf{u} = A\dot{\mathbf{q}} + \tfrac{1}{2}B\mathbf{q}. \tag{4.18}$$

Putting (4.18) in (4.17) we find that

$$K(\mathbf{q}) = J(\mathbf{q}), \tag{4.19}$$

where $J(\mathbf{q})$ is the functional defined in (3.2). Thus we have

THEOREM 4.2. *The functional* $K(\mathbf{q})$ *in* (4.17) *is stationary at* $\mathbf{q}^*$.

In this theorem we have recovered the Euler–Lagrange principle of Theorem 3.1. Secondly, using (4.14) we can define a functional $G(\mathbf{u})$ by

$$G(\mathbf{u}) = I(\mathbf{u}, \mathbf{q}), \tag{4.20}$$

where $\mathbf{q}$ is such that the second canonical equation (4.8) holds identically, that is

$$\mathbf{q} = (C - \tfrac{1}{4}BA^{-1}B)^{-1}(\mathbf{f} - \dot{\mathbf{u}} - \tfrac{1}{2}BA^{-1}\mathbf{u}). \tag{4.21}$$

Putting (4.21) in (4.20) we find that

$$\begin{aligned} G(\mathbf{u}) = &-\tfrac{1}{2}\langle \mathbf{u}, A^{-1}\mathbf{u}\rangle_c - \tfrac{1}{2}\langle \mathbf{v}, D^{-1}\mathbf{v}\rangle_c - \langle \mathbf{u}, \mathbf{a}\rangle_0 \\ &+ \langle D^{-1}\mathbf{v}, \mathbf{u} - (A\mathbf{b} + \tfrac{1}{2}B\mathbf{a})\rangle_0 , \end{aligned} \tag{4.22}$$

where

$$D = C - \tfrac{1}{4}BA^{-1}B, \qquad \mathbf{v} = \mathbf{f} - \dot{\mathbf{u}} - \tfrac{1}{2}BA^{-1}\mathbf{u}. \tag{4.23}$$

From its construction we have

THEOREM 4.3. *The functional* $G(\mathbf{u})$ *in* (4.22) *is stationary at* $\mathbf{u}^*$.

This completes our canonical description of the class of problems in Section 2.

The ideas employed here can also be used to obtain variational principles for problems in diffusion theory and other areas of mathematical physics.

REFERENCES

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