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A new multilinear insight on Littlewood's 4/3-inequality

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Abstract

We unify Littlewood's classical 4/3-inequality (a forerunner of Grothendieck's inequality) together with its m-linear extension due to Bohnenblust and Hille (which originally settled Bohr's absolute convergence problem for Dirichlet series) with a scale of inequalties of Bennett and Carl in ℓ_p-spaces (which are of fundamental importance in the theory of eigenvalue distribution of power compact operators). As an application we give estimates for the monomial coefficients of homogeneous ℓ_p-valued polynomials on c_0.

Keywords: Multilinear theory; Local Banach space theory; Summing operators

1. Introduction

In 1930 Littlewood settled a long-standing question of Daniell. Motivated through his analysis of Daniell's problem Littlewood in [30, Theorem 1] proved an inequality nowadays sometimes

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cited as Littlewood's 4/3-inequality: for every bilinear form $A : c_0 \times c_0 \to \mathbb{C}$ (c_0 the Banach space of all scalar zero sequences) the following holds:

$$\left(\sum_{i,j=1}^{\infty} \left|A(e_i, e_j)\right|^{4/3}\right)^{3/4} \leqslant \sqrt{2}\|A\|, \tag{1}$$

and the exponent 4/3 is optimal; here as usual the norm of A is given by

$$\|A\| = \sup\left\{\left|A(x_1, x_2)\right|: \|x_i\|_\infty \leqslant 1\right\}.$$

In 1931 Bohnenblust and Hille proved [4, Theorem I] that for each $m \in \mathbb{N}$ and every m-linear mapping $A : c_0 \times \cdots \times c_0 \to \mathbb{C}$ the following holds:

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \left|A(e_{i_1}, \dots, e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leqslant 2^{\frac{m-1}{2}} \|A\|, \tag{2}$$

and showed that the exponent $\frac{2m}{m+1}$ is optimal. Using this they answered Harald Bohr's so called absolute convergence problem for Dirichlet series which had been open for over 15 years (see below). Inequality (2) was overlooked for long time and re-discovered by Davie and Kaijser in [11] and [25] (in fact with the constant $2^{\frac{m-1}{2}}$ given in (2) which is better than the original one from [4]).

More recently, in [6, Theorem 3.2] the following vector-valued variant was proved. Fix some $1 \leqslant p \leqslant \infty$ and m. Then the optimal exponent $1 \leqslant r \leqslant \infty$ for which there is a constant $C_p > 0$ satisfying

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \left\|A(e_{i_1}, \dots, e_{i_m})\right\|_p^r\right)^{1/r} \leqslant C_p^m \|A\|, \tag{3}$$

for every m-linear mapping $A : c_0 \times \cdots \times c_0 \to \ell_p$, is given by

$$r = \begin{cases} 2 & \text{if } p \leqslant 2, \\ p & \text{if } p \geqslant 2. \end{cases}$$

Note that the case $m = 1$ goes back to Orlicz [33]. In the beginning of the 1970s Bennett [2] and Carl [9] independently proved the following. Define for given $1 \leqslant p \leqslant q \leqslant \infty$ the number

$$r = \begin{cases} \frac{2}{1+2(\frac{1}{p}-\max\{\frac{1}{q},\frac{1}{2}\})} & \text{if } p \leqslant 2, \\ p & \text{if } p \geqslant 2. \end{cases}$$

Then there is a constant $C_{p,q} > 0$ such that for each linear operator $A : c_0 \to \ell_p$ we have

$$\left(\sum_{i=1}^{\infty} \left\|A(e_i)\right\|_q^r\right)^{1/r} \leqslant C_{p,q} \|A\|. \tag{4}$$

Again the given exponent r cannot be improved. An easy calculation shows that the case $p = 1$ and $q = 4/3$ is again nothing else than Littlewood's 4/3-inequality (1). The so called Bennett–Carl inequalities (4) are crucial within the theory of summing operators (today at the heart of modern Banach space theory, see [20]), and have deep applications within the theory of eigenvalue distribution of power compact operators in Banach spaces (see e.g. [26,35]).

Clearly, Eqs. (3) and (4) have the same flavour as Littlewoods's 4/3 inequality (1) and its multilinear extension (2) of Bohnenblust and Hille. But still, there are some obvious differences between them: in (1) and its improvement (2) *scalar-valued bilinear and multilinear mappings* are considered, whereas in (3) there are *vector-valued multilinear operators* and in (4) *linear operators*. Also, while in (2) the optimal exponent highly *depends on the degree* m, in (3) the optimal exponent *is valid for every* m.

Our aim in this article is to give a unified vision of all these inequalities that allows to look at each one of them as a particular case of a general situation. The following theorem is our main result.

Theorem 1. *Given* $m \in \mathbb{N}$ *and* $1 \leqslant p \leqslant q \leqslant \infty$, *define*

$$\rho = \begin{cases} \frac{2m}{m+2(\frac{1}{p}-\max\{\frac{1}{q},\frac{1}{2}\})} & \text{if } p \leqslant 2, \\ p & \text{if } p \geqslant 2. \end{cases}$$

Then there exists a constant $C > 0$ *such that for every* m*-linear mapping* $A : c_0 \times \cdots \times c_0 \to \ell_p$ *the following holds*:

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left\| A(e_{i_1}, \ldots, e_{i_m}) \right\|_q^{\rho} \right)^{1/\rho} \leqslant C \|A\|. \tag{5}$$

Moreover, ρ *is best possible.*

This inequality covers all four previous inequalities: (2) follows from the case $p = 1$ and $q = 2$ (consider in (5) only m-linear mappings which have their range in the span of the first basis vector e_1), (3) is the case $p = q$ in (5) and (4) is the case $m = 1$. For related results see [1,6,8,10,34,36,42].

Although we believe that our main results from Theorems 1 and 4 are of independent interest we here want to sketch the application which originally motivated these results, and which will be presented in the forthcoming paper [15]. Bohr showed in [5] that the width of the strip in $\mathbb{C}$ on which a Dirichlet series $\sum a_n/n^s$, $s \in \mathbb{C}$, converges uniformly but not absolutely, is at most 1/2, and Bohnenblust and Hille in [4] were able to prove that this bound is even optimal. Given a Banach space Y, denote by $T(Y)$ the supremum of all such width taken over all Dirichlet series $\sum a_n/n^s$ with coefficients a_n in Y. The main result in [19] proves that $T(Y) = 1 - 1/\mathrm{Cot}(Y)$, where $\mathrm{Cot}(Y)$ denotes the optimal cotype of Y. The inequality in (3) (and a more general version in the setting of spaces with cotype) turned out to be crucial for the proof of this result.

Similarly, let $T_m(Y)$ be the supremum of the width of all strips of uniform but not absolute convergence, the supremum now taken with respect to all m-homogeneous Dirichlet polynomials, i.e. series $\sum a_n/n^s$ where the only coefficients $a_n \in Y$ different from 0 are those with indices $n = p^\alpha$ satisfying $|\alpha| = m$ (where p is the sequence of primes and α is a multi-index). The results in [19] as a byproduct show that $T(Y)$ always equals $T_m(Y)$ whenever Y is infinite-dimensional. For finite-dimensional Y, however, the situation is drastically different—inequality (18) (a polynomial version of (2)) is used in [4] to prove that $T_m(\mathbb{C}) = \frac{m-1}{2m}$ which as a consequence even allows to prove that $T_m(Y) = \frac{m-1}{2m}$ whenever $\dim Y < \infty$. The question then is

> Is it possible to give a unified vision of the formulas $T_m(Y) = 1 - 1/\mathrm{Cot}(Y)$ (Y infinite-dimensional) and $T_m(Y) = \frac{m-1}{2m}$ (Y finite-dimensional)? Or more vaguely, why do the m-homogeneous Dirichlet polynomials in infinite-dimensional case disappear?

In the same way as (2) and (3) play an important role in [4,19], so also using Theorem 1 we are able to give a complete answer in the ℓ_p-case in [15]: For $1 \leqslant p \leqslant q \leqslant \infty$ we consider m-homogeneous Dirichlet polynomials $\sum a_n/n^s$ in ℓ_q whose coefficients $a_n \in \ell_p$ and for each one of them the difference between the abscissa of uniform convergence in ℓ_p and that of absolute convergence in ℓ_q. We then define $T_m(p,q)$ to be the maximal width of these strips in $\mathbb{C}$. This number somehow measures how much the summability of a homogeneous Dirichlet polynomial in ℓ_p improves when we move from ℓ_p to a bigger ℓ_q. Then

$$T_m(p,q) = \begin{cases} \frac{m-2(1/p-\max\{1/q,1/2\})}{2m} & \text{if } 1 \leqslant p \leqslant 2, \\ \frac{1}{p'} & \text{if } 2 \leqslant p. \end{cases}$$

Let us give a brief review of the contain of this article. After some preliminaries given in Section 1, Section 2 is devoted to the proof of Theorem 1. The main step is Lemma 3, a result given in terms of summing operators and injective tensor product. The optimality of the given exponent ρ in (5) follows from random techniques. The main result in Section 3 is Theorem 4, a 'symmetrization' of Theorem 1 which replaces the m-linear mappings $A : c_0 \times \cdots \times c_0 \to \ell_p$ by m-homogeneous polynomials $P : c_0 \to X$, and the matrix entries of A by the coefficients $c_\alpha(P)$ in the monomial series expansion $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(P) z^\alpha$ of P. For scalar-valued m-homogeneous polynomials $P : c_0 \to \mathbb{C}$, the case $p = 1$ and $q = 2$ recovers the important result [4, Section 2] due to Bohnenblust and Hille (see also [37, Theorem III-1]). Finally, in Section 4 we try to integrate our study into the theory of summing operators in a more systematic way. We introduce the notion of $(r,1)$-summing operators $v : X \to Y$ of order m and Bohnenblust–Hille indices. This allows us to reformulate some deep facts from local Banach space theory into interesting new Littlewood–Bohnenblust–Hille type inequalities.

2. Preliminaries

Standard notation and notions from Banach space theory are used, as presented e.g. in [28,29]. All Banach spaces X are over the real or complex field $\mathbb{K}$, their duals are denoted by X^* and their open unit balls by B_X. Given a Banach space X, the space of all sequences (x_n) in X such that $\sum_n \|x_n\|^p < \infty$ is denoted $\ell_p(X)$. As usual $\ell_p(\mathbb{K}) = \ell_p$ and $\ell_p^n = (\mathbb{K}^n, \|\ \|_p)$. By a Banach sequence space we will mean a Banach space X of scalar sequences such that $\ell_1 \subseteq X \subseteq \ell_\infty$ satisfying that if $x \in \mathbb{K}^{\mathbb{N}}$ and $y \in X$ are such that $|x| \leqslant |y|$ then $x \in X$ and $\|x\| \leqslant \|y\|$. A Banach sequence space X is symmetric whenever a given scalar sequence x belongs to X if and only if its decreasing rearrangement $x^\star$ does, and in this case they have the same norm. The Banach space of all (bounded) linear operators between two Banach spaces X and Y is denoted by $\mathcal{L}(X,Y)$, and the Banach space of all (bounded) m-linear mappings from $X \times \cdots \times X$ to Y by $\mathcal{L}(^m X, Y)$.

We refer to [12] for all needed background on the metric theory of tensor products, and [21,22] for whatever is used on polynomials and symmetric tensor products. Let us recall that the injective norm $\|\cdot\|_\varepsilon$ of an element $z=\sum_k x_k\otimes y_k$ (a fixed finite representation) in the tensor product $X\otimes Y$ of two Banach spaces is given by

$$\|z\|_\varepsilon=\sup_{\substack{\|x^*\|_{X^*}\leqslant 1\\ \|y^*\|_{Y^*}\leqslant 1}}\left|\sum_k x^*(x_k)y^*(y_k)\right|.$$

As usual, we write $X\otimes_\varepsilon Y$ for the injective tensor product of X and Y, and $\bigotimes_\varepsilon^m X:=X\otimes_\varepsilon\cdots\otimes_\varepsilon X$ for the mth full injective tensor product. A function $P:X\to Y$ between two Banach spaces is said to be an m-homogeneous polynomial if there is an m-linear mapping $\varphi:\prod_{k=1}^m X\to Y$ such that $P(x)=\varphi(x,\dots,x)$ for all $x\in X$. We denote by $\mathcal{P}(^mX,Y)$ the vector space of all m-homogeneous continuous polynomials $P:X\to Y$ which together with the norm $\|P\|:=\sup_{\|x\|_X\leqslant 1}\|P(x)\|_Y$ forms a Banach space. It is well known [21, Proposition 1.8] that the norms of an m-homogeneous polynomial and of the associated m-linear mapping are related in the following way:

$$\|P\|\leqslant\|\varphi\|\leqslant c(m,X)\|P\|, \tag{6}$$

where $c(m,X)$ denotes the polarization constant of E [21, Defintion 1.40] that satisfies $1\leqslant c(m,X)\leqslant\frac{m^m}{m!}$.

It will be often more convenient to think in terms of symmetric tensor products instead of spaces of polynomials. We write $\bigotimes_{\varepsilon_s}^{m,s}X$ for the mth symmetric injective tensor product. Let us recall that $\bigotimes^{m,s}X$ can be realized as the range of the symmetrization operator

$$\sigma_m:\bigotimes^m X\to\bigotimes^m X,\quad \sigma_m(\otimes y_k):=\frac{1}{m!}\sum_{\pi\in\Pi_m}\otimes y_{\pi(k)}, \tag{7}$$

where Π_m stands for the group of all permutations of $\{1,\dots,m\}$. The following isometric equality will be frequently used: for every finite-dimensional Banach space X and any Banach space Y

$$\bigotimes_\varepsilon^m X^*\otimes_\varepsilon Y=\mathcal{L}\big(^mX,Y\big),\quad \big(x_1^*\otimes\cdots\otimes x_m^*\big)\otimes y\rightsquigarrow\Big[z\rightsquigarrow\prod_k x_k^*(x)y\Big], \tag{8}$$

$$\bigotimes_{\varepsilon_s}^{m,s} X^*\otimes_\varepsilon Y=\mathcal{P}\big(^mX,Y\big),\quad \big(x^*\otimes\cdots\otimes x^*\big)\otimes y\rightsquigarrow\big[x\rightsquigarrow x^*(x)^m y\big]. \tag{9}$$

For all needed information on the theory of summing operators as well as local Banach space theory see [20,40]. Given $1\leqslant p,q\leqslant\infty$ and some operator $v\in\mathcal{L}(X,Y)$, the infimum over all $c>0$ such that for each choice of finitely many $x_1,\dots,x_n\in X$ we have

$$\left(\sum_{i=1}^n\|vx_i\|^p\right)^{1/p}\leqslant c\sup_{\|x^*\|\leqslant 1}\left(\sum_{i=1}^n\big|x^*(x_i)\big|^q\right)^{1/q},$$

is called the (p, q)-summing norm of v and denoted $\pi_{p,q}(v)$. Then v is said to be (p, q)-summing whenever $\pi_{p,q}(v) < \infty$. Operators that are $(1, 1)$-summing are usually called summing. Standard arguments easily show that an operator is $(r, 1)$-summing if and only if there is a constant $C > 0$ such that for every linear operator $A : c_0 \to X$ we have

$$\left(\sum_{i=1}^{\infty} \left\| vA(e_i)\right\|^r\right)^{1/r} \leqslant C\|A\|, \tag{10}$$

and $\pi_{r,1}(v)$ here is the best constant. We will frequently use the following simple reformulation of the summing norm in terms of tensor products, an immediate consequence of (8): for each operator $v \in \mathcal{L}(X, Y)$

$$\pi_{p,1}(v) = \left\| \mathrm{id} \otimes v : \ell_1 \otimes_\varepsilon X \to \ell_p(Y)\right\| = \sup_n \left\| \mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon X \to \ell_p^n(Y)\right\|. \tag{11}$$

Finally, we recall another well established notion from local Banach space theory [20, Chapter 11]. Let $2 \leqslant p < \infty$. A Banach space X is said to have cotype p whenever there is some constant $C > 0$ such that for each choice of finitely many vectors $x_1, \dots, x_n \in X$ we have

$$\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \leqslant C\left(\int \left\|\sum_{i=1}^{n} \varepsilon_i(\omega)x_i\right\|^2 d\omega\right)^{1/2},$$

where ε_i are independent Bernoulli random variables; as usual, the best such C is denoted by $\mathrm{C}_p(X)$. It is well known that ℓ_p has cotype $\max\{p, 2\}$.

3. Proof of the main result

The proof is done by induction on the degree m, and clearly the case $m = 1$ is valid by the Bennett–Carl inequality (4). We follow ideas of Kaijser's re-proof [25] of the Bohnenblust–Hille result, and give the proof in terms of summing norms and tensor products.

Note first that by (10) the exponent r given in (4) is the optimal number for which the identity $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ is $(r, 1)$-summing, and in fact this was the original context in which this result was stated.

We start with two lemmas of independent interest.

Lemma 2. *Let $v : X \to Y$ be an $(r, 1)$-summing operator for some $1 \leqslant r < \infty$ where Y is a cotype 2 space. Then*

$$\pi_{r,1}\big(\mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \ell_2^{n^m}(Y)\big) \leqslant \big(\sqrt{2}\,\mathrm{C}_2(Y)\big)^m \pi_{r,1}(v).$$

Proof. We proceed by induction. We consider first the case $m = 1$. Take $x_1, \dots, x_N \in \ell_1^n \otimes_\varepsilon X$, each $x_k = \sum_{i=1}^n e_i \otimes x_k(i)$ where $x_k(i) \in X$. We want

$$\left(\sum_{k=1}^{N} \|vx_k\|^r_{\ell_2^n(Y)}\right)^{1/r} \leqslant \sqrt{2}\,\mathrm{C}_2(Y) \sup_{\gamma \in B_{(\ell_1^n \otimes_\varepsilon X)^*}} \sum_{k=1}^{N} \big|\gamma(x_k)\big|, \tag{12}$$

and start dealing with the left-hand side of this inequality. Applying first that Y has cotype 2, second the well-known fact that $(\int \|\sum_i x_i\varepsilon_i(\omega)\|^2\,d\omega)^{1/2} \leqslant \sqrt{2}\int\|\sum_i x_i\varepsilon_i(\omega)\|\,d\omega$ (see e.g. [20, Theorem 11.1 and p. 227]), then the (continuous) Minkowski inequality and last that v is $(r,1)$-summing we get

$$\begin{aligned}
\left(\sum_{k=1}^N \|vx_k\|^r_{\ell_2^n(Y)}\right)^{1/r} &= \left(\sum_{k=1}^N\left(\left(\sum_{i=1}^n \|vx_k(i)\|_Y^2\right)^{1/2}\right)^r\right)^{1/r}\\
&\leqslant \sqrt{2}\,\mathrm{C}_2(Y)\left(\sum_{k=1}^N\left(\int\left\|\sum_{i=1}^n vx_k(i)\varepsilon_i(\omega)\right\|_Y d\omega\right)^r\right)^{1/r}\\
&\leqslant \sqrt{2}\,\mathrm{C}_2(Y)\int\left(\sum_{k=1}^N\left\|\sum_{i=1}^n vx_k(i)\varepsilon_i(\omega)\right\|_Y^r\right)^{1/r} d\omega\\
&\leqslant \sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v)\int \sup_{x^*\in B_{X^*}}\sum_{k=1}^N\left|x^*\left(\sum_{i=1}^n x_k(i)\varepsilon_i(\omega)\right)\right| d\omega\\
&\leqslant \sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v)\sup_{\lambda\in B_{\ell_\infty^N}}\sup_{\eta\in B_{\ell_\infty^n}}\left\|\sum_{k=1}^N\sum_{i=1}^n x_k(i)\eta(i)\lambda_k\right\|_X .
\end{aligned}$$

On the other hand, we have, for the right-hand side of the inequality (12),

$$\begin{aligned}
\sup_{\gamma\in B_{(\ell_1^n\otimes_\varepsilon X)^*}}\sum_{k=1}^N\left|\gamma(x_k)\right| &= \sup_{\lambda\in B_{\ell_\infty^N}}\left\|\sum_{k=1}^N x_k\lambda_k\right\|_{\ell_1^n\otimes_\varepsilon X}\\
&= \sup_{\lambda\in B_{\ell_\infty^N}}\sup_{\substack{\eta\in B_{\ell_\infty^n}\\ x^*\in B_{X^*}}}\left|\sum_{i=1}^n\sum_{k=1}^N \eta(i)x^*\big(x_k(i)\big)\lambda_k\right|\\
&= \sup_{\lambda,\eta}\sup_{x^*\in B_{X^*}}\left|x^*\left(\sum_{i=1}^n\sum_{k=1}^N \eta(i)x_k(i)\lambda_k\right)\right|\\
&= \sup_{\lambda,\eta}\left\|\sum_{k=1}^N\sum_{i=1}^n x_k(i)\eta(i)\lambda_k\right\|_X .
\end{aligned}$$

This shows that

$$\pi_{r,1}\big(\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon X\to\ell_2^n(Y)\big)\leqslant\sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v). \tag{13}$$

Let us assume that

$$\pi_{r,1}\big(\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon \overset{m-1}{\cdots}\otimes_\varepsilon \ell_1^n\otimes_\varepsilon X\to\ell_2^{n^{m-1}}(Y)\big)\leqslant\big(\sqrt{2}\,\mathrm{C}_2(Y)\big)^{m-1}\pi_{r,1}(v).$$

Define $V = \ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X$, $U = \ell_2^{n^{m-1}}(Y)$ and $w = \mathrm{id} \otimes v : V \to U$. It is easily seen that $\ell_2^{n^{m-1}}(Y)$ has cotype 2 with $\mathrm{C}_2(\ell_2^{n^{m-1}}(Y)) \leqslant \mathrm{C}_2(Y)$. Then by (13)

$$\pi_{r,1}\big(\mathrm{id} \otimes w : \ell_1^n \otimes_\varepsilon V \to \ell_2^n(U)\big) \leqslant \sqrt{2}\,\mathrm{C}_2\big(\ell_2^{n^{m-1}}(Y)\big)\pi_{r,1}(w) \leqslant \sqrt{2}\,\mathrm{C}_2(Y)\big(\sqrt{2}\,\mathrm{C}_2(Y)\big)^{m-1}\pi_{r,1}(v),$$

which completes the proof. □

The second lemma is based on complex interpolation and will handle the case $p \leqslant 2$ in (5).

Lemma 3. *Let $m \in \mathbb{N}$, Y a Banach space with cotype 2, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leqslant r \leqslant 2$. Define*

$$\rho = \frac{2m}{m + 2(1/r - 1/2)}.$$

Then there is some $C > 0$ such that for every m-linear mapping $A : c_0 \times \cdots \times c_0 \to X$ the following holds

$$\Bigg(\sum_{i_1,\ldots,i_m=1}^{\infty} \big\|vA(e_{i_1},\ldots,e_{i_m})\big\|_Y^\rho\Bigg)^{1/\rho} \leqslant C\|A\|.$$

Let us note that our proof shows $C \leqslant (\sqrt{2}\,\mathrm{C}_2(Y))^{m-1}\pi_{r,1}(v)$.

Proof. By (8) we have to show that there exists some constant $C > 0$ such that, for all n,

$$\big\|\mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \ell_\rho^{n^m}(Y)\big\| \leqslant C.$$

The case $m = 1$ is simply the fact that v is $(r,1)$-summing.

First, we only consider the case $m = 2$ (here our proof appears to be a bit more transparent than in the general argument given later).

On one hand, we have from Lemma 2 that

$$\pi_{r,1}\big(\mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon X \to \ell_2^n(Y)\big) \leqslant \sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v),$$

hence by (11)

$$\big\|\mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \ell_r^n\big(\ell_2^n(Y)\big)\big\| \leqslant \sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v). \tag{14}$$

Given $1 \leqslant s \leqslant 2$ and elements $y(k,l) \in Y$ with $k = 1,\ldots,M$ and $l = 1,\ldots,N$, we have by Minkowski's inequality that

$$\left(\sum_{l=1}^{N}\|y(\cdot,l)\|^2_{\ell_s^M(Y)}\right)^{1/2}=\left(\sum_{l=1}^{N}\left(\sum_{k=1}^{M}\|y(k,l)\|_Y^s\right)^{2/s}\right)^{1/2}\leqslant\left(\sum_{k=1}^{M}\left(\sum_{l=1}^{N}\|Y(k,l)\|_Y^2\right)^{2/s}\right)^{1/s}$$
$$=\left(\sum_{k=1}^{M}\|y(k,\cdot)\|^s_{\ell_2^M(Y)}\right)^{1/s}.$$

This shows that the operator

$$T:\ell_s^M\big(\ell_2^N(Y)\big)\to\ell_2^N\big(\ell_s^M(Y)\big),\quad \big(\big(y(k,l)\big)_{l=1}^N\big)_{k=1}^M\rightsquigarrow\big(\big(y(k,l)\big)_{k=1}^M\big)_{l=1}^N$$

has norm $\leqslant 1$.

On the other hand, we can consider the operator

$$S:\ell_1^n\otimes_\varepsilon\ell_1^n\to\ell_1^n\otimes_\varepsilon\ell_1^n,\quad S(e_i\otimes e_j)=e_j\otimes e_i.$$

Clearly $\|S\|=1$, and by composition we obtain from (14)

$$\Big\|\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\xrightarrow{S\otimes\mathrm{id}_X}\ell_1^n\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\xrightarrow{\mathrm{id}\otimes v}\ell_r^n\big(\ell_2^n(Y)\big)\xrightarrow{T}\ell_2^n\big(\ell_r^n(Y)\big)\Big\|$$
$$\leqslant\sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v). \tag{15}$$

Now we interpolate (14) and (15) with the complex method and $\theta=1/2$ (see e.g. [3, Chapter 5]) and get

$$\Big\|\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\to\big[\ell_r^n\big(\ell_2^n(Y)\big),\ell_2^n\big(\ell_r^n(Y)\big)\big]_{1/2}\Big\|\leqslant\sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v).$$

But

$$\big[\ell_r^n\big(\ell_2^n(Y)\big),\ell_2^n\big(\ell_r^n(Y)\big)\big]_{1/2}=\big[\ell_r^n,\ell_2^n\big]_{1/2}\big(\big[\ell_2^n,\ell_r^n\big]_{1/2}(Y)\big)=\ell_\mu^n\big(\ell_\nu^n(Y)\big),$$

where $\frac{1}{\mu}=\frac{1/2}{r}+\frac{1/2}{2}$ and $\frac{1}{\nu}=\frac{1/2}{2}+\frac{1/2}{r}$, which finally as desired gives

$$\Big\|\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\to\ell^{n^2}_{\frac{4r}{2+r}}(Y)\Big\|\leqslant\sqrt{2}\,\mathrm{C}_2(Y)\pi_{r,1}(v).$$

We proceed now by induction on m and use the notation $\rho=\rho_{m,r}$. Let us assume that the result is true for $m-1$. From Lemma 2 we have

$$\pi_{r,1}\big(\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon\overset{m-1}{\cdots}\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\to\ell_2^{n^{m-1}}(Y)\big)\leqslant\big(\sqrt{2}\,\mathrm{C}_2(Y)\big)^{m-1}\pi_{r,1}(v),$$

hence

$$\Big\|\mathrm{id}\otimes v:\ell_1^n\otimes_\varepsilon\overset{m}{\cdots}\otimes_\varepsilon\ell_1^n\otimes_\varepsilon X\to\ell_r^n\big(\ell_2^{n^{m-1}}(Y)\big)\Big\|\leqslant\big(\sqrt{2}\,\mathrm{C}_2(Y)\big)^{m-1}\pi_{r,1}(v). \tag{16}$$

On the other hand, we consider the operator

$$S : \ell_1^n \otimes_\varepsilon \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \to \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \otimes_\varepsilon \ell_1^n$$
$$e_{i_1} \otimes (e_{i_2} \otimes \cdots \otimes e_{i_m}) \rightsquigarrow (e_{i_2} \otimes \cdots \otimes e_{i_m}) \otimes e_{i_1}.$$

Again $\|S\| = 1$ and we compose

$$\begin{array}{c} \ell_1^n \otimes_\varepsilon \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \otimes_\varepsilon X \\ \Big\downarrow {\scriptstyle (S \otimes \mathrm{id}_X)} \\ \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \\ \Big\downarrow {\scriptstyle \text{induction}} \\ \ell_{\rho_{m-1,r}}^{n^{m-1}}\big(\ell_2^n(Y)\big) \\ \Big\downarrow {\scriptstyle T} \\ \ell_2^n\big(\ell_{\rho_{m-1,r}}^{n^{m-1}}(Y)\big), \end{array}$$

to get

$$\big\| \mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \ell_2^n\big(\ell_{\rho_{m-1,r}}^{n^{m-1}}(Y)\big) \big\| \leqslant \big(\sqrt{2}\, \mathrm{C}_2(Y)\big)^{m-1} \pi_{p,1}(v). \tag{17}$$

Complex interpolation of (16) and (17) with $\theta = 1/m$ gives

$$\begin{aligned} &\big\| \mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \big[\ell_r^n\big(\ell_2^{n^{m-1}}(Y)\big), \ell_2^n\big(\ell_{\rho_{m-1,r}}^{n^{m-1}}(Y)\big)\big]_{1/m} \big\| \\ &\quad \leqslant \big(\sqrt{2}\, \mathrm{C}_2(Y)\big)^{m-1} \pi_{p,1}(v). \end{aligned}$$

But again

$$\big[\ell_r^n\big(\ell_2^{n^{m-1}}(Y)\big), \ell_2^n\big(\ell_{\rho_{m-1,r}}^{n^{m-1}}(Y)\big)\big]_{1/m} = \big[\ell_r^n, \ell_2^n\big]_{1/m}\big(\big[\ell_2^{n^{m-1}}, \ell_{\rho_{m-1,r}}^{n^{m-1}}\big]_{1/m}(Y)\big) = \ell_\mu^n\big(\ell_\nu^{n^{m-1}}(Y)\big),$$

where

$$\frac{1}{\mu} = \frac{1/m}{r} + \frac{1 - 1/m}{2} \quad \text{and} \quad \frac{1}{\nu} = \frac{1/m}{2} + \frac{1 - 1/m}{\rho_{m-1,r}}.$$

This completes the proof. □

We can finally give the proof of Theorem 1. Three different cases are considered.

3.1. The case $1 \leqslant p \leqslant q \leqslant 2$

We know from the Bennett–Carl inequalities (4) that the embedding $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ is $(r, 1)$-summing, where $1/r = 1/p - 1/q - 1/2$. Using this in Lemma 3 together with the fact that ℓ_q for $q \leqslant 2$ has cotype 2, we get (5). In order to see optimality, we assume that r is such that

$$\sup_n \big\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n \to \ell_r^{n^m}\big(\ell_q^n\big) \big\| = c < \infty$$

(see again (8)). We take families of independent standard Gaussian random variables $(g_{i_1,\dots,i_{m+1}})$, $(g_{i_1,\dots,i_m})$ and (g_k). By Chevét's inequalities (see e.g. [40, (43.2)] for the bilinear version and [16, Lemma 6] for the m-linear version) we have for all n

$$\begin{aligned} &\int \Big\| \sum_{i_1,\dots,i_{m+1}} g_{i_1,\dots,i_{m+1}}(\omega)\, e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_{m+1}} \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n} d\omega \\ &\quad \leqslant C_1 \bigg(\int \Big\| \sum_{i_1,\dots,i_m} g_{i_1,\dots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n} d\omega \big\| \mathrm{id} : \ell_2^n \to \ell_p^n \big\| \\ &\qquad + \big\| \mathrm{id} : \ell_2^{n^m} \to \otimes_\varepsilon{}^m \ell_1^n \big\| \int \Big\| \sum_k g_k e_k \Big\|_{\ell_p^n} d\omega \bigg). \end{aligned}$$

First of all, using [16, Lemma 6] we have

$$\begin{aligned} &\int \Big\| \sum_{i_1,\dots,i_m} g_{i_1,\dots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n} d\omega \\ &\quad \leqslant K \int \Big\| \sum_k g_k e_k \Big\|_{\ell_1^n} d\omega \big\| \mathrm{id} : \ell_2^n \to \ell_1^n \big\|^{m-1}. \end{aligned}$$

Now, it is known (see e.g. [17, (4)]) that $\int \| \sum_k g_k e_k \|_{\ell_p^n} d\omega \leqslant \kappa n^{1/p}$ for all $1 \leqslant p < \infty$. On the other hand, $\| \mathrm{id} : \ell_2^{n^m} \to \bigotimes_\varepsilon^m \ell_1^n \| = \| \mathrm{id} : \ell_2^n \to \ell_1^n \|^m$ and $\| \mathrm{id} : \ell_2^n \to \ell_p^n \| = n^{1/p - 1/2}$ for all $1 \leqslant p \leqslant 1$. This altogether gives

$$\begin{aligned} &\int \Big\| \sum_{i_1,\dots,i_{m+1}} g_{i_1,\dots,i_{m+1}}(\omega)\, e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_{m+1}} \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n} d\omega \\ &\quad \leqslant C_2 \big(n \cdot n^{(m-1)/2} \cdot n^{1/p - 1/2} + n^{m/2 + 1/p} \big) \leqslant C_3 n^{m/2 + 1/p}. \end{aligned}$$

Now it is a well-known fact that the Bernoulli averages are dominated by the Gaussian averages (see e.g. [20, 12.11]). Hence there is some $C > 0$ such that for all n

$$\int \Big\| \sum_{i_1,\dots,i_{m+1}} \varepsilon_{i_1,\dots,i_{m+1}}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_{m+1}} \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n} d\omega \leqslant C n^{m/2 + 1/p},$$

which means that for each n there exists a choice of signs $\varepsilon_{i_1,\dots,i_{m+1}} = \pm 1$ such that

$$\left\| z_n := \sum_{i_1,\dots,i_{m+1}} \varepsilon_{i_1,\dots,i_{m+1}} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_{m+1}} \right\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n} \leqslant C n^{m/2+1/p}.$$

By our assumption we for all n have

$$\|z_n\|_{\ell_r^{n^m}(\ell_q^n)} \leqslant c \|z_n\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n \otimes_\varepsilon \ell_p^n} \leqslant c C n^{m/2+1/p}.$$

But, on the other hand,

$$\begin{aligned} \|z_n\|_{\ell_r^{n^m}(\ell_q^n)} &= \left\| \sum_{i_1,\dots,i_m} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes \left(\sum_{i_{m+1}} \varepsilon_{i_1,\dots,i_{m+1}} e_{i_{m+1}} \right) \right\|_{\ell_r^{n^m}(\ell_q^n)} \\ &= \left(\sum_{i_1,\dots,i_m} \left\| \sum_{i_{m+1}} \varepsilon_{i_1,\dots,i_{m+1}} e_{i_{m+1}} \right\|_{\ell_q^n}^r \right)^{1/r} = \left(n^m \cdot n^{r/q} \right)^{1/r}. \end{aligned}$$

All in all, we conclude that there is some $D > 0$ such that for all n

$$\left(n^m \cdot n^{r/q} \right)^{1/r} \leqslant D n^{m/2+1/p},$$

which implies the desired inequality $m/r + 1/q \leqslant m/2 + 1/p$.

3.2. The case $1 \leqslant p < 2 < q \leqslant \infty$

We first factor $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ through ℓ_2. By (4) we know that $\mathrm{id} : \ell_p \hookrightarrow \ell_2$ is $(p, 1)$-summing. We can then apply Lemma 3 combined with $\|\cdot\|_q \leqslant \|\cdot\|_2$, in order to get the inequality in (5).

The fact that the exponent ρ is best possible now will follow by a careful analysis of techniques developed in [4, Section 2]. Without loss of generality we may assume that $q = \infty$ since all exponents satisfying (5) in the case $1 \leqslant p < 2 < q \leqslant \infty$ will also do this in the case $1 \leqslant p < 2$ and $q = \infty$.

Let r be an exponent satisfying (5) for $1 \leqslant p < 2$ and $q = \infty$. Fix n and consider an $n \times n$ matrix $(a_{jk})_{j,k}$ satisfying

$$\sum_{t=1}^n a_{jt} \bar{a}_{kt} = n \delta_{jk} \quad \text{and} \quad |a_{jk}| = 1 \quad \text{for all } j, k$$

(take, e.g. $a_{jk} = e^{2\pi i (j-k)/n}$). With this we define $\varphi : \ell_\infty^n \times \overset{m}{\cdots} \times \ell_\infty^n \times \ell_2^n \to \mathbb{C}$ by

$$\varphi(x_1, \dots, x_m, y_{m+1}) = \sum_{i_1,\dots,i_m=1}^n a_{i_1 i_2} \cdots a_{i_{m-1} i_m} x_1(i_1) \cdots x_m(i_m) y_{m+1}(i_m).$$

Let us see that $\|\varphi\| \leqslant n^{m/2}$; indeed, if $x_1, \dots, x_m \in B_{\ell_\infty^n}$ and $y_{m+1} \in B_{\ell_2^n}$ then, using the Cauchy–Schwarz inequality and the first condition of the matrix $(a_{jk})_{j,k}$ we have

$$
\begin{aligned}
&\bigl|\varphi(x_1,\dots,x_m,y_{m+1})\bigr| \\
&\quad = \biggl|\sum_{i_m}\biggl(\sum_{i_1,\dots,i_{m-1}} a_{i_1 i_2}\cdots a_{i_{m-1} i_m} x_1(i_1)\cdots x_m(i_m)\biggr) y_{m+1}(i_m)\biggr| \\
&\quad \leqslant \biggl(\sum_{i_m}\biggl|\sum_{i_1,\dots,i_{m-1}} a_{i_1 i_2}\cdots a_{i_{m-1} i_m} x_1(i_1)\cdots x_m(i_m)\biggr|^2\biggr)^{1/2} \\
&\quad = \biggl(\sum_{i_m}\bigl|x_m(i_m)\bigr|^2\biggl|\sum_{i_1,\dots,i_{m-1}} a_{i_1 i_2}\cdots a_{i_{m-1} i_m} x_1(i_1)\cdots x_{m-1}(i_{m-1})\biggr|^2\biggr)^{1/2} \\
&\quad \leqslant \biggl(\sum_{i_m}\biggl|\sum_{i_1,\dots,i_{m-1}} a_{i_1 i_2}\cdots a_{i_{m-1} i_m} x_1(i_1)\cdots x_{m-1}(i_{m-1})\biggr|^2\biggr)^{1/2} \\
&\quad = \biggl(\sum_{i_m}\sum_{\substack{i_1,\dots,i_{m-1}\\ j_1,\dots,j_{m-1}}} a_{i_1 i_2}\bar a_{j_1 j_2}\cdots a_{i_{m-1} i_m}\bar a_{j_{m-1} i_m} x_1(i_1)\overline{x_1(j_1)}\cdots x_{m-1}(i_{m-1})\overline{x_{m-1}(j_{m-1})}\biggr)^{1/2} \\
&\quad = \biggl(\sum_{\substack{i_1,\dots,i_{m-1}\\ j_1,\dots,j_{m-1}}} a_{i_1 i_2}\bar a_{j_1 j_2}\cdots a_{i_{m-2} i_{m-1}}\bar a_{j_{m-2} j_{m-1}} x_1(i_1)\overline{x_1(j_1)}\cdots x_{m-1}(i_{m-1})\overline{x_{m-1}(j_{m-1})} \\
&\qquad \times \biggl(\sum_{i_m} a_{i_{m-1} i_m}\bar a_{j_{m-1} i_m}\biggr)\biggr)^{1/2} \\
&\quad = n^{1/2}\biggl(\sum_{\substack{i_1,\dots,i_{m-1}\\ j_1,\dots,j_{m-1}}} a_{i_1 i_2}\bar a_{j_1 j_2}\cdots a_{i_{m-2} i_{m-1}}\bar a_{j_{m-2} j_{m-1}} \\
&\qquad \times x_1(i_1)\overline{x_1(j_1)}\cdots x_{m-2}(i_{m-2})\overline{x_{m-2}(j_{m-2})}\bigl|x_{m-1}(i_{m-1})\bigr|^2\biggr)^{1/2} \\
&\quad = n^{1/2}\biggl(\sum_{i_{m-1}}\bigl|x_{m-1}(i_{m-1})\bigr|^2\biggl|\sum_{i_1,\dots,i_{m-2}} a_{i_1 i_2}\cdots a_{i_{m-2} i_{m-1}} x_1(i_1)\cdots x_{m-2}(i_{m-2})\biggr|^2\biggr)^{1/2} \\
&\quad \leqslant n^{1/2}\biggl(\sum_{i_{m-1}}\biggl|\sum_{i_1,\dots,i_{m-2}} a_{i_1 i_2}\cdots a_{i_{m-2} i_{m-1}} x_1(i_1)\cdots x_{m-2}(i_{m-2})\biggr|^2\biggr)^{1/2}.
\end{aligned}
$$

Repeating this argument we finally end up in

$$
\begin{aligned}
\bigl|\varphi(x_1,\dots,x_m,y_{m+1})\bigr| &\leqslant \bigl(n^{1/2}\bigr)^{m-2}\biggl(\sum_{i_2}\sum_{i_1,j_1} a_{i_1 i_2}\overline{a_{j_1 i_2}} x_1(i_1)\overline{x_1(j_1)}\biggr)^{1/2} \\
&= \bigl(n^{1/2}\bigr)^{m-2}\biggl(\sum_{i_1,j_1} x_1(i_1)\overline{x_1(j_1)}\biggl(\sum_{i_2} a_{i_1 i_2}\overline{a_{j_1 i_2}}\biggr)\biggr)^{1/2} \\
&= \bigl(n^{1/2}\bigr)^{m-2} n^{1/2}\biggl(\sum_{i_1}\bigl|x_1(i_1)\bigr|^2\biggr)^{1/2} \leqslant \bigl(n^{1/2}\bigr)^{m}.
\end{aligned}
$$

This φ induces an m-linear mapping $\tilde{\varphi}: \ell_\infty^n \times \cdots \times \ell_\infty^n \to \ell_2^n$ with $\|\tilde{\varphi}\| = \|\varphi\|$. Composing with the inclusion we define an m-linear mapping

$$A: \ell_\infty^n \times \cdots \times \ell_\infty^n \to \ell_2^n \hookrightarrow \ell_p^n$$

with norm $\|A\| \leqslant \|\tilde{\varphi}\| \, \|\mathrm{id}: \ell_2^n \hookrightarrow \ell_p^n\| \leqslant n^{m/2} n^{1/p-1/2}$. Then by our assumption on r,

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \left\| A(e_{i_1}, \ldots, e_{i_m}) \right\|_\infty^r \right)^{1/r} \leqslant C_m n^{m/2} n^{1/p-1/2}.$$

Let us compute now the left-hand side. Each $A(e_{i_1}, \ldots, e_{i_m})$ is a vector in ℓ_p^n whose kth component is given by

$$\begin{aligned} A(e_{i_1}, \ldots, e_{i_m})(k) &= \varphi(e_{i_1}, \ldots, e_{i_m}, e_k) \\ &= \sum_{j_1,\ldots,j_m=1}^{n} a_{j_1 j_2} \cdots a_{j_{m-1} j_m} e_{i_1}(j_1) \cdots e_{i_m}(j_m) e_k(j_m), \end{aligned}$$

and this is $a_{i_1 i_2} \cdots a_{i_{m-1} i_m}$ if $i_m = k$ and 0, otherwise. Hence $A(e_{i_1}, \ldots, e_{i_m})$ has all its entries but the i_mth equal to 0. Then $\|A(e_{i_1}, \ldots, e_{i_m})\|_\infty = 1$ (since $|a_{jk}| = 1$ for all j, k) and we have $n^{m/r} \leqslant C_m n^{m/2} n^{1/p-1/2}$ for every n. This gives as desired

$$r \geqslant \frac{2m}{m + 2(\frac{1}{p} - \frac{1}{2})}.$$

3.3. The case $2 \leqslant p \leqslant q \leqslant \infty$

We have by (3) that if $A \in \mathcal{L}({}^m \ell_\infty^n, \ell_p)$ then

$$\left(\sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_q^p \right)^{1/p} \leqslant \left(\sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_p^p \right)^{1/p} \leqslant C_p^m \|A\|,$$

i.e. for each m the exponent $\rho = p$ satisfies (5). Assume conversely that the exponent r satisfies (5) for m. Then an easy argument shows that r satisfies (5) for $m = 1$, in other terms: $\mathrm{id}: \ell_p \hookrightarrow \ell_q$ is $(r, 1)$-summing. From the optimality in the Bennett–Carl inequalities (4) we get that $r \geqslant p$.

This completes the proof of Theorem 1. □

4. Bohnenblust–Hille type results for polynomials

Every m-homogeneous polynomial P defined on a Banach sequence space X with values in some Banach space Y has a monomial series expansion $\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha^{(n)}(P) z^\alpha$ whenever it is restricted to any finite-dimensional section X_n (the span of the first n basis vectors e_i) of X.

And clearly we have $c_\alpha^{(n)}(P) = c_\alpha^{(n+1)}(P)$ for $\alpha \in \mathbb{N}_0^n \subset \mathbb{N}_0^{n+1}$. Thus there is a unique family $(c_\alpha(P))_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ in Y such that for all $n \in \mathbb{N}$ and all $z \in X_n$

$$P(z) = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha(P) z^\alpha.$$

The power series $\sum_\alpha c_\alpha(P) z^\alpha$ is called the monomial expansion of P, and $c_\alpha = c_\alpha(P)$ are its monomial coefficients.

We are interested in controlling the r-norm of the coefficients $c_\alpha(P)$. Bohnenblust and Hille in [4, Section 3] used their inequality (2) to show that for each $m \in \mathbb{N}$ there exists a constant $C > 0$ such that for every m-homogeneous polynomial $P : c_0 \to \mathbb{C}$

$$\Bigg(\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} \big|c_\alpha(P)\big|^{\frac{2m}{m+1}}\Bigg)^{\frac{m+1}{2m}} \leqslant C\|P\|, \tag{18}$$

and by a highly non-trivial argument they even proved that this exponent $\frac{2m}{m+1}$ is optimal. The following theorem is the main contribution in this section. It is a polynomial analog of Theorem 1 and generalizes the preceding inequality. We will use modern techniques from the metric theory symmetric tensor products to show optimality—arguments very different from the original ones given by Bohnenblust and Hille in the scalar case.

Theorem 4. *Given $m \in \mathbb{C}$ and $1 \leqslant p \leqslant q \leqslant \infty$, define*

$$\rho = \begin{cases} \frac{2m}{m+2(1/p-\max\{1/q,1/2\})} & \text{if } p \leqslant 2, \\ p & \text{if } p \geqslant 2. \end{cases}$$

Then there exists a constant $C > 0$ such that for every m-homogeneous polynomial $P : c_0 \to \ell_p$ we have

$$\Bigg(\sum_{i_1,\dots,i_m=1}^{\infty} \big\|c_\alpha(P)\big\|_q^\rho\Bigg)^{1/\rho} \leqslant C\|P\|.$$

Moreover, ρ is best possible.

This result turns out to be an immediate consequence of the next, independently interesting, lemma combined with Theorem 1.

Lemma 5. *Let E be a Banach sequence space, $v : X \to Y$ an operator, $1 \leqslant r < \infty$ and $m \in \mathbb{N}$. Consider the following two statements*:

(a) *There is $C_{\mathrm{mult}} > 0$ such that for every m-linear mapping $A : E \times \cdots \times E \to X$*

$$\Bigg(\sum_{i_1,\dots,i_m} \big\|vA(e_{i_1},\dots,e_{i_m})\big\|_Y^r\Bigg)^{1/r} \leqslant C_{\mathrm{mult}}\|A\|.$$

(b) *There is* $C_{\mathrm{pol}} > 0$ *such that for every* m*-homogeneous polynomial* $P : E \to X$

$$\Big(\sum_{|\alpha|=m} \big\| v c_\alpha(P)\big\|_Y^r\Big)^{1/r} \leqslant C_{\mathrm{pol}}\|P\|.$$

Then (a) *always implies* (b) *with* $C_{\mathrm{pol}} \leqslant (m!)^{1-1/r} c(m,E) C_{\mathrm{mult}}$. *Conversely, if* E *is symmetric, then* (b) *implies* (a) *with* $C_{\mathrm{mult}} \leqslant m! C_{\mathrm{pol}}$.

Proof. We first show that (a) implies (b). Following [16, Section 2] (see also [18, Section 3]) we consider the following three index sets:

$$\mathcal{M}(m,n) = \{1,\ldots,n\}^m,$$
$$\mathcal{J}(m,n) = \big\{\boldsymbol{i} = (i_1,\ldots,i_m) \in \mathcal{M}(m,n):\ i_1 \leqslant \cdots \leqslant i_m\big\},$$
$$\Lambda(m,n) = \big\{\alpha \in \mathbb{N}_0^n:\ |\alpha| = m\big\}.$$

In $\mathcal{M}(m,n)$ we define the following equivalence relation: $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation $\pi \in \Pi_m$ such that $i_k = j_{\pi(k)}$ for all $k = 1,\ldots,m$. Clearly, the equivalence class of a given index $[\boldsymbol{i}]$ has at most $|\Pi_m| = m!$ elements; also $\mathcal{M}(m,n) = \dot{\bigcup}_{\boldsymbol{i}\in\mathcal{J}(m,n)}[\boldsymbol{i}]$. Moreover, there is a one-to-one correspondence between $\mathcal{J}(m,n)$ and $\Lambda(m,n)$ defined in the following terms. If $\boldsymbol{i} \in \mathcal{J}(m,n)$ there is an associated multi-index $\alpha_{\boldsymbol{i}}$ given by $\alpha_r = |\{k:\ i_k = r\}|$ (i.e. α_1 is the number of 1's in $\boldsymbol{i}$, α_2 is the number of 2's, ...). If $\alpha \in \Lambda(m,n)$ then we define $\boldsymbol{i}_\alpha = (1, \overset{\alpha_1}{\cdots}, 1, 2, \overset{\alpha_2}{\cdots}, 2, \ldots, n \overset{\alpha_n}{\cdots}, n) \in \mathcal{J}(m,n)$. Note that $\mathrm{card}[\boldsymbol{i}_\alpha] = m!/\alpha!$.

Let now $P : E_n \to X$ be an m-homogeneous polynomial (where E_n denotes the span of the first n basis vectors e_k in E), and $A : E_n \times \cdots \times E_n \to X$ its associated symmetric m-linear mapping. We show now that the monomial coefficients $c_\alpha(P)$ of P and the coefficients $a_{i_1,\ldots,i_m} = A(e_{i_1},\ldots,e_{i_m})$ defining A are related in the following way:

$$c_\alpha = \frac{1}{\mathrm{card}[\boldsymbol{i}_\alpha]} a_{\boldsymbol{i}_\alpha}; \tag{19}$$

indeed,

$$\sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} a_{i_1,\ldots,i_m} z_{i_1}\cdots z_{i_m} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,n)} \sum_{\boldsymbol{j}\in[\boldsymbol{i}]} a_{\boldsymbol{j}} z_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,n)} \mathrm{card}[\boldsymbol{i}] a_{\boldsymbol{i}} z_{\boldsymbol{i}}$$
$$= \sum_{\alpha\in\Lambda(m,n)} \mathrm{card}[\boldsymbol{i}_\alpha] c_\alpha z^\alpha.$$

Then, since $1 \leqslant \mathrm{card}[\boldsymbol{i}_\alpha] \leqslant m!$, we have

$$\Big(\sum_{|\alpha|=m} \big\|v c_\alpha(P)\big\|^r\Big)^{1/r} = (m!)^{1-1/r}\Big(\sum_{|\alpha|=m} \frac{\|v c_\alpha(P)\|^r}{(m!)^{(1-1/r)r}}\Big)^{1/r}$$
$$= (m!)^{1-1/r}\Big(\sum_{|\alpha|=m} \frac{\|v c_\alpha(P)\|^r}{(m!)^{r-1}}\Big)^{1/r}$$
$$\leqslant (m!)^{1-1/r}\Big(\sum_{|\alpha|=m} \frac{\|v c_\alpha(P)\|^r}{(\mathrm{card}[\boldsymbol{i}_\alpha])^{r-1}}\Big)^{1/r}$$

and hence by (6) and (19)

$$\begin{aligned}\Big(\sum_{|\alpha|=m}\big\|vc_\alpha(P)\big\|^r\Big)^{1/r} &\leqslant (m!)^{1-1/r}\Big(\sum_{|\alpha|=m}\operatorname{card}[\boldsymbol{i}_\alpha]\Big\|\frac{vc_\alpha(P)}{\operatorname{card}[\boldsymbol{i}_\alpha]}\Big\|^r\Big)^{1/r}\\ &= (m!)^{1-1/r}\Big(\sum_{|\alpha|=m}\operatorname{card}[\boldsymbol{i}_\alpha]\Big\|\frac{vc_\alpha(P)}{\operatorname{card}[\boldsymbol{i}_\alpha]}\Big\|^r\Big)^{1/r}\\ &= (m!)^{1-1/r}\Big(\sum_{|\alpha|=m}\big\|vA(e_{i_1},\dots,e_{i_m})\big\|^r\Big)^{1/r}\\ &\leqslant (m!)^{1-1/r}C_{\mathrm{mult}}\|A\| \leqslant (m!)^{1-1/r}C_{\mathrm{mult}}c(m,E)\|P\|.\end{aligned}$$

For the proof of the second statement we assume that the Banach sequence space E is symmetric, and deduce from our assumption (b) by (9) that

$$\sup_n\Big\|\mathrm{id}\otimes v : \bigotimes\nolimits_{\varepsilon_s}^{m,s} E_n^*\otimes_\varepsilon X \to \ell_r^{d(m,n)}(Y)\Big\| \leqslant C_{\mathrm{pol}}, \tag{20}$$

here $d(m,n) = \dim\bigotimes_r^{m,s}\ell_r^n = \operatorname{card}\mathcal{J}(m,n) = \binom{m+n-1}{n-1}$. We will use a technique that was first considered in [7,13], later used in [16,17,23] and finally presented in its more general form in [14]. For each fixed $n\in\mathbb{N}$ and every $i=1,\dots,m$ we consider mappings

$$\begin{array}{ll} I_i:\mathbb{C}^n\to\mathbb{C}^{mn} & P_i:\mathbb{C}^{mn}\to\mathbb{C}^n\\ \displaystyle\sum_{j=1}^n\lambda_j e_j \rightsquigarrow \sum_{j=1}^n\lambda_j e_{n(i-1)+j} & \displaystyle\sum_{j=1}^{mn}\lambda_j e_j \rightsquigarrow \sum_{j=1}^n\lambda_{n(i-1)+j} e_j. \end{array}$$

On the other hand, there are the natural embedding and projection (see (7))

$$\iota_m : \bigotimes^{m,s}\mathbb{C}^{mn}\to\bigotimes^m\mathbb{C}^{mn} \quad\text{and}\quad \sigma_m:\bigotimes^m\mathbb{C}^{mn}\to\bigotimes^{m,s}\mathbb{C}^{mn}.$$

From all this it can be easily deduced that the following diagram is commutative (see [22]):

$$\begin{array}{ccc} \bigotimes^m\mathbb{C}^n & \xrightarrow{\ \bigotimes^m \mathrm{id}_n\ } & \bigotimes^m\mathbb{C}^n\\ {\scriptstyle I_1\otimes\cdots\otimes I_m}\downarrow & & \uparrow{\scriptstyle m!P_1\otimes\cdots\otimes P_m}\\ \bigotimes^m\mathbb{C}^{mn} & & \bigotimes^m\mathbb{C}^{mn}\\ {\scriptstyle\sigma_m}\downarrow & & \uparrow{\scriptstyle\iota_m}\\ \bigotimes^{m,s}\mathbb{C}^{mn} & \xrightarrow{\ \bigotimes^{m,s}\mathrm{id}_{mn}\ } & \bigotimes^{m,s}\mathbb{C}^{mn} \end{array} \tag{21}$$

Define, for any $1 \leqslant r \leqslant \infty$ and any natural number N, on $\bigotimes_r^{m,s} \ell_r^N \otimes Y$ the norm induced by $\ell_r^{d(m,N)}(Y)$, and denote the resulting Banach space by

$$\bigotimes_r^{m,s} \ell_r^N \otimes_r Y = \ell_r^{d(m,N)}(Y).$$

Tensorizing and putting appropriate norms we obtain in (21):

$$\begin{array}{ccc}
\bigotimes_\varepsilon^m E_n^* \otimes_\varepsilon X & \xrightarrow{\ \bigotimes^m \mathrm{id} \otimes v\ } & \bigotimes_r^m \ell_r^n \otimes_r Y \\
\Big\downarrow {\scriptstyle I_1 \otimes \cdots \otimes I_m \otimes \mathrm{id}_X} & & \Big\uparrow {\scriptstyle m! P_1 \otimes \cdots \otimes P_m \otimes \mathrm{id}_Y} \\
\bigotimes_\varepsilon^m E_{mn}^* \otimes_\varepsilon X & & \bigotimes_r^m \ell_r^{mn} \otimes_r Y \\
\Big\downarrow {\scriptstyle \sigma_m \otimes \mathrm{id}_X} & & \Big\uparrow {\scriptstyle \iota_m \otimes \mathrm{id}_Y} \\
\bigotimes_{\varepsilon_s}^{m,s} E_{mn}^* \otimes_\varepsilon X & \xrightarrow{\ \bigotimes^{m,s} \mathrm{id} \otimes v\ } & \bigotimes_r^{m,s} \ell_r^{mn} \otimes_r Y
\end{array} \tag{22}$$

We now conclude from the metric mapping property of the injective norm, our assumption from (20), the fact that E is symmetric and the very definitions that

$$\Big\| I_1 \otimes \cdots \otimes I_m \otimes \mathrm{id}_X : \bigotimes_\varepsilon^m E_n^* \otimes_\varepsilon X \to \bigotimes_\varepsilon^m E_{mn}^* \otimes_\varepsilon X \Big\| \leqslant 1,$$

$$\Big\| \sigma_m \otimes \mathrm{id}_X : \bigotimes_\varepsilon^m E_{mn}^* \otimes_\varepsilon X \to \bigotimes_{\varepsilon_s}^{m,s} E_{mn}^* \otimes_\varepsilon X \Big\| \leqslant 1,$$

$$\Big\| \mathrm{id} \otimes v : \bigotimes_{\varepsilon_s}^{m,s} E_{mn}^* \otimes_\varepsilon X \to \ell_r^{d(m,mn)}(Y) \Big\| \leqslant C_{\mathrm{pol}},$$

$$\big\| \iota_m \otimes \mathrm{id}_Y : \ell_r^{d(m,mn)}(Y) \to \ell_r^{(mn)^m}(Y) \big\| \leqslant 1,$$

$$\big\| m! P_1 \otimes \cdots \otimes P_m \otimes \mathrm{id}_Y : \ell_r^{(mn)^m}(Y) \to \ell_r^{n^m}(Y) \big\| \leqslant m!.$$

Hence we finally deduce from (22) that

$$\sup_n \Big\| \mathrm{id} \otimes v : \bigotimes_\varepsilon^m E_n \otimes_\varepsilon X \to \ell_r^{n^m}(Y) \Big\| \leqslant m! C_{\mathrm{pol}},$$

which by (8) finishes the proof. □

For $E = c_0$, $m \in \mathbb{N}$ and $v = \mathrm{id}_{\mathbb{C}}$ let now C_{mult} and C_{pol} be the optimal constants in (a) and (b), respectively. Then we know that the optimal exponent in (a) is $\frac{2m}{m+1}$ and $C_{\mathrm{mult}} \leqslant 2^{\frac{m-1}{2}}$. Moreover, by Harris [24] (see also [32,41]) we have

$$c(m, E) \leqslant \frac{m^{m/2}(m+1)^{(m+1)/2}}{2^m m!},$$

hence in (18)

$$C_{\text{pol}} \leqslant (\sqrt{2})^{m-1} \frac{m^{m/2}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

Using Blei's theory of p-Sidon sets Queffélec in [37, Theorem III-1] reproves (18) in the complex case. He obtains the following upper bound for C_{pol}:

$$C_{\text{pol}} \leqslant \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} \frac{m^{m/2}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

The second constant is better than the first one, since $\frac{2}{\sqrt{\pi}} < \sqrt{2}$. However, if we use in the proof of Lemma 2 the Khintchine-type inequality for Steinhaus random variables with constant $\frac{2}{\sqrt{\pi}}$ from Sawa's paper [38] instead of the classical Khintchine inequality (just as Queffélec does) and proceed as above we get the same constant as in [37].

5. The Bohnenblust–Hille index

In view of (10), we say that an operator $v : X \to Y$ between Banach spaces is $(r, 1)$-summing of order m if there exist a constant $C > 0$ such that for every continuous m-linear $A : c_0 \times \cdots \times c_0 \to X$ the following holds:

$$\left(\sum_{i_1,\ldots,i_m} \left\| vA(e_{i_1}, \ldots, e_{i_m})\right\|_Y^r\right)^{1/r} \leqslant C\|A\|.$$

By $\pi^m_{r,1}(v)$ we denote the best of these constants, and it is easily checked that the class of all $(r, 1)$-summing operators of order m together with the norm

$$\pi^m_{r,1}(v) = \sup_n \left\| \mathrm{id} \otimes v : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \otimes_\varepsilon X \to \ell_r^{n^m}(Y)\right\|$$

forms a Banach operator ideal in the sense of Pietsch (for the latter reformulation of $\pi^m_{r,1}$ use (8)). Clearly, $\pi^1_{r,1}(v) = \pi_{r,1}(v)$. Finally, the mth Bohnenblust–Hille index of v is defined through

$$BH_m(v) = \inf\bigl\{r\colon\ v \text{ is } (r, 1)\text{-summing of order } m\bigr\}.$$

With a straightforward proof we have monotonicity

$$BH_1(v) \leqslant \cdots \leqslant BH_m(v),$$

and of course the name Bohnenblust–Hille index is motivated through (2):

$$BH_m(\mathrm{id}_{\mathbb{C}}) = \frac{2m}{m+1}.$$

An immediate consequence yields for every non-zero $v : X \to Y$

$$\frac{2m}{m+1} \leqslant BH_m(v).$$

The question for which operators we here even have equality is settled by the following simple observation.

Proposition 6. *An operator $v : X \to Y$ is summing if and only if it is $(\frac{2m}{m+1}, 1)$-summing of order m for every m. In particular,*

$$BH_m(v) = \frac{2m}{m+1}.$$

Proof. Clearly, only one implication has to be proved. Assume that v is summing and consider some m-linear mapping $A : c_0 \times \cdots \times c_0 \to X$. If $x^* \in X^*$ with $\|x^*\| \leqslant 1$ then $x^* \circ A \in \mathcal{L}(^m c_0, \mathbb{C})$ and by (2)

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left| x^* A(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant C \|x^* \circ A\| \leqslant C\|A\|,$$

for some constant $C > 0$. On the other hand, since v is summing, it is $(\frac{2m}{m+1}, \frac{2m}{m+1})$-summing, and therefore

$$\begin{aligned} &\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left\| vA(e_{i_1}, \ldots, e_{i_m}) \right\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\ &\leqslant \pi_{\frac{2m}{m+1}, \frac{2m}{m+1}}(v) \sup_{x^* \in B_{X^*}} \left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left| x^* A(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\ &\leqslant \pi_{\frac{2m}{m+1}, \frac{2m}{m+1}}(v) C_m \|A\|. \end{aligned}$$

This completes the proof. □

The next result—partly a consequence of the preceding one—gives a precise description of Bohnenblust–Hille indices for identities on Banach spaces X. We write

$$\operatorname{Cot}(X) := \inf\{2 \leqslant p \leqslant \infty \mid X \text{ has cotype } p\}$$

for the optimal cotype of X.

Proposition 7.

$$BH_m(\mathrm{id}_X) = \begin{cases} \frac{2m}{m+1} & \text{if } \dim X < \infty, \\ \operatorname{Cot}(X) & \text{if } \dim X = \infty. \end{cases}$$

Proof. The proof of this proposition only translates several results well known in the literature to our new setting: if X is finite-dimensional we have that id_X is summing, and then Proposition 6 gives the result. Assume that X is infinite-dimensional, and recall that $BH_1(\mathrm{id}_X)$ is the infimum over all r such that id_X is $(r,1)$-summing. By the Dvoretzky–Rogers theorem (see [20, Theorem 10.5]) we necessarily have $BH_1(\mathrm{id}_X) \geqslant 2$. Hence by a fundamental result of Maurey and Pisier [31, Théorème 1.1] (see also Talagrand [39] and [20, p. 304]) we have $BH_1(\mathrm{id}_X) = \mathrm{Cot}(X)$ which by monotonicity yields the lower bound for $BH_m(\mathrm{id}_X)$. The remaining inequality is proved in [6, Theorem 3.2] which in our language states that id_X is $(q,1)$-summing of order m and $BH_m(\mathrm{id}_X) \leqslant \mathrm{Cot}(X)$. This completes the proof. □

For ℓ_p-spaces X the preceding proposition appears as the particular case $p = q$ of our main result Theorem 1:

$$BH_m(\mathrm{id} : \ell_p \hookrightarrow \ell_q) = \begin{cases} \frac{2m}{m+2(\frac{1}{p}-\max\{\frac{1}{q},\frac{1}{2}\})} & \text{if } p \leqslant 2, \\ p & \text{if } p \geqslant 2; \end{cases}$$

note that this infimum is attained. For special operators between special spaces this estimate by Lemma 3 extends to a more general result.

Proposition 8. *Assume that v is an operator with values in a cotype 2 space. Then*

$$BH_1(v) \leqslant BH_m(v) \leqslant \frac{2m}{m + 2(\frac{1}{BH_1(v)} - \frac{1}{2})}.$$

As an application we give a multilinear extension of a famous result of Kwapień from [27, (1.1)] (see also [20, p. 208]) which shows that $1/BH_1(v) = 1 - |1/p - 1/2|$ for every operator $v : \ell_1 \to \ell_p$. For $p = 2$ this extends Grothendieck's famous theorem.

Corollary 9. *Every operator $v : \ell_1 \to \ell_q$ with $1 \leqslant q \leqslant 2$ is $(\frac{2m}{m+2-2/q}, 1)$-summing of order m.*

We say that an operator $v : X \to Y$ is polynomially $(r,1)$-summing of order m if there exist a constant $C > 0$ such that for every continuous m-homogeneous polynomial $P : c_0 \to X$ the following holds

$$\left(\sum_{|\alpha|=m} \left\| v c_\alpha(P) \right\|_Y^r \right)^{1/r} \leqslant C \|P\|,$$

and define

$$BH_m^{\mathrm{pol}}(v) = \inf\bigl\{r\colon\ v \text{ is polynomially } (r,1)\text{-summing of order } m\bigr\}.$$

Then all results in this final section transfer to this new notion since the following proposition holds true as an immediate consequence of Lemma 5.

Proposition 10. *An operator $v \neq 0$ is $(r, 1)$-summing of order m if and only if it is polynomially $(r, 1)$-summing of order m. In particular, for every m we have*

$$BH_m^{\mathrm{pol}}(v) = BH_m(v).$$

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