Instability of standing waves for Klein–Gordon–Zakharov equations with different propagation speeds in three space dimensions

Zaihui Gan \(^{a,b,*}\), Jian Zhang \(^{a}\)

\(^a\) College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, PR China
\(^b\) Mathematical College, Sichuan University, Chengdu 610064, PR China

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Abstract

This paper is concerned with the standing wave for Klein–Gordon–Zakharov equations with different propagation speeds in three space dimensions. The existence of standing wave with the ground state is established by applying an intricate variational argument and the instability of the standing wave is shown by applying Pagne and Sattinger’s potential well argument and Levine’s concavity method.

Keywords: Standing wave; Instability; Klein–Gordon–Zakharov equation; Ground state; Homogeneous Sobolev space

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\(^*\) Corresponding author.
E-mail address: ganzaihui2008cn@yahoo.com.cn (Z. Gan).

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1. Introduction

In the present paper, we consider the instability of standing wave for the Klein–Gordon–Zakharov equations with different propagation speeds in three space dimensions:

\[ \begin{align*}
\phi_{tt} - \Delta \phi + \phi &= -\psi \phi, \quad t \geq 0, \quad x \in \mathbb{R}^3, \\
\psi_{tt} - c^2 \Delta \psi &= \Delta |\phi|^2, \quad t \geq 0, \quad x \in \mathbb{R}^3.
\end{align*} \tag{1}
\]

The propagation speed in Eq. (1) is normalized as unit, while that in Eq. (2) is denoted by \( c \).

Equations (1) and (2) describe the interaction of the Langmuir wave and the ion acoustic wave in a plasma (see Dendy \[1, \text{Chapter 6}\] and Zakharov \[2\]). The function \( \phi \) denotes the fast time scale component of electric field raised by electrons and the function \( \psi \) denotes the deviation of ion density from its equilibrium. The functions \( \phi \) and \( \psi \) are originally real vector valued and real scalar valued, respectively. In this paper, however, we take two functions \( \phi \) and \( \psi \) as complex scalar valued, because it does not matter what kind of value the functions \( \phi \) and \( \psi \) take in our argument (see Ozawa, Tsutaya, Tsutsumi \[3\]). From a physical point of view, the propagation speed in Eq. (1) is about one thousand times as large as that in Eq. (2) (see Dendy \[1, \text{Chapter 6}\]), so that it is natural to assume the following condition:

\[ 0 < c < 1. \tag{\text{H1}} \]

Many authors have been studying the problem of stability and instability of standing waves for nonlinear wave equations (see \[4–13\]). For the Cauchy problem of Eqs. (1) and (2), when the Cauchy data are sufficiently small, Ozawa, Tsutaya and Tsutsumi \[3\] got the global existence of the Cauchy problem for (1) and (2). In the case of \( c = 1 \), Ozawa, Tsutaya and Tsutsumi \[14,15\] got the similar results on the Cauchy problem of Eqs. (1) and (2). In the present paper, in terms of the characteristics of the ground state and the local theory \[3\], we are interested in studying instability of the standing waves for Eqs. (1) and (2), which originates in \[4,16\].

If a pair of real functions

\[ (u, v) = (u(x), v(x)), \quad x \in \mathbb{R}^3, \]

verify the semilinear elliptic system

\[ \begin{align*}
-\Delta u + u &= -uv, \quad (u, v) \in H^1(R^3) \times L^2(R^3), \\
-c^2 \Delta v &= \Delta |u|^2, \quad (u, v) \in H^1(R^3) \times L^2(R^3) \\
\end{align*} \tag{3} \]

and

\[ (u, v) \in H^1(R^3) \times L^2(R^3) \setminus \{(0, 0)\}, \]

then

\[ \phi(t, x) = u(x), \quad \psi(t, x) = v(x), \quad t \geq 0, \quad x \in \mathbb{R}^3, \]

verify (1)–(2), which are standing wave solutions of (1)–(2).

From the physical viewpoint, an important role is played by the ground state solution of (3). We recall that a solution \((u, v)\) of (3) is termed as a ground state if it has some minimal action among all solutions of (3).
2. Principal results

For \((u, v) \in H^1(R^3) \times L^2(R^3)\), we define the action \(S(u, v)\) of the solution \((u, v)\) of (3) as follows:

\[
S(u, v) = \|\nabla u\|_{L^2(R^3)}^2 + \|u\|_{L^2(R^3)}^2 + \frac{c^2}{2} \|v\|_{L^2(R^3)}^2 + \text{Re} \int R^3 v|u|^2 \, dx.
\] (4)

In addition, we define the functional \(R(u, v)\) by

\[
R(u, v) = 2\|\nabla u\|_{L^2(R^3)}^2 + 2\|u\|_{L^2(R^3)}^2 + c^2 \|v\|_{L^2(R^3)}^2 + 3 \text{Re} \int R^3 v|u|^2 \, dx,
\] (5)

and define the set

\[
M = \{(u, v) \in H^1(R^3) \times L^2(R^3) \setminus \{(0, 0)\} : R(u, v) = 0\}.\] (6)

Now we consider the constrained variational problem

\[
\inf_{(u, v) \in M} S(u, v) = d.
\] (7)

Firstly about the standing wave of (1)–(2), we have the following existence theorem associated with the ground state.

**Theorem 2.1.** There exists \((D, Q) \in M\) such that

(1) \(S(D, Q) = \inf_{(u, v) \in M} S(u, v) = d\);
(2) \((D, Q)\) is a ground state solution of (3).

For the evolution problem (1)–(2), we impose the following initial conditions on (1)–(2):

\[
\begin{align*}
\phi(0, x) &= \phi_0(x), & \phi_t(0, x) &= \phi_1(x), & x \in R^3, \\
\psi(0, x) &= \psi_0(x), & \psi_t(0, x) &= \psi_1(x), & x \in R^3.
\end{align*}
\] (8)

Now we define the energy \(E\) for the data in (8) by

\[
E(t) = \|\nabla \phi\|_{L^2(R^3)}^2 + \|\phi\|_{L^2(R^3)}^2 + \|\phi_t\|_{L^2(R^3)}^2 + \frac{c^2}{2} \|\psi\|_{L^2(R^3)}^2
+ \frac{1}{2} \|\psi_t\|_{H^{-1}(R^3)}^2 + \text{Re} \int R^3 |\psi|^2 \, dx = E(0),
\] (9)

and for \(s < 0\), the homogeneous Sobolev space \(\dot{H}^s\) on \(R^3\) is defined by

\[
\dot{H}^s(R^3) = \{v \in S'(R^3) : \|v\|_{\dot{H}^s(R^3)} < \infty\}, \quad \|v\|_{\dot{H}^s(R^3)} = \|\xi|^s \hat{v}\|_{L^2(R^3)}.
\]

Here, \(S'\) denotes the Schwartz slowly increasing distribution space and \(\hat{v}(\xi)\) denotes the Fourier transform of \(v(x)\) in the spatial variables (see [3]).

From Theorem 2.1, we have
Lemma 2.1. Let \((D(x), Q(x))\) be the ground state of Eq. (3) for \(N = 3\). Then
\[
S(D, Q) = \min_{(u, v) \in M} S(u, v).
\] (10)

Remark 2.1. It is obvious that \(R(D, Q) = 0\) and
\[
S(D, Q) = -\frac{1}{2} \text{Re} \int_{\mathbb{R}^3} Q |D|^2 \, dx
= \frac{1}{3} \left( \left\| \nabla D \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| D \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{e^2}{2} \left\| Q \right\|_{L^2(\mathbb{R}^3)}^2 \right) .
\] (11)

On the characterization of the standing wave of (1)–(2) with minimal action, we further have the following instability theorem which originates in [4,16].

Theorem 2.2. Let \((D, Q)\) be a ground state solution of Eq. (3). Then for any \(\varepsilon > 0\), there exists \((\phi_0, \psi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\), such that
\[
\left\| \phi_0 - D \right\|_{H^1(\mathbb{R}^3)} < \varepsilon, \quad \left\| \psi_0 - Q \right\|_{L^2(\mathbb{R}^3)} < \varepsilon
\]
and with the property: the solution \((\phi, \psi)\) of the Cauchy problem for (1)–(2) corresponding to the initial data
\[
\begin{cases}
\phi(0, x) = \phi_0(x), & \phi_1(0, x) = 0, \\
\psi(0, x) = \psi_0(x), & \psi_1(0, x) = 0
\end{cases}
\] (12)
is defined for \(0 < T < \infty\), such that \((\phi, \psi) \in C([0, T), H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))\) and
\[
\lim_{t \to T} \left( \left\| \phi \right\|_{H^1(\mathbb{R}^3)} + \left\| \psi \right\|_{L^2(\mathbb{R}^3)} \right) = \infty.
\] (13)

Remark 2.2. This theorem shows the instability of the standing wave of (1)–(2) with minimal action. In fact, this theorem shows that for any neighborhood in \(H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\), the solution \((\phi, \psi)\) of (1)–(2) with (12) goes away from the orbit of the standing wave associated with \((D, Q)\) to infinity in a finite time.

Before we prove Theorems 2.1 and 2.2, we first give three propositions.

Proposition 2.1. Assume that (H1) holds. Let \((\phi_0, \phi_1, \psi_0, \psi_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)\). Then there exists a unique solution \((\phi(t, x), \psi(t, x))\) of the Cauchy problem (1)–(2) and (8) on a maximal time interval \([0, T]\) for some \(T \in (0, \infty)\) (maximal existence time) such that \((\phi, \psi) \in C([0, T); H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \cap C^1([0, T); L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))\) and either \(T = \infty\) or else \(T < \infty\) and
\[
\lim_{t \to T} \left( \left\| \phi \right\|_{L^2(\mathbb{R}^3)} + \left\| \psi \right\|_{H^{-1}(\mathbb{R}^3)} \right) = \infty.
\] (14)

Furthermore, one has that \(\forall t \in [0, T), (\phi(t, x), \psi(t, x))\) satisfies the conservation law of the energy:
$$E(t) = \| \nabla \phi \|^2_{L^2(\mathbb{R}^3)} + \| \phi \|^2_{L^2(\mathbb{R}^3)} + \| \phi_t \|^2_{L^2(\mathbb{R}^3)} + \frac{\epsilon^2}{2} \| \psi \|^2_{L^2(\mathbb{R}^3)}$$
$$+ \frac{1}{2} \| \psi_t \|^2_{H^{-1}(\mathbb{R}^3)} + \text{Re} \int_{\mathbb{R}^3} \psi |\phi|^2 \, dx = E(0). \quad (15)$$

**Proposition 2.2.** For \((u, v) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \setminus \{(0,0)\}\) and \(\lambda > 0\) let \(u_\lambda(x) = \lambda u(x), v_\lambda(x) = \lambda v(x)\). Then there exists a unique \(\mu\) (depending on \((u, v)\)) such that \(R(u_\mu, v_\mu) = 0\). Moreover, \(R(u_\lambda, v_\lambda) > 0\) for \(\lambda \in (0, \mu)\), \(R(u_\lambda, v_\lambda) < 0\) for \(\lambda \in (\mu, \infty)\), and for \(\forall \lambda > 0\), \(S(u_\mu, v_\mu) \geq S(u_\lambda, v_\lambda)\).

**Proof.** It just suffices to write down the expression of \(R(u_\lambda, v_\lambda)\) and \(S(u_\lambda, v_\lambda)\). By (4) and (5), we have

$$R(u_\lambda, v_\lambda) = 2\lambda^2 \left( \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \| u \|^2_{L^2(\mathbb{R}^3)} + \frac{\epsilon^2}{2} \| v \|^2_{L^2(\mathbb{R}^3)} \right)$$
$$+ 3\lambda^3 \text{Re} \int_{\mathbb{R}^3} v |u|^2 \, dx. \quad (16)$$

$$S(u_\lambda, v_\lambda) = \lambda^2 \left( \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \| u \|^2_{L^2(\mathbb{R}^3)} + \frac{\epsilon^2}{2} \| v \|^2_{L^2(\mathbb{R}^3)} \right)$$
$$+ \lambda^3 \text{Re} \int_{\mathbb{R}^3} v |u|^2 \, dx. \quad (17)$$

From the definition of \(M\) (\(M\) is not a empty set), there must exist a unique \(\mu > 0\) such that \(R(u_\mu, v_\mu) = 0\).

Moreover,

\[ R(u_\lambda, v_\lambda) > 0 \quad \text{for} \ \lambda \in (0, \mu); \quad R(u_\lambda, v_\lambda) < 0 \quad \text{for} \ \lambda \in (\mu, \infty). \]

Because

\[ \frac{d}{d\lambda} S(u_\lambda, v_\lambda) = 2\lambda \left( \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \| u \|^2_{L^2(\mathbb{R}^3)} + \frac{\epsilon^2}{2} \| v \|^2_{L^2(\mathbb{R}^3)} \right) + 3\lambda^2 \text{Re} \int_{\mathbb{R}^3} v |u|^2 \, dx \]

\[ = \lambda^{-1} R(u_\lambda, v_\lambda), \]

noting that \(R(u_\mu, v_\mu) = 0\), it follows that \(S(u_\mu, v_\mu) \geq S(u_\lambda, v_\lambda), \ \forall \lambda > 0\).

This completes the proof of this proposition. \(\square\)

**Proposition 2.3.** Let \(E(0) < S(D, Q)\). Put

\[ K_1 = \{(u, v) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : R(u, v) < 0, S(u, v) < S(D, Q)\}. \]

Then \(K_1\) is invariant under the flow generated by the Cauchy problem (1)–(2) and (8).
Proof. Let \((\phi_0(x), \psi_0(x)) \in K_1\) and \((\phi(t), \psi(t))\) be the solution of (1)–(2) and (8). By (4) and (15), one has
\[
S(\phi, \psi) \leq E(t) = E(0) < S(D, Q), \quad t \in [0, T).
\]
Equation (18)

To check \((\phi(t), \psi(t)) \in K_1\), we need to prove
\[
R(\phi(t), \psi(t)) < 0, \quad t \in [0, T).
\]
Equation (19)

We proceed as follows. If (19) is not true, by continuity, because of \(R(\phi_0, \psi_0) < 0\), there would exist a \(\bar{t} > 0\) such that \(R(\phi(\bar{t}), \psi(\bar{t})) = 0\). It follows that \((\phi(\bar{t}), \psi(\bar{t})) \in M\). This is impossible from (10) and (18). Thus (19) is true for \(t \in [0, T)\). So \(K_1\) is invariant under the flow generated by the Cauchy problem (1)–(2) and (8).}

\[\Box\]

3. Standing wave with ground state

In this section, we prove Theorem 2.1 by an intricate variational argument which originates in [4, 16–18].

Proposition 3.1. \(S\) is bounded below on \(M\).

Proof. From (4)–(6), on \(M\) one has
\[
S(u, v) = \frac{1}{3} \left( \|\nabla u\|_{L^2(R^3)}^2 + \|u\|_{L^2(R^3)}^2 + \frac{c^2}{2} \|v\|_{L^2(R^3)}^2 \right); \tag{20}
\]
it follows that \(S(u, v) > 0\) on \(M\). So \(S\) is bounded below on \(M\). \[\Box\]

Now we begin to solve the variational problem (7).

Since Proposition 3.1, we may let
\[
\{(u_n, v_n): n \in N\} \subset M
\]
be a minimizing sequence for (7), that is
\[
S(u_n, v_n) \to \inf_{(u, v) \in M} S(u, v) \quad (n \to \infty). \tag{21}
\]

Let \(u^*, v^*\) denote the Schwarz spherical rearrangement of functions \(u\) and \(v\), respectively. We recall that \(u^*, v^*\) are spherically symmetric, nonincreasing (with respect to \(|x|\)) functions. The symmetrization has the following properties:
\[
\int_{R^3} |\nabla u^*|^2 \, dx \leq \int_{R^3} |\nabla u|^2 \, dx, \quad \int_{R^3} |\nabla v^*|^2 \, dx \leq \int_{R^3} |\nabla v|^2 \, dx, \tag{22}
\]
\[
\int_{R^3} |u^*|^\sigma \, dx = \int_{R^3} |u|^\sigma \, dx, \quad \int_{R^3} |v^*|^\sigma \, dx = \int_{R^3} |v|^\sigma \, dx \quad \text{for } \sigma > 1. \tag{23}
\]

Furthermore, it is straightforward to check that
\[
(u_\lambda)^* = (u^*)_\lambda, \quad (v_\lambda)^* = (v^*)_\lambda, \tag{24}
\]
where as in Proposition 2.2, $u_\lambda(x) = \lambda u(x)$, $v_\lambda(x) = \lambda v(x)$.

Now for the minimizing sequence $\{(u_n, v_n): n \in \mathbb{N}\}$, we let
\[ D_n = (u_n^*)_{\mu_n}, \quad Q_n = (v_n^*)_{\mu_n}, \]
where $\mu_n > 0$ is uniquely determined by
\[ R(D_n, Q_n) = R[(u_n^*)_{\mu_n}, (v_n^*)_{\mu_n}] = 0. \] (25)

In view of (24), one also has
\[ D_n = [(u_n)_{\mu_n}]^*, \quad Q_n = [(v_n)_{\mu_n}]^* \]
and therefore by (20), (22) and (23), one has
\[ S(D_n, Q_n) \leq S[(u_n)_{\mu_n}, (v_n)_{\mu_n}] \leq S(u_n, v_n). \] (26)
The right-hand side inequality in (26) is a consequence of Proposition 2.2, since $R(u_n, v_n) = 0$ (note that $\mu = 1$ in this case). Thus
\[ \{(D_n, Q_n): n \in \mathbb{N}\} \subset M \]
and by (26),
\[ S(D_n, Q_n) \leq S(u_n, v_n). \]

Therefore $\{(D_n, Q_n): n \in \mathbb{N}\}$ is also a minimizing sequence for (7).

From (20) and (21), one knows that $\|D_n\|_{H^1(\mathbb{R}^3)}$ and $\|Q_n\|_{L^2(\mathbb{R}^3)}$ are all bounded for all $n \in \mathbb{N}$. Then there exists a subsequence
\[ \{Q_{nk}: k \in \mathbb{N}\} \subset \{Q_n: n \in \mathbb{N}\} \]
such that
\[ Q_{nk} \rightharpoonup Q_\infty \text{ weakly in } L^2(\mathbb{R}^3). \] (27)

It is of course that
\[ D_{nk} \rightharpoonup D_\infty \text{ weakly in } H^1(\mathbb{R}^3). \] (28)

Thus we extract a subsequence $\{(D_{nk}, Q_{nk}): m \in \mathbb{N}\}$ from $\{(D_n, Q_n): n \in \mathbb{N}\}$ such that (27) and (28) hold. For simplicity, we still denote $\{D_{nk}, Q_{nk}: m \in \mathbb{N}\}$ by $\{(D_n, Q_n): n \in \mathbb{N}\}$.

Now we need to use Strauss’ compactness lemma (see [18]), that is, for $2 < \sigma < 6$, the imbedding
\[ H^1_{radial}(\mathbb{R}^3) \hookrightarrow L^\sigma(\mathbb{R}^3) \] is compact, (29)
where \( H^1_{\text{radial}}(\mathbb{R}^3) = \{ f(x) \in H^1(\mathbb{R}^3) : f(x) = f(|x|) \text{ is a function of } |x| \text{ alone} \} \).

Thus from (28), one has
\[
D_n \rightarrow D_\infty \text{ strongly in } L^4(\mathbb{R}^3). 
\] (30)

Now we assert that \((D_\infty, Q_\infty) \neq (0, 0)\). We get this fact by contradiction. If \((D_\infty, Q_\infty) \equiv (0, 0)\), then from (27) and (30),
\[
D_n \rightarrow 0 \text{ strongly in } L^4(\mathbb{R}^3), \quad Q_n \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^3). 
\]

Moreover, from the Hölder’s inequality
\[
\text{Re} \int_{\mathbb{R}^3} Q_n |D_n|^2 \, dx \leq \text{Re} \left( \int_{\mathbb{R}^3} |Q_n|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^3} |D_n|^4 \, dx \right)^{1/2} = \|Q_n\|_{L^2(\mathbb{R}^3)} \|D_n\|_{L^4(\mathbb{R}^3)}^2.
\]

So
\[
\text{Re} \int_{\mathbb{R}^3} Q_n |D_n|^2 \, dx \rightarrow 0, \quad n \rightarrow \infty.
\]

Since \((D_n, Q_n) \in M, R(D_n, Q_n) = 0\) implies that
\[
2\|\nabla D_n\|_{L^2(\mathbb{R}^3)}^2 + 2\|D_n\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0, \quad n \rightarrow \infty.
\]

Therefore
\[
\|\nabla D_n\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0, \quad n \rightarrow \infty. \quad (31)
\]

On the other hand, from \((D_n, Q_n) \in M, \) one has
\[
2\|\nabla D_n\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q_n\|_{L^2(\mathbb{R}^3)}^2 \leq -3 \text{Re} \int_{\mathbb{R}^3} Q_n |D_n|^2 \, dx. \quad (32)
\]

From (32), one has
\[
2\|\nabla D_n\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q_n\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{3}{2} (\|Q_n\|_{L^2(\mathbb{R}^3)}^2 + \|D_n\|_{L^4(\mathbb{R}^3)}^4). \quad (33)
\]

By the Cagliardo–Nirenberg inequality, \(D_n\) verify
\[
\|D_n\|_{L^4(\mathbb{R}^3)} \leq C \|\nabla D_n\|_{L^2(\mathbb{R}^3)}^{3/4} \|D_n\|_{L^2(\mathbb{R}^3)}^{1/4}, \quad (34)
\]

where \(C > 0\) denotes various positive constants. The boundedness of \(\|D_n\|_{H^1(\mathbb{R}^3)}\) and \(\|Q_n\|_{L^2(\mathbb{R}^3)}\) shows that
\[
\|D_n\|_{L^2(\mathbb{R}^3)} \leq C, \quad \|Q_n\|_{L^2(\mathbb{R}^3)} \leq C.
\]

Thus from (34), one has
\[
\|D_n\|_{L^4(\mathbb{R}^3)} \leq C \|\nabla D_n\|_{L^2(\mathbb{R}^3)}^3. \quad (35)
\]

Therefore (33) and (35) yield that
\[
2\|\nabla D_n\|_{L^2(R^3)}^2 + c^2\|Q_n\|_{L^2(R^3)}^2 \leq 3\left(\|Q_n\|_{L^2(R^3)}^2 + C\|\nabla D_n\|_{L^2(R^3)}^3\right)
\leq C\left(2\|\nabla D_n\|_{L^2(R^3)}^2 + c^2\|Q_n\|_{L^2(R^3)}^2\right)^{3/2},
\]
which is contradictory with (31). So \((D, Q) \neq (0, 0)\).

Now we take \(D = (D_n)_{\mu}, Q = (Q_n)_{\mu}\) with \(\mu > 0\) uniquely determined from the condition
\[
R(D, Q) = R[(D_n)_{\mu}, (Q_n)_{\mu}] = 0.
\]

From (27), (28) and (30), thus one gets
\[
\begin{cases}
(D_n)_{\mu} \rightarrow D, \text{ strongly in } L^4(R^3); \\
(D_n)_{\mu} \rightharpoonup D, \text{ weakly in } H^1(R^3); \\
(Q_n)_{\mu} \rightharpoonup Q, \text{ weakly in } L^2(R^3).
\end{cases}
\]

Since \(R(D_n, Q_n) = 0\), Proposition 2.2 shows that
\[
S[(D_n)_{\mu}, (Q_n)_{\mu}] \leq S(D, Q).
\]

Hence, using (36), (37), one has
\[
S(D, Q) \leq \lim_{n \to \infty} S[(D_n)_{\mu}, (Q_n)_{\mu}] \leq \lim_{n \to \infty} S(D_n, Q_n) = \inf_{(u,v) \in M} S.(38)
\]

As \((D, Q) \neq (0, 0)\) and \(R(D, Q) = 0\), one has \((D, Q) \in M\). Therefore from (38), \((D, Q)\) solves the minimization problem
\[
S(D, Q) = \min_{(u,v) \in M} S(u, v). (39)
\]

Thus we proved (1) of Theorem 2.1.

Now we prove (2) of Theorem 2.1.

Since \((D, Q)\) is a solution of the problem (39), there exist a Lagrange multiplier \(\Lambda\) such that
\[
\delta_D[S + \Lambda R] = 0, \quad \delta_Q[S + \Lambda R] = 0,
\]
where \(\delta_u G\) denotes the variation of \(G(u, v)\) about \(u\). By the formula
\[
\delta_u G(u, v) = \frac{\partial}{\partial \eta} G(u + \eta \delta u, v)|_{\eta=0},
\]
we get
\[
\begin{cases}
\delta_D[S + \Lambda R] = (2 + 4\Lambda) \int_{R^3} (-\Delta u \cdot \delta u + u \delta u) \, dx + (2 + 6\Lambda) \Re \int_{R^3} uv \delta u \, dx, \\
\delta_Q[S + \Lambda R] = c^2(1 + 2\Lambda) \int_{R^3} |u|^2 \delta v \, dx + (1 + 3\Lambda) \Re \int_{R^3} |Q|^2 \delta v \, dx,
\end{cases}
\]
where \(\delta u\) denotes the variation of \(u\). By (40), one has
\[
\begin{cases}
(2 + 4\Lambda) \int_{R^3} (|\nabla D|^2 + |D|^2) \, dx + (2 + 6\Lambda) \Re \int_{R^3} |D|^2 Q \, dx = 0, \\
c^2(1 + 2\Lambda) \int_{R^3} |Q|^2 \, dx + (1 + 3\Lambda) \Re \int_{R^3} |D|^2 Q \, dx = 0.
\end{cases}
\]
From the second equation of (3), we have that $v$ is a real function. Hence by (42), we have
\[
\begin{aligned}
&\left\{ (2 + 4\Lambda) \int_{\mathbb{R}^3} |\nabla D|^2 + |D|^2 \, dx + (2 + 6\Lambda) \int_{\mathbb{R}^3} |D|^2 \, Q \, dx = 0, \\
&c^2(1 + 2\Lambda) \int_{\mathbb{R}^3} |Q|^2 \, dx + (1 + 3\Lambda) \int_{\mathbb{R}^3} |D|^2 \, Q \, dx = 0.
\end{aligned}
\]
(43)

From $R(D, Q) = 0$, one has
\[
2\|\nabla D\|_{L^2(\mathbb{R}^3)}^2 + 2\|D\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q\|_{L^2(\mathbb{R}^3)}^2 = 0.
\]
(44)

By (42) and (44), one gets
\[
\Lambda = 0.
\]

Thus from (43), one implies that
\[
\begin{aligned}
&\left\{ \int_{\mathbb{R}^3} (\nabla D \nabla \bar{D} + D \bar{D}) \, dx + 2 \int_{\mathbb{R}^3} D \bar{D} \, Q \, dx = 0, \\
&c^2 \int_{\mathbb{R}^3} Q^2 \, dx + \int_{\mathbb{R}^3} |D|^2 \, Q \, dx = 0.
\end{aligned}
\]
That is
\[
\begin{aligned}
&-\Delta D + D + QD = 0, \\
c^2\Delta Q + \Delta |D|^2 \, dx = 0.
\end{aligned}
\]
Therefore $(D, Q)$ is a solution of (3). Noting that (39), then $(D, Q)$ is a ground state solution of (3).

Thus we get the proof of (2) of Theorem 2.1.

So far, we completed the proof of Theorem 2.1.

4. Instability of standing wave

In this section, we prove Theorem 2.2 according to Theorem 2.1.

For the initial data (12), by (4) and (15), one has
\[
E(0) = \mathcal{S}(\phi_0, \psi_0).
\]
(45)

Now take
\[
\phi_0(x) = \lambda D(x), \quad \psi_0(x) = \lambda Q(x), \quad \lambda > 1.
\]
(46)

For any $\epsilon > 0$, one can always take a $\lambda$ with $\lambda > 1$ such that
\[
\|\phi_0 - D\|_{H^1(\mathbb{R}^3)} = (\lambda - 1)\|D\|_{H^1(\mathbb{R}^3)} < \epsilon, \\
\|\psi_0 - Q\|_{L^2(\mathbb{R}^3)} = (\lambda - 1)\|Q\|_{L^2(\mathbb{R}^3)} < \epsilon.
\]

Since $\lambda > 1$, by (46), Proposition 2.2 yields that
\[
\begin{aligned}
R(\phi_0, \psi_0) &< R(D, Q) = 0, \\
S(\phi_0, \psi_0) &< S(D, Q) = \frac{1}{2} (\|\nabla D\|_{L^2(\mathbb{R}^3)}^2 + 2\|D\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q\|_{L^2(\mathbb{R}^3)}^2).
\end{aligned}
\]
(47)

From (45), it follows that
\[
E(0) < S(D, Q) = \frac{1}{3} \left( \|\nabla D\|_{L^2(\mathbb{R}^3)}^2 + 2\|D\|_{L^2(\mathbb{R}^3)}^2 + c^2\|Q\|_{L^2(\mathbb{R}^3)}^2 \right).
\]
(48)
Therefore, (47), (48) and Proposition 2.3 imply that

\[ R(\phi(t), \psi(t)) < 0 \quad \text{for } t \in [0, T). \]

Since \((\phi(t), \psi(t))\) is a solution of (1)–(2) and (12) on \([0, T)\), we put

\[ J(t) = 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)}. \]

(49)

Thus one has

\[ J'(t) = \int_{\mathbb{R}^3} \left[ 2(\phi_t \bar{\phi} + \phi \bar{\phi_t}) + (\Delta^{-1/2} \psi_t \Delta^{-1/2} \bar{\psi} + \Delta^{-1/2} \psi \Delta^{-1/2} \bar{\psi_t}) \right] dx, \]

(50)

\[ J''(t) = 2(2\|\phi_t\|^2_{L^2(\mathbb{R}^3)} + \|\psi_t\|^2_{H^{-1}(\mathbb{R}^3)}) - 2R(\phi, \psi). \]

(51)

On the other hand, from (15), one has that

\[ J''(t) = 5(2\|\phi_t\|^2_{L^2(\mathbb{R}^3)} + \|\psi_t\|^2_{H^{-1}(\mathbb{R}^3)}) + 2(\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + \|\phi\|^2_{L^2(\mathbb{R}^3)})
+ c^2\|\psi\|^2_{L^2(\mathbb{R}^3)} - 6E(0). \]

(52)

From (51) and \(R(\phi, \psi) < 0\), \(J(t)\) is a convex function of \(t\). It follows that if there exists a time \(t_1\) such that \(J'(t)|_{t=t_1} > 0\), then \(J(t)\) is increasing for all \(t > t_1\) (within the interval of existence). In that case, the quantity \(2\|\phi_t\|^2_{L^2(\mathbb{R}^3)} + \|\psi_t\|^2_{H^{-1}(\mathbb{R}^3)} - 6E(0)\) will eventually become positive, and will remain positive thereafter. Thus for \(t\) large enough from (52), we would have

\[ J''(t) \geq 5(2\|\phi_t\|^2_{L^2(\mathbb{R}^3)} + \|\psi_t\|^2_{H^{-1}(\mathbb{R}^3)}). \]

(53)

In view of (49), (50) and (53), using the Hölder’s inequality, one has

\[ J(t)J''(t) \geq \frac{5}{4}(J'(t))^2. \]

(54)

Since

\[ \left[ J^{-1/4}(t) \right]'' = -\frac{1}{4}J^{-9/4}(t) \left[ J(t)J''(t) - \frac{5}{4}(J'(t))^2 \right], \]

from (54), we see that

\[ \left[ J^{-1/4}(t) \right]'' \leq 0. \]

Therefore \(J^{-1/4}(t)\) is concave for sufficiently large \(t\), and there exists a finite time \(T^*\) such that

\[ \lim_{t \to T^*} J^{-1/4}(t) = 0. \]

In other words,

\[ \lim_{t \to T^*} J(t) = \infty. \]

Thus one has \(T < \infty\) and

\[ \lim_{t \to T^-} (\|\phi\|^2_{H^1(\mathbb{R}^3)} + \|\psi\|^2_{L^2(\mathbb{R}^3)}) = \infty. \]
The proof of Theorem 2.2 will be completed once we have shown that for some \( t_1 \),
\[
\frac{d}{dt} \left( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \right) > 0.
\]
We prove this by contradiction. Suppose that for all \( t \),
\[
\frac{d}{dt} \left( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \right) \leq 0. \tag{55}
\]
Then since \( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} > 0 \) and is convex, \( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \) must tend to a finite, nonnegative limit \( A \) as \( t \to \infty \). By Proposition 2.3, we assert that \( A > 0 \). Therefore one has, as \( t \to \infty \),
\[
\frac{d}{dt} \left( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \right) \to 0,
\]
\[
\frac{d^2}{dt^2} \left( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \right) \to 0.
\]
Thus from (51) we get
\[
\lim_{t \to \infty} \left( 2\|\phi_t\|^2_{L^2(\mathbb{R}^3)} + \|\psi_t\|^2_{H^{-1}(\mathbb{R}^3)} \right) = 0. \tag{56}
\]
Recalling (51), we conclude that
\[
R(\phi, \psi) \to 0 \quad \text{as} \quad t \to \infty. \tag{57}
\]
Now for any fixed \( t > 0 \), because of \( R(\phi, \psi) < 0 \), there exists \( 0 < \mu < 1 \) such that
\[
R(\mu \phi, \mu \psi) = 0.
\]
Furthermore, one can easily check that
\[
S(\phi, \psi) - S(\mu \phi, \mu \psi) = \left| \nabla \phi \right|^2_{L^2(\mathbb{R}^3)} + \left| \nabla \psi \right|^2_{L^2(\mathbb{R}^3)} + \frac{c^2}{2} \left| \nabla \psi \right|^2_{L^2(\mathbb{R}^3)}
\]
\[
+ \mu^3 \left| \nabla \phi \right|^2_{L^2(\mathbb{R}^3)} + \frac{3}{2} \left| \nabla \psi \right|^2_{L^2(\mathbb{R}^3)}
\]
\[
\geq \left| \nabla \phi \right|^2_{L^2(\mathbb{R}^3)} + \left| \nabla \psi \right|^2_{L^2(\mathbb{R}^3)} + \frac{c^2}{2} \left| \nabla \psi \right|^2_{L^2(\mathbb{R}^3)}
\]
\[
+ \frac{3}{2} \left| \nabla \phi \right|^2_{L^2(\mathbb{R}^3)}
\]
\[
= \frac{1}{2} R(\phi, \psi). \tag{58}
\]
By (10), (57) and (58), we may conclude that
\[
S(\phi, \psi) \geq S(\mu \phi, \mu \psi) \geq S(D, Q) \quad \text{as} \quad t \to \infty. \tag{59}
\]
This is impossible from Proposition 2.3. So the supposition (55) is false. That is
\[
(d/dt) \left( 2\|\phi\|^2_{L^2(\mathbb{R}^3)} + \|\psi\|^2_{H^{-1}(\mathbb{R}^3)} \right) > 0 \quad \text{for some} \quad t_1 > 0.
\]
Thus we completed the proof of Theorem 2.2.
References