Chromatic numbers of integer distance graphs

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Abstract

An integer distance graph is a graph $G(D)$ with the set of integers as vertex set and with an edge joining two vertices $u$ and $v$ if and only if $|u - v| \in D$ where $D$ is a subset of the positive integers. We determine the chromatic number $\chi(D)$ of $G(D)$ if $D$ is a 4-element set of the form $D = \{x, y, x + y, y - x\}$, $x < y$, or if $D$ is an arithmetical progression $D = \{a + kd : k = 0, 1, 2, \ldots\}$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

If $S$ is a subset of the $d$-dimensional Euclidean space, $S \subseteq \mathbb{R}^d$, then the distance graph $G(S, D)$ is defined as the graph $G$ with vertex set $V(G) = S$ and two vertices $u$ and $v$ are adjacent if and only if their distance $d(u, v)$ is an element of the so-called distance set $D$ which is a subset of the set of positive real numbers, $D \subseteq \mathbb{R}_+$. In case that $V(G) = \mathbb{Z}$, the set of all integers, and $D$ is a subset of the set of positive integers, $D \subseteq \mathbb{N}$, the graph $G(\mathbb{Z}, D) = G(D)$ is called integer distance graph.

A coloring $f : V(G) \rightarrow \{f_1, f_2, \ldots\}$ of $G$ is an assignment of colors to the vertices of $G$ such that $f(u) \neq f(v)$ for all adjacent vertices $u$ and $v$. The minimum number of colors necessary to color the vertices of $G$ is the chromatic number $\chi(G)$ of $G$. If such a minimum does not exist we write $\chi(G) = \infty$.

For a distance set $D = \{d_1, d_2, \ldots\} \subseteq \mathbb{N}$ we write $G = G(D) = G(d_1, d_2, \ldots)$ and $\chi(G(D)) = \chi(D) = \chi(d_1, d_2, \ldots)$.

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Coloring problems on distance graphs are motivated by the famous Hadwiger–Nelson unit distance plane coloring problem which asks for the minimum number of colors necessary to color the points of the Euclidean plane (i.e., $V(G)=\mathbb{R}^2$) such that pairs of points of unit distance (i.e., $D=\{1\}$) are colored differently. In [8], Hadwiger gives a tiling of the plane in seven sets of congruent hexagons such that no set contains two points of distance 1. On the other hand, there exist 4-chromatic unit distance graphs in the plane (see e.g. [12]). Therefore, we have $4\leq\chi(G(\mathbb{R}^2,\{1\}))\leq7$. No substantial progress has been made on the problem till now.

Integer distance graphs were introduced by Eggleton et al. [5]. For example, $\chi(\mathbb{P})=4$ is proved in [5] where $\mathbb{P}$ is the set of primes. Of course, $\chi(D)\leq\chi(\mathbb{P})=4$ for prime distance sets $D\subseteq\mathbb{P}$. In [6] the problem is posed to characterize all prime distance sets $D\subseteq\mathbb{P}$ such that $\chi(D)=4$. The problem is solved for 3- and 4-element prime distance sets $D$ (see [5,7,13,16]) and is unsolved for finite prime distance sets $D$ of cardinality $\geq5$.

More recently, the chromatic numbers of more general integer distance graphs have been discussed. Obviously, if $D$ contains only odd integers then $\chi(D)=2$ (color all vertices alternately with two colors). Since $|D|+1$ is a trivial upper bound for $\chi(D)$ if $D$ is finite (see [3,17]) we have for 2-element distance sets $\chi(D)=2$ if $D$ contains two odd vertices and $\chi(D)=3$ if $D$ consists of two coprime vertices of distinct parity.

The chromatic numbers of integer distance graphs for arbitrary 3-element distance sets $D$ are completely determined [3,14,18]. If the greatest common divisor of $D$ is 1 which is no loss of generality (see Lemma 2) and $|D|=3$ then $\chi(D)=4$ if and only if $D=\{1,2,3n\}$ or if $D=\{x,y,x+y\}$ and $x\not\equiv y\pmod{3}$ [14,18], and $\chi(D)\leq3$ for all other 3-element distance sets.

For integer distance graphs $G(D)$ with distance set $D$ of cardinality at least 4 there are only some known partial results for $\chi(D)$. For example, $\chi(1,2,3,4n)=5$. Also $\chi(D)$ is determined for $D=\{2,3,s,s+u\}$ and for $D=\{x,y,x+y,y-x\}$ for many pairs $(s,u)$ and $(x,y)$, respectively (see [15,10]).

If the distance set $D$ is of a certain shape then some more general results are proved. For example, if $D=\{1,2,\ldots,n\}\setminus\{m,2m,\ldots,sm\}$ then $\chi(D)=m$ if $n<(s+1)m$ and $[(n+sm+1)/s+1] \leq \chi(D) \leq [(n+sm+1)/s+1]+1$ if $n\geq(s+1)m$ (see [4,11]). The cases when $\chi(D)$ coincides with the lower bound and when with the upper bound are determined in [9] (see also [1,11]).

In [2] the chromatic numbers $\chi(D)$ are determined for distance sets $D$ which are totally ordered by divisibility: If $D=\{d_1,d_2,d_3,\ldots\}$ and $d_id_{i+1}$ for $i=1,2,3,\ldots$ then $\chi(D)\leq4$. A complete characterization of all such distance sets with respect to the chromatic number of the corresponding integer distance graph is given.

In this paper we determine in Section 2 the chromatic numbers $\chi(D)$ of integer distance graphs $G(D)$ if $D$ consists of two different positive integers together with their sum and their (positive) difference. In Section 3 we consider integer distance graphs with finite or infinite distance sets which are arithmetical sequences. We determine the chromatic numbers of all such graphs.
2. $\chi(x, y, x + y, y - x)$

If $D = \{2, 3, 5, 8\}$ then $\chi(D) = 5$ [10]. The distance set $D$ is of the form $\{2, 3, s, t\}$, $3 < s < t$. It is conjectured that there is no other $D$ of this form such that the chromatic number $\chi(D)$ of the corresponding integer distance graph $G(D)$ equals the upper bound 5 for 4-element distance sets (see [10]). The set $D = \{2, 3, 5, 8\}$ is also of the form $\{x, y, x + y, y - x\}$ i.e., $D$ consists of two elements $x, y$ as well as their sum $x + y$ and their difference $y - x$.

In this section we determine the chromatic numbers of all integer distance graphs $G(D)$ with distance sets $D = \{x, y, x + y, y - x\} \subseteq \mathbb{N}$, $x < y$. Since $\chi(d_1, d_2, \ldots, d_r) = \chi(nd_1, nd_2, \ldots, nd_r)$, $n \in \mathbb{N}$, for finite distance sets ([13], see also Lemma 2 in Section 3) we assume without loss of generality that the greatest common divisor of $x$ and $y$ is 1. Theorem 1 extends a result of [10].

**Theorem 1.** Let $D = \{x, y, x + y, y - x\}$, $x < y$, $\gcd(x, y) = 1$, $(x, y) \neq (1, 2)$.

(a) If $x$ and $y$ have distinct parity then $\chi(D) = 4$.

(b) If $x$ and $y$ both are odd then $\chi(D) = 5$.

We use the following lemma in the proof of Theorem 1 (see also [13]).

**Lemma 1.** If $D^n = \{d \in D: n|d\}$ then

$$\chi(D) \leq \min_{n \in \mathbb{N}} n(|D^n| + 1).$$

**Proof.** Partition the vertices of $G(D)$ into residue classes $V_i = \{v \in V(G) = \mathbb{Z}: v \equiv i \pmod{n}\}$, $i = 0, 1, \ldots, n-1$, with respect to an arbitrary positive integer $n$. If $u_i$ and $w_i$ are vertices of $V_i$ then $u_i$ is adjacent to $w_i$ iff $|u_i - w_i| \in D^n$. Since $|D| + 1$ is an upper bound of $\chi(D)$ if $D$ is finite (see [3,17]) and since the induced subgraphs $\langle V_0 \rangle, \langle V_1 \rangle, \ldots, \langle V_{n-1} \rangle$ are pairwise isomorphic and therefore isomorphic to $G((1/n)D^n)$ which has distance set $\{(d/n) : d \in D^n\}$ we obtain $\chi((1/n)D^n) \leq |D^n| + 1$. This implies that the vertices of each $V_i$, $i = 0, 1, \ldots, n-1$, can be colored with $|D^n| + 1$ colors. The number of residue classes is $n$ and therefore all vertices of $G(D)$ are to be colored by $n(|D^n| + 1)$ colors. □

**Proof of Theorem 1.** We consider the subset $V_z = \{z, z + x, z + y, z + x + y\} \subseteq V(G)$ where $z$ is an arbitrary integer. The subgraph of $G(D)$ induced by $V_z$ is isomorphic to the complete graph on four vertices, $\langle V_z \rangle \cong K_4$, since all distances between the vertices of $V_z$ are in $D$ (see Fig. 1). This implies $\chi(D) \geq 4$.

(a) Since $x$ and $y$ are of different parity we get that $x + y$ and $y - x$ are odd i.e., there is exactly one even element in $D$. By application of Lemma 1 choosing $n = 2$ we obtain $\chi(D) \leq 4$.

(b) Assume that there exists a 4-coloring $f$ of the vertices of $G(D)$. 

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We consider the induced subgraphs $\langle V_z \rangle$ and $\langle V_{z+x} \rangle$ of $G(D)$ which are both isomorphic to $K_4$ according to the remark at the beginning of the proof i.e., we need four colors to color each of these subgraphs (see Fig. 2).

Since $z+x$ and $z+x+y$ are common vertices of $\langle V_z \rangle$ and $\langle V_{z+x} \rangle$ we get $\{f(z), f(z+y)\} = \{f(z+2x), f(z+2x+y)\}$ for the remaining vertices of the two subgraphs. Replacing $z$ by $z+(k-2)x$, $k \in \mathbb{Z}$, we obtain $\{f(z+kx), f(z+kx+y)\} = \{f(z+(k-2)x), f(z+(k-2)x+y)\}$ and therefore

$$\{f(z+kx), f(z+kx+y)\} = \begin{cases} \{f(z), f(z+y)\} & \text{if } k \text{ is even,} \\ \{f(z+x), f(z+x+y)\} & \text{if } k \text{ is odd.} \end{cases} \tag{1}$$

If we consider the induced subgraphs $\langle V_z \rangle$ and $\langle V_{z+y} \rangle$ of $G(D)$ instead of $\langle V_z \rangle$ and $\langle V_{z+x} \rangle$ then we analogously obtain (by formal exchanging of $x$ and $y$)

$$\{f(z+ky), f(z+x+ky)\} = \begin{cases} \{f(z), f(z+x)\} & \text{if } k \text{ is even,} \\ \{f(z+y), f(z+x+y)\} & \text{if } k \text{ is odd.} \end{cases} \tag{2}$$

If we set $k = y$ in (1) and $k = x$ in (2), respectively, then we obtain $f(z+xy) \in \{f(z+x), f(z+x+y)\}$ and $f(z+xy) \in \{f(z+y), f(z+x+y)\}$, respectively, since $x$ and $y$ are odd. The vertices $z+x, z+y, z+x+y$ are colored differently by assumption since they are pairwise adjacent. This implies $f(z+xy) = f(z+x+y)$.

Replacing $z$ by $z-x$ in (1) and (2) (recall that $z$ is an arbitrary integer) we get

$$\{f(z+(k-1)x), f(z+(k-1)x+y)\}$$

$$= \begin{cases} \{f(z-x), f(z-x+y)\} & \text{if } k \text{ is even,} \\ \{f(z), f(z+y)\} & \text{if } k \text{ is odd,} \end{cases} \tag{3}$$

and

$$\{f(z-x+ky), f(z+ky)\} = \begin{cases} \{f(z-x), f(z)\} & \text{if } k \text{ is even,} \\ \{f(z-x+y), f(z+y)\} & \text{if } k \text{ is odd.} \end{cases} \tag{4}$$
If we set \( k - 1 = y \) in (3) and \( k = x \) in (4), respectively, then we get \( f(z + xy) \in \{ f(z - x), f(z - x + y) \} \) and \( f(z + xy) \in \{ f(z - x + y), f(z + y) \} \), respectively. Again, \( z - x, z - x + y, z + y \) are colored differently by assumption and therefore \( f(z + xy) = f(z - x + y) \).

Together we have

\[
f(z + xy) = f(z + x + y) = f(z - x + y).
\]

By multiple repetition we obtain \( f(z - x + y) = f(z + x + y) = f(z + 3x + y) = f(z + 5x + y) = \cdots \). Choosing \( z = 2x - y \) in this equality chain for the arbitrary integer \( z \) results in \( f(x) = f(3x) = f(5x) = \cdots = f(yx) = \cdots \).

On the other hand, if we choose \( z = 0 \) in (5) we get \( f(xy) = f(x + y) \). That implies \( f(x) = f(x + y) \) which is a contradiction since the vertices \( x \) and \( x + y \) are adjacent. Therefore, \( \chi(D) \geq 5 \).

On the other hand, we obtain \( \chi(D) \leq 5 \) by applying Lemma 1 with \( n = 1 \) which concludes the proof. \( \square \)

If \( D = \{1, 2, 3, 4n\}, n \in \mathbb{N} \), then \( \chi(D) = 5 \). Are there any other 4-element distance sets besides \( D = \{1, 2, 3, 4n\}, n \in \mathbb{N} \), and \( D = \{x, x + y, y + x, y\} \), \( x < y \), \( \gcd(x, y) = 1 \), \( x \equiv y \equiv 1 \pmod{2} \), of Theorem 1 such that the chromatic number of the corresponding integer distance graph is 5?

### 3. Arithmetical sequences as distance sets

In [10] the chromatic numbers of integer distance graphs \( G(D) \) are determined where \( D \) is an arbitrary sequence of consecutive integers.

**Theorem 2** (Kemnitz and Kolberg [10]). If \( D = \{a, a + 1, \ldots, a + t\} \), \( a, t \in \mathbb{N} \), and \( k \in \mathbb{Z}, k \geq 0 \), chosen in such a way that \( ka < t \leq (k + 1)a \) then \( \chi(D) = k + 3 \).

The condition on \( k \) in the theorem implies \( k = \lceil t/a \rceil - 1 \) and therefore \( \chi(D) = \lceil t/a \rceil + 2 \).

A coloring \( f : \mathbb{Z} \rightarrow \{f_1, f_2, \ldots, f_s\} \) is called \( p \)-periodic if \( f(v) = f(v + p) \) for all vertices \( v \in V(G) = \mathbb{Z} \).

**Proof.** Assigning the colors

\[
\begin{array}{c}
\underbrace{f_1 \cdots f_1}_a \underbrace{f_2 \cdots f_2}_a \underbrace{f_{k+2} \cdots f_{k+2}}_a \underbrace{f_{k+3} \cdots f_{k+3}}_a
\end{array}
\]

to \( (k + 3)a \) consecutive vertices of \( G(D) \) and continuing this coloring periodically then we obtain a coloring of \( G(D) \) which uses \( k + 3 \) colors. Therefore, \( \chi(D) \leq k + 3 \).

Consider the induced subgraph \( G = \langle \{0, 1, 2, \ldots, 2a + t - 1\} \rangle \) of \( G(D) \). Two vertices \( u \) and \( v \) of \( G \) are adjacent in \( G \) if and only if \( a \leq |u - v| \leq a + t \). An independent set \( I_u \) of \( G \) which contains a vertex \( u \) can contain in addition only elements of the sets...
Let $V_1 = \{u+1, u+2, \ldots, u+a-1\}$ and $V_2 = \{u-1, u-2, \ldots, u-a+1\}$ where the vertices are considered modulo $2a + t$. Since each vertex $u + k \in V_1$, $0 < k < a$, is adjacent to the vertex $u - a + k \in V_2$ (namely, the distance between these vertices is $a$ or $a + t$ which are both elements of $D$), the cardinality of $I_u$ is at most $a = |V_1| + 1 = |V_2| + 1$. Therefore, the independence number $\alpha(G)$ of $G$ is at most $a$. Since $\chi(G) \geq |V(G)|/\alpha(G)$ we obtain

$$\chi(D) \geq \chi(G) \geq \frac{2a + t}{a} > \frac{2a + ka}{a} = k + 2$$

which concludes the proof. □

The distance set $D$ of Theorem 2 is an arithmetical progression $D = \{a + kd : k = 0, 1, 2, \ldots, t\}$ with $d = 1$. In Theorem 3 we determine the chromatic numbers of integer distance graphs $G(D)$ where the distance set $D$ is an arithmetical sequence with arbitrary $d$ and $t$ (finite or infinite). The following lemma guarantees that we can focus our attention on the case that the greatest common divisor of $a$ and $d$ is 1 without loss of generality (see [13] for a finite version of Lemma 2).

**Lemma 2.** Let $D = \{d_1, d_2, \ldots\}$ and $n|d_i$ for $i = 1, 2, \ldots$. Then

$$\chi(d_1, d_2, \ldots) = \chi\left(\frac{d_1}{n}, \frac{d_2}{n}, \ldots\right).$$

**Proof.** Define $V_i = \{v \in \mathbb{Z} : v \equiv i \pmod{n}\}, i = 0, 1, \ldots, n-1$. If $u$ and $v$ are adjacent vertices of $G(D)$ then $|u-v| \in D$ and therefore $|u-v|$ is divisible by $n$ which implies that $u$ and $v$ belong to the same residue class $V_i$. Therefore, there do not exist edges of $G(D)$ between vertices of different residue classes which implies that $G(D)$ consists of the $n$ induced subgraphs $\langle V_0 \rangle, \langle V_1 \rangle, \ldots, \langle V_{n-1} \rangle$ which are mutually isomorphic. Since $\langle V_0 \rangle$ is isomorphic to $G(d_1/n, d_2/n, \ldots)$ we have $\chi(\langle V_0 \rangle) = \chi(d_1/n, d_2/n, \ldots)$. Because of the isomorphism of the subgraphs $\langle V_0 \rangle, \langle V_1 \rangle, \ldots, \langle V_{n-1} \rangle$ and the fact that there are no edges between any two of them we obtain $\chi(D) = \chi(d_1/n, d_2/n, \ldots)$. □

**Theorem 3.** Let $D = \{a, a + d, a + 2d, \ldots\} \subseteq \mathbb{N}$ be an arithmetical sequence (finite or infinite) where $a$ is a positive and $d$ a non-negative integer such that $\gcd(a, d) = 1$. Then

$$\chi(D) = \begin{cases} \left\lfloor \frac{D-1}{a} \right\rfloor + 2 & \text{if } d = 1, \\ 2 & \text{if } d \text{ is even or } |D| = 1, \\ 3 & \text{otherwise}. \end{cases}$$

In the proof of Theorem 3 we use the concept of the so-called circulant graphs.

If $D = \{d_1, d_2, \ldots, d_r\}$ is a set of positive integers then the circulant graph $G = G_n^a(D) = G_n^a(d_1, d_2, \ldots, d_r)$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is defined in such a way that the vertices $v_1, v_2, \ldots, v_n$ are arranged on a circle and vertices $v_i$ and $v_j$ are adjacent if and only if there exists an element $d \in D$ such that $j - i$ is congruent to $d$ or
to $-d$ modulo $n$. Observe that circulant graphs are regular and that they may contain loops, namely if the order $n$ of $G$ divides an element of the set $D$. Fig. 3 provides as an example the circulant graph $G_8^c(1,2)$.

We obtain other examples of circulant graphs $G_n^{c}(D)$ if we choose $V(G_n^{c}(D)) = \{0,1,2,\ldots,n-1\} \subseteq V(G(D))$ of an integer distance graph $G(D)$ where two vertices $i$ and $j$ are adjacent in $G_n^{c}(D)$ if and only if there exists an element $d \in D$ such that $j-i \equiv \pm d \pmod{n}$. In a way, integer distance graphs themselves are infinite circulant graphs.

For any given set $D$ and any fixed positive integer $n$ there exist the integer distance graph $G_n^{c}(D)$ and the corresponding circulant graph $G_n^{c}(D)$ if we choose $\mathcal{V}(G_n^{c}(D)) = \{0;1;2;\ldots;n-1\} \subseteq \mathcal{V}(G(D))$ of an integer distance graph $G(D)$ where two vertices $i$ and $j$ are adjacent in $G_n^{c}(D)$ if and only if there exists an element $d \in D$ such that $j-i \equiv \pm d \pmod{n}$. In a way, integer distance graphs themselves are infinite circulant graphs.

Proof of Theorem 3. (a) If $|D| = 1$ (e.g. if $d = 0$) then $\chi(G) = 2$.

(b) If $d = 1$ and $|D|$ is finite then $\chi(G) = \lceil |D| - 1/a \rceil + 2$ according to Theorem 2.

(c) If $d = 1$ and $|D|$ is infinite then $\chi(G) = \infty$.

In the remaining cases we have $|D| \geq 2$ and $d \geq 2$.

(d) If $|D| \geq 2$ and $d \geq 2$ is even then $a$ is odd since $\gcd(a,d) = 1$ and therefore all elements of $D$ are odd which implies $\chi(G) = 2$.

(e) If $|D| \geq 2$ and $d \geq 2$ is odd then $D' = \{a,a+d\} \subseteq D$ contains elements of distinct parity which implies that $\chi(D) \geq \chi(D') = 3$.

Since $a \equiv a+d \equiv a+2d \equiv \cdots \pmod{d}$ and $\gcd(a,d) = 1$ it follows that $d$ does not divide any element of the distance set $D$. This implies that the circulant graph $G_n^{c}(D)$ contains no loops and therefore $\chi(G_n^{c}(D)) \geq \chi(G(D))$.

Since all elements of $D$ are congruent to $a$ modulo $d$ two vertices $u$ and $v$ of $G_n^{c}(D)$ are adjacent if and only if $u-v \equiv \pm a \pmod{d}$. Therefore, each vertex $w$ of $G_n^{c}(D)$
has at most two neighbors, namely \( w + a \) and \( w - a \) modulo \( d \), and, moreover, exactly two neighbors since \( d \) is odd. Therefore,

\[
\chi(D) \leq \chi(G_d^2(D)) \leq 4(G_d^2(D)) + 1 = 3. \quad \square
\]

In [2] the chromatic numbers \( \chi(D) \) of integer distance graphs are determined whose distance sets \( D = \{d_1, d_2, d_3, \ldots \} \) are divisibility chains i.e., \( d_i | d_{i+1} \) for \( i = 1, 2, 3, \ldots \). In particular, this set of graphs contains those integer distance graphs whose distance sets \( D \) are geometric sequences \( D = \{a, aq, aq^2, aq^3, \ldots \} \). It holds that \( \chi(D) = 2 \) if \( q \) is odd and \( \chi(D) = 3 \) if \( q \) is even [2].

References

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