ACADEMIC PRESS

Journal of Algebra 254 (2002) 279–299

JOURNAL OF Algebra

www.academicpress.com

Hilbert series of quadratic algebras associated with pseudo-roots of noncommutative polynomials

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Received 12 September 2001

Communicated by Efim Zelmanov

Abstract

The quadratic algebras Q_n are associated with pseudo-roots of noncommutative polynomials. We compute the Hilbert series of the algebras Q_n and of the dual algebras $Q_n^!$.
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Introduction

Let $P(x) = x^n - a_1x^{n-1} + \cdots + (-1)^n a_n$ be a polynomial over a ring R. Two classical problems concern the polynomial $P(x)$: investigation of the solutions of the equation $P(x) = 0$ and the decomposition of $P(x)$ into a product of irreducible polynomials.

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PII: S0021-8693(02)00081-9

In the commutative case relations between these two problems are well known: when R is a commutative division algebra, x is a central variable, and the equation $P(x)=0$ has roots $x_1,\dots,x_n$, then

$$P(x)=(x-x_n)\dots(x-x_2)(x-x_1). \tag{0.1}$$

In noncommutative case relations between the two problems are highly nontrivial. They were investigated by Ore [11] and others. ([10] is a good source for references; see also the book [3] where matrix polynomials are considered.) More recently, some of the present authors have obtained results [6,7,15] which are important for the present work. For a division algebra R, I. Gelfand and V. Retakh [6–8] studied connections between the coefficients of $P(x)$ and a generic set of solutions $x_1,\dots,x_n$ of the equation $P(x)=0$. They showed that for any ordering $I=(i_1,\dots,i_n)$ of $\{1,\dots,n\}$ one can construct elements y_k, $k=1,\dots,n$, depending on $x_{i_1},\dots,x_{i_k}$ such that

$$\begin{aligned} a_1 &= y_1+y_2+\dots+y_n,\\ a_2 &= \sum_{i<j} y_j y_i,\\ &\ \ \vdots\\ a_n &= y_n\dots y_2 y_1. \end{aligned} \tag{0.2}$$

These formulas are equivalent to the decomposition

$$P(t)=(t-y_n)\dots(t-y_2)(t-y_1) \tag{0.3}$$

where t is a central variable. Formula (0.3) can be viewed as a noncommutative analog of formula (0.1). A decomposition of $P(x)$ for a noncommutative variable x is more complicated (see [7]).

The element y_k, which is defined to be the conjugate of x_{i_k} by a Vandermonde quasideterminant involving $x_{i_1},\dots,x_{i_k}$, is a rational function in $x_{i_1},\dots,x_{i_k}$; it is symmetric in $x_{i_1},\dots,x_{i_{k-1}}$. (Quasideterminants were introduced and studied in [4,5,8]. We do not need the explicit formula for y_k here.) It was shown in [15] that the polynomials in y_k for a fixed ordering I which are symmetric in x_l can be written as polynomials in the symmetric functions $a_1,\dots,a_n$ given by formulas (0.2). Thus these are the natural noncommutative symmetric functions.

It is convenient for our purposes to use the notation $y_k=x_{A_k,i_k}$ where $A_k=\{i_1,\dots,i_{k-1}\}$ for $k=2,\dots,n$, $A_1=\emptyset$. In the generic case there are $n!$ decompositions of type (0.3). Such decompositions are given by products of linear polynomials $t-x_{A,i}$ where $A\subset\{1,\dots,n\}$, $i\in\{1,\dots,n\}$, $i\notin A$. It is natural to call the elements $x_{A,i}$ *pseudo-roots* of the polynomial $P(x)$. Note that elements $x_{\emptyset,i}=x_i$, $i=1,\dots,n$, are roots of the polynomial $P(x)$.

In [9] I. Gelfand, V. Retakh, and R. Wilson introduced the algebra Q_n of all pseudo-roots of a generic noncommutative polynomial. It is defined by generators

$x_{A,i}$, $A \subset \{1,\ldots,n\}$, $i \in \{1,\ldots,n\}$, $i \notin A$, and relations

$$x_{A\cup i,j} + x_{A,i} - x_{A\cup j,i} - x_{A,j}, \tag{0.4a}$$

$$x_{A\cup i,j} \cdot x_{A,i} - x_{A\cup j,i} \cdot x_{A,j}, \quad i,j \in \{1,\ldots,n\} \setminus A. \tag{0.4b}$$

In [9] a natural homomorphism e of Q_n into the free skew field generated by $x_1,\ldots,x_n$ was constructed. We believe that the map e is an embedding.

We consider the algebra Q_n as a universal algebra of pseudo-roots of a noncommutative polynomial of degree n. Our philosophy is the following: the algebraic operations of addition, subtraction and multiplication are cheap, but the operation of division is expensive. For our problem we cannot use the "cheap" free associative algebra generated by $x_1,\ldots,x_n$, but to use the gigantic free skew field is too expensive. So, we suggest to use an "affordable intermediate" algebra Q_n.

Relations (0.4) show (see [9]) that we may define a linearly independent set of generators

$$r_A = x_{A\setminus\{a_1\},a_1} + x_{A\setminus\{a_1,a_2\},a_2} + \cdots + x_{\emptyset,a_k}$$

for all nonempty $A = \{a_1,\ldots,a_k\} \subseteq \{1,\ldots,n\}$. These generators satisfy the quadratic relations

$$\Big\{r(A)\big(r\big(A\setminus\{i\}\big) - r\big(A\setminus\{j\}\big)\big) + \big(r\big(A\setminus\{i\}\big) - r\big(A\setminus\{j\}\big)\big)r\big(A\setminus\{i,j\}\big) \\ - r\big(A\setminus\{i\}\big)^2 + r\big(A\setminus\{j\}\big)^2 \,\Big|\, i,j \in A \subseteq \{1,\ldots,n\}\Big\}.$$

Another linearly independent set of generators in Q_n, $\{u_A \mid \emptyset \neq A \subseteq \{1,\ldots,n\}\}$, supersymmetric to $\{r_A \mid \emptyset \neq A \subseteq \{1,\ldots,n\}\}$, was used in [2] for a construction of noncommutative algebras related to simplicial complexes.

As a quadratic algebra Q_n has a dual quadratic algebra $Q_n^!$, see [14]. A study of this algebra is of an independent interest. In Section 5 we describe generators and relations for the algebra $Q_n^!$.

In this paper we compute the Hilbert series of the quadratic algebras Q_n and $Q_n^!$. Recall that if $W = \sum_{i\geqslant 0} W_i$ is a graded vector space with $\dim W_i$ finite for all i then the Hilbert series of W is defined by

$$H(W,t) = \sum_{i\geqslant 0} (\dim W_i)t^i.$$

Any quadratic algebra A has a natural graded structure $A = \sum_{i\geqslant 0} A_i$ where A_i is the span of all products of i generators. If A is finitely generated then the subspaces A_i are finite-dimensional and the Hilbert series $H(A,t)$ of A is defined. Note that the Hilbert series $H(A^!,t)$ is also defined for the dual algebra $A^!$.

Recall that if A and $A^!$ are Koszul algebras then $H(A,t)H(A^!,-t) = 1$ (see [14]). The converse is not true but the counter-examples are rather superficial (see [12,13]).

The following two theorems, which are the main results of this paper, show that the quadratic algebras Q_n satisfy this necessary condition for the Koszulity of Q_n.

Theorem 1. $H(Q_n, t) = \dfrac{1-t}{1-t(2-t)^n}$.

Theorem 2. $H(Q_n^!, t) = \dfrac{1+t(2+t)^n}{1+t}$.

In the course of proving Theorem 1 we develop results (cf. Lemma 4.4) which describe the structure of Q_n in terms of Q_{n-1}. These results appear to be of independent interest. We use these results to compute (Corollary 4.9) the Hilbert series of Q_n in terms of the Hilbert series of Q_{n-1}. While proving Theorem 2 we determine (Proposition 6.4) a basis for the dual algebra $Q_n^!$.

We begin, in Section 1, by recalling, from [9], the construction of Q_n (as a quotient of the tensor algebra $T(V)$ for an appropriate vector space V) and developing notation for certain important elements of $T(V)$. We also note that Q_n has a natural filtration. In Section 2 we study the associated graded algebra $\operatorname{gr} Q_n$, obtaining a presentation for $\operatorname{gr} Q_n$. In view of the basis theorem for Q_n (in [9]) it is easy to determine a basis for $\operatorname{gr} Q_n$. We next, in Section 3, define certain important subalgebras of Q_n which we denote $Q_n(1)$ and $Q_n(\hat{1})$. We show that the structures of these algebras are closely related to the structure of Q_{n-1}. In Section 4 we use these facts to prove Theorem 1 by induction on n. We then begin the study of the dual algebra $Q_n^!$, recalling generalities about the algebra and finding the space of defining relations in Section 5 and constructing a basis for $Q_n^!$ in Section 6. The proof of Theorem 2, contained in Section 7, is then straightforward.

1. Generalities about Q_n

The quadratic algebra Q_n is defined in [9]. Here we recall one presentation of Q_n and develop some notation. Let V denote the vector space over a field F with basis $\{v(A) \mid \emptyset \neq A \subseteq \{1, \ldots, n\}\}$ and $T(V)$ denote the tensor algebra on V. The symmetric group on $\{1, \ldots, n\}$ acts on V by $\sigma(v(A)) = v(\sigma(A))$ and hence also acts on $T(V)$.

Note that

$$T(V) = \sum_{i \geqslant 0} T(V)_i$$

where

$$T(V)_i = \operatorname{span}\bigl\{v(A_1) \ldots v(A_i) \bigm| \emptyset \neq A_1, \ldots, A_i \subseteq \{1, \ldots, n\}\bigr\}$$

is a graded algebra. Each $T(V)_i$ is finite-dimensional.

Also, defining

$$T(V)_{(j)} = \operatorname{span}\bigl\{v(A_1)\dots v(A_i) \bigm| i \geqslant 0,\ |A_1| + \cdots + |A_i| \leqslant j\bigr\}$$

gives an increasing filtration

$$F.1 = T(V)_{(0)} \subset T(V)_{(1)} \subset \cdots$$

of $T(V)$.

Note that

$$T(V)_{(j)} = \sum_{i \geqslant 0} T(V)_i \cap T(V)_{(j)}.$$

Let $\emptyset \neq B \subseteq A \subseteq \{1,\dots,n\}$ and write $B = \{b_1,\dots,b_k\}$ where $b_1 > b_2 > \cdots > b_k$. Let $\operatorname{Sym}(B)$ denote the group of all permutations of B. When convenient we will write $A \setminus b_1 \dots \setminus b_k$ in place of $A \setminus \{b_1,\dots,b_k\}$. Define $\mathcal{V}(A:B)$ to be

$$\sum_{\sigma \in \operatorname{Sym}(B)} \operatorname{sgn}(\sigma)\sigma \Biggl\{ v(A)v(A\setminus b_1)v(A\setminus b_1\setminus b_2)\dots v(A\setminus b_1 \dots \setminus b_{k-1}) \\ + \sum_{u=1}^{k-1} (-1)^u \bigl\{ v(A\setminus b_1)\dots v(A\setminus b_1\dots\setminus b_{u-1}) v(A\setminus b_1 \dots \setminus b_u)^2 \\ \times v(A\setminus b_1 \dots \setminus b_{u+1}) \dots v(A\setminus b_1 \dots \setminus b_{k-1}) \bigr\} \\ + (-1)^k v(A\setminus b_1)\dots v(A\setminus b_1 \dots \setminus b_k) \Biggr\}.$$

Let $\mathcal{Q} = \operatorname{span}\{\mathcal{V}(A:B) \mid B \subseteq A \subseteq \{1,\dots,n\},\ |B| = 2\}$ and let $\langle\mathcal{Q}\rangle$ denote the ideal in $T(V)$ generated by $\mathcal{Q}$. Denote the quotient $T(V)/\langle\mathcal{Q}\rangle$ by Q_n. Since $\mathcal{Q} \subseteq T(V)_2$, Q_n is a quadratic algebra. Q_n is, of course, graded:

$$Q_n = \sum_{i \geqslant 0} Q_{n,i}, \quad \text{where } Q_{n,i} = \bigl(T(V)_i + \langle\mathcal{Q}\rangle\bigr)\big/\langle\mathcal{Q}\rangle.$$

Defining

$$Q_{n,(j)} = \bigl(T(V)_{(j)} + \langle\mathcal{Q}\rangle\bigr)\big/\langle\mathcal{Q}\rangle$$

gives an increasing filtration

$$F.1 = Q_{n,(0)} \subset Q_{n,(1)} \subset \cdots$$

of Q_n. Note that

$$Q_{n,(j)} = \sum_{i \geqslant 0} Q_{n,i} \cap Q_{n,(j)}.$$

Let $r(A)$ denote $v(A) + \langle\mathcal{Q}\rangle$ and $\mathcal{R}(A:B)$ denote $\mathcal{V}(A:B) + \langle\mathcal{Q}\rangle$.

Note that if $|B| = 2$ then $\mathcal{R}(A:B) = 0$ (in Q_n).

2. The associated graded algebra gr Q_n

Let $\mathcal{X} = \operatorname{span}\{v(A)(v(A \setminus i) - v(A \setminus j)) \mid i, j \in A \subseteq \{1, \ldots, n\}\}$.

$$X_n = \bigl(T(V) + \langle \mathcal{X} \rangle\bigr)/\langle \mathcal{X} \rangle.$$

Let $x(A)$ denote $v(A) + \langle \mathcal{X} \rangle$.

Note that X_n is graded

$$X_n = \sum_{i \geqslant 0} X_{n,i}, \quad \text{where } X_{n,i} = \bigl(T(V)_i + \langle \mathcal{X} \rangle\bigr)/\langle \mathcal{X} \rangle$$

and has an increasing filtration

$$F.1 = X_{n,(0)} \subset X_{n,(1)} \subset \cdots \quad \text{where } X_{n,(j)} = \bigl(T(V)_{(j)} + \langle \mathcal{X} \rangle\bigr)/\langle \mathcal{X} \rangle.$$

A *string* is a finite sequence $\mathcal{B} = (B_1, \ldots, B_l)$ of nonempty subsets of $\{1, \ldots, n\}$. We call $l = l(\mathcal{B})$ the *length* of $\mathcal{B}$ and $|\mathcal{B}| = \sum_{i=1}^{l} |B_i|$ the *degree* of $\mathcal{B}$. Let S denote the set of all strings. If $\mathcal{B} = (B_1, \ldots, B_l)$ and $\mathcal{C} = (C_1, \ldots, C_m) \in S$ define $\mathcal{BC} = (B_1, \ldots, B_l, C_1, \ldots, C_m)$ and $x(\mathcal{B}) = x(B_1) \ldots x(B_l)$. For any set $W \subseteq S$ of strings we will denote $\{x(\mathcal{B}) \mid \mathcal{B} \in W\}$ by $x(W)$. Note that S contains the empty string $\emptyset$. Let $S^\circ = S \setminus \{\emptyset\}$. For any subset $U \subseteq S$ let $U^\circ = U \cap S^\circ$.

We recall from [9], the definition of $Y \subseteq S$. Let $\emptyset \neq A = \{a_1, \ldots, a_l\} \subseteq \{1, \ldots, n\}$ where $a_1 > a_2 > \cdots > a_l$ and $j \leqslant |A|$. Then we write $(A : j) = (A, A \setminus a_1, \ldots, A \setminus a_1 \setminus \cdots \setminus a_{j-1})$, a string of length j.

Consider the following condition on a string $(A_1 : j_1) \ldots (A_s : j_s) \in S$:

$$\text{if} \quad 2 \leqslant i \leqslant s \text{ and } A_i \subseteq A_{i-1} \quad \text{then} \quad |A_i| \neq |A_{i-1}| - j_{i-1}. \tag{2.1}$$

Let $Y = \{(A_1 : j_1) \ldots (A_s : j_s) \in S \mid (2.1) \text{ is satisfied}\}$. It is proved in [9] that $r(Y)$ is a basis for Q_n.

Suppose $\mathcal{B} = (B_1, \ldots, B_l)$ is a string. Recall, from [9], that we may define by induction a sequence of integers $n(\mathcal{B}) = (n_1, n_2, \ldots, n_t)$, $1 = n_1 < n_2 < \cdots < n_t = l + 1$, as follows:

- $n_1 = 1$,
- $n_{k+1} = \min(\{j > n_k \mid B_j \nsubseteq B_{n_k} \text{ or } |B_j| \neq |B_{n_k}| + n_k - j\} \cup \{l + 1\})$,
- and t is the smallest i such that $n_i = l + 1$.

We call $n(\mathcal{B})$ the *skeleton* of $\mathcal{B}$.

Let $\mathcal{B} = (B_1, \ldots, B_l)$ be a string with skeleton $(n_1 = 1, n_2, \ldots, n_t = l + 1)$. Define $\mathcal{B}^\vee$ to be the string $(B_{n_1}, n_2 - n_1)(B_{n_2}, n_3 - n_2) \ldots (B_{n_{t-1}}, n_t - n_{t-1})$. Note that $l(\mathcal{B}^\vee) = l(\mathcal{B})$ and $|\mathcal{B}^\vee| = |\mathcal{B}|$.

Proposition 2.1. $x(\mathcal{B}) = x(\mathcal{B}^\vee)$.

Proof. If $t = 1$ then $l = 0$ so $\mathcal{B} = \mathcal{B}^\vee$ is the empty string and $x(\mathcal{B}) = x(\mathcal{B}^\vee) = 1$. Assume $t = 2$, so $\mathcal{B}^\vee = (B_1, l)$. We will proceed by induction on l. If $l = 1$ then

$\mathcal{B} = (B_1) = \mathcal{B}^\vee$ so there is nothing to prove. If $l = 2$, then $\mathcal{B} = (B_1, B_1 \setminus i)$ for some i and $\mathcal{B}^\vee = (B_1, B_1 \setminus j)$ for some j. Since $x(B_1)x(B_1 \setminus i) = x(B_1)x(B_1 \setminus j)$ by the defining relations, the result holds in this case.

Now assume $l > 2$ and that the result holds for all $\mathcal{C} = (C_1, \ldots, C_k)$ with skeleton $(1, k+1)$ and $k < l$. We have $\mathcal{B} = (B_1, \ldots, B_{l-1})(B_l)$ so $x(\mathcal{B}) = x(B_1, \ldots, B_{l-1})x(B_l)$. Since the skeleton of $(B_1, \ldots, B_{l-1})$ is $(1, l)$ the induction assumption applies and shows that $x(B_1, \ldots, B_{l-1}) = x(B_1, l-1)$. Let b denote the largest element of B_1. Then since $(B_1, l-1) = (B_1)(B_1 \setminus b, l-2)$ we have $x(\mathcal{B}) = x(B_1, \ldots, B_{l-1})x(B_l) = x(B_1, l-1)x(B_l) = x(B_1)x(B_1 \setminus b, l-2)x(B_l)$. If $b \notin B_l$ the induction assumption shows that this is $x(B_1)x(B_1 \setminus b, l-1) = x(B_1, l)$ as required. So we may assume $b \in B_l$ and then, since $|B_l| < |B_1|$, we may find $c \in B_1$, $c \neq b$, $c \notin B_l$. Then by the induction assumption $x(B_1, l-1)$ $x(B_l) = x(B_1)x(B_1 \setminus c, l-2)x(B_l)$ and, again by the induction assumption, this is equal to $x(B_1)x(B_1 \setminus c, l-1)$.

Write $(B_1)(B_1 \setminus c, l-1) = (B_1, C_2, \ldots, C_l)$ and note that, as $l > 2$, the largest element of B_1 is not in C_l. Then by the previous case $x(B_1, C_2, \ldots, C_l) = x((B_1, C_2, \ldots, C_l)^\vee)$. But $x(\mathcal{B}) = x(B_1, C_2, \ldots, C_l)$ and $(B_1, C_2, \ldots, C_l)^\vee = (B_1, l)$ proving the result in case $t = 2$.

Finally, suppose $t > 2$ and suppose $n(\mathcal{B}) = (n_1, \ldots, n_t)$. We proceed by induction on t. Let $\mathcal{B}' = (B_1, \ldots, B_{n_2-1})$ and $\mathcal{B}'' = (B_{n_2}, \ldots, B_l)$. Note that $n(\mathcal{B}') = (1, n_2)$ and $n(\mathcal{B}'') = (n_2, \ldots, n_t)$, and so, by induction, $x(\mathcal{B}') = x(\mathcal{B}'^\vee)$ and $x(\mathcal{B}'') = x(\mathcal{B}''^\vee)$. Then $x(\mathcal{B}) = x(\mathcal{B}')x(\mathcal{B}'') = x(\mathcal{B}'^\vee)x(\mathcal{B}''^\vee) = x(\mathcal{B}^\vee)$, proving the proposition. □

Let $\operatorname{gr} Q_n$ denote the associated graded algebra of Q_n. For any string $\mathcal{B}$ let $\bar{r}(\mathcal{B})$ denote the element $r(\mathcal{B}) + Q_{n,|\mathcal{B}|-1}$ of $\operatorname{gr} Q_n$.

For any set S of strings write $\bar{r}(S) = \{\bar{r}(\mathcal{B}) \mid \mathcal{B} \in S\}$.

Lemma 2.2. *$\bar{r}(Y)$ is a basis for* $\operatorname{gr} Q_n$.

Proof. This follows from the fact that $r(Y) \cap Q_{n,i}$ is a basis for $Q_{n,i}$ (Theorem 1.3.8 and Proposition 1.4.1 of [9]). □

Corollary 2.3. *The linear map $\phi : X_n \to \operatorname{gr} Q_n$ defined by $\phi(x(\mathcal{B})) = \bar{r}(\mathcal{B})$ is an isomorphism of algebras.*

Proof. Since Q_n is generated by $\{r(A) \mid \emptyset \neq A \subseteq \{1, \ldots, n\}\}$, $\operatorname{gr} Q_n$ is generated by $\{\bar{r}(A) \mid \emptyset \neq A \subseteq \{1, \ldots, n\}\}$. If $i > j$

$$\begin{aligned} 0 &= \mathcal{R}\big(A : \{i, j\}\big) \\ &= r(A)\big(r(A \setminus i) - r(A \setminus j)\big) + \big(r(A \setminus i) - r(A \setminus j)\big)r(A \setminus i \setminus j) \\ &\quad - r(A \setminus i)^2 + r(A \setminus j)^2, \end{aligned}$$

we have

$$r(A)\big(r(A\setminus i)-r(A\setminus j)\big)\in Q_{n,2|A|-2}$$

and so $\bar r(A)(\bar r(A\setminus i)-\bar r(A\setminus j))=0$ in $\operatorname{gr} Q_n$. Consequently there is a homomorphism from X_n into $\operatorname{gr} Q_n$ that takes $x(A)$ into $\bar r(A)$. Since the generating set $\{\bar r(A)\mid \emptyset\neq A\subseteq\{1,\dots,n\}\}$ is contained in the image of this map, the map is onto. Note that $Y=\{\mathcal B\mid \mathcal B=\mathcal B^\vee\}$. Thus by Proposition 2.1, X_n is spanned by $x(Y)$. Since the image of this set is the linearly independent set $\bar r(Y)$, the map is injective. □

3. The subalgebras $Q_n(1)$ and $Q_n(\hat 1)$

Let $Q_n(\hat 1)$ denote the subalgebra of Q_n generated by $\{r(A)\mid \emptyset\neq A\subseteq\{2,\dots,n\}\}$. Let

$$\begin{aligned}
S(1)&=\big\{\mathcal B=(B_1,\dots,B_l)\in S\ \big|\ 1\in B_i \text{ for all } i\big\},\\
S(1)^\dagger&=\big\{\mathcal B=(B_1,\dots,B_l)\in S(1)\ \big|\ |B_i|>1 \text{ for all } i\big\},\quad\text{and}\\
S(\hat 1)&=\big\{\mathcal B=(B_1,\dots,B_l)\in S\ \big|\ B_1,\dots,B_l\subseteq\{2,\dots,n\}\big\}.
\end{aligned}$$

Let $Y(1)=Y\cap S(1)$, $Y(1)^\dagger=Y\cap S(1)^\dagger$, and $Y(\hat 1)=Y\cap S(\hat 1)$.

Let $Y_{(n-1)}$ denote $\{(B_1,\dots,B_l)\in Y(1)\mid B_1,\dots,B_l\subseteq\{1,\dots,n-1\}\}$.

Lemma 3.1. *Q_{n-1} is isomorphic to $Q_n(\hat 1)$.*

Proof. For any subset $A\subseteq\{1,\dots,n-1\}$, let $A+1$ denote $\{a+1\mid a\in A\}$, a subset of $\{2,\dots,n\}$. Clearly there is a homomorphism from Q_{n-1} into $Q_n(\hat 1)$ that takes $r(A)$ into $r(A+1)$. This map is injective since the "$r(Y)$-basis" for Q_{n-1} maps into a subset of $r(Y)\subseteq Q_n$. Since the generators for $Q_n(\hat 1)$ are contained in the image of this map, it is onto. □

Corollary 3.2. *$Y(\hat 1)$ is a basis for $Q_n(\hat 1)$.*

Let $Q_n(1)$ denote the subalgebra of Q_n generated by $\{r(A)\mid 1\in A\subseteq\{1,\dots,n\}\}$.

Lemma 3.3. *The map from $\operatorname{gr} Q_n(\hat 1)$ into $\operatorname{gr} Q_n(1)$ that takes $\bar r(A)$ into $\bar r(A\cup\{1\})$ is an injective homomorphism and $\bar r(Y(1)^\dagger)$ is a basis for the image.*

Proof. $\operatorname{gr} Q_n(\hat 1)$ has generators $\{\bar r(A)\mid \emptyset\neq A\subseteq\{2,\dots,n\}\}$ and relations $\{\bar r(A)(\bar r(A\setminus i)-\bar r(A\setminus j))\mid i,j\in A\subseteq\{2,\dots,n\}\}$. Since

$$\bar r\big(A\cup\{1\}\big)\big(\bar r\big(A\setminus i\cup\{1\}\big)-\bar r\big(A\setminus j\cup\{1\}\big)\big)=0\quad\text{in } \operatorname{gr} Q_n(1),$$

the required homomorphism exists. Since the homomorphism maps $\bar{r}(Y(\hat{1}))$ injectively to $\bar{r}(Y(1)^{\dagger})$, a subset of $\bar{r}(Y)$, the homomorphism is injective and $\bar{r}(Y(1)^{\dagger})$ is a basis for the image. □

Lemma 3.4. (a) $\bar{r}(Y(1))$ *is a basis for* $\operatorname{gr} Q_n(1)$.
(b) $r(Y(1))$ *is a basis for* $Q_n(1)$.

Proof. (a) Since $\bar{r}(Y(1)) \subseteq \bar{r}(Y)$ it is linearly independent. Hence it is sufficient to show that $\bar{r}(Y(1))$ spans $\operatorname{gr} Q_n(1)$. But $\operatorname{gr} Q_n(1)$ is spanned by the elements $\bar{r}(\mathcal{B})$ where $\mathcal{B} = (B_1, \dots, B_l)$, $1 \in B_1, \dots, B_l$. By Proposition 2.1 $\bar{r}(\mathcal{B}) = \bar{r}(\mathcal{B}^{\vee})$ where $(n_1, \dots, n_t)$ is the skeleton of $\mathcal{B}$ and

$$\mathcal{B}^{\vee} = (B_{n_1}, n_2 - n_1)(B_{n_2}, n_3 - n_2) \dots (B_{n_{t-1}}, n_t - n_{t-1}).$$

Since $1 \in B_j$ for each j, $\mathcal{B}^{\vee} \in Y(1)$ giving the result.
Part (b) is immediate from (a). □

If A and B are algebras, let $A * B$ denote the free product of A and B (cf. [1, Chapter 3, Section 5, Exercise 6]). Thus there exist homomorphisms $\alpha : A \to A * B$ and $\beta : B \to A * B$ such that if G is any associative algebra and $\mu : A \to G$, $\nu : B \to G$ are homomorphisms then there exists a unique homomorphism $\lambda : A * B \to G$ such that $\lambda\alpha = \mu$ and $\lambda\beta = \nu$. Furthermore, if A and B have identity element 1, $\{1\} \cup \Gamma_A$ is a basis for A and $\{1\} \cup \Gamma_B$ is a basis for B then $A * B$ has a basis consisting of 1 and all products $g_1 \dots g_m$ or $g_2 \dots g_{m+1}$ where $n \geqslant 1$ and $g_t \in \alpha(\Gamma_A)$ if t is even and $g_t \in \beta(\Gamma_B)$ if t is odd.

Lemma 3.5. $\operatorname{gr} Q_n(1)$ *is isomorphic to* $\operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$.

Proof. Let $\alpha : \operatorname{gr} Q_{n-1} \to \operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$ and $\beta : F[\bar{r}(1)] \to \operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$ be the homomorphisms occurring in the definition of $\operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$.

If $\emptyset \neq A = \{a_1, \dots, a_k\} \subseteq \{1, \dots, n-1\}$ define

$$\delta(A) = \{1, 1 + a_1, \dots, 1 + a_k\}.$$

Then define a map $\mu : \{\bar{r}(A) \mid \emptyset \neq A \subseteq \{1, \dots, n-1\}\} \to \operatorname{gr} Q_n(1)$ by

$$\mu\bigl(\bar{r}(A)\bigr) = \bar{r}\bigl(\delta(A)\bigr).$$

In view of Lemma 2.2, μ extends to a linear map

$$\mu : \operatorname{gr} Q_{n-1} \to \operatorname{gr} Q_n(1).$$

By Corollary 2.3, μ preserves the defining relations for $\operatorname{gr} Q_{n-1}$ and so is a homomorphism. Lemma 3.4 implies that μ is injective. Note that $\bar{r}(1) \in \operatorname{gr} Q_n(1)$ generates a subalgebra isomorphic to the polynomial algebra $F[\bar{r}(1)]$. Thus there is an injection

$$\nu : F\bigl[\bar{r}(1)\bigr] \to \operatorname{gr} Q_n(1).$$

Consequently there is a homomorphism

$$\lambda : \operatorname{gr} Q_{n-1} * \Gamma\big[\bar{r}(1)\big] \to \operatorname{gr} Q_n(1)$$

such that $\lambda\alpha = \mu$ and $\lambda\beta = \nu$. We claim that λ is an isomorphism.

Let $\mathcal{T}$ denote the set of all strings $\mathcal{G}_1 \dots \mathcal{G}_n$ or $\mathcal{G}_2 \dots \mathcal{G}_{n+1}$ where $\mathcal{G}_i = \mathcal{B}_i \in Y(1)^\dagger$ if i is odd and $\mathcal{G}_i = \{1\}^{j_i}$ if i is even. Note that $\mathcal{T} \subseteq S(1)$. Define

$$\Phi : \mathcal{T} \to Y(1)$$

by $\Phi(\mathcal{B}) = \mathcal{B}^\vee$. Define $\Psi : Y(1) \to \mathcal{T}$ by $\Psi((A, j)) = (A, j)$ if $j < |A|$, $\Psi((A, j)) = (A, j-1)\{1\}$ if $j = |A|$, and

$$\Psi\big((A_1, j_1) \dots (A_s, j_s)\big) = \Psi\big((A_1, j_1)\big) \dots \Psi\big((A_s, j_s)\big)$$

if $(A_1, j_1) \dots (A_s, j_s)$ satisfies (2.1). Then Φ and Ψ are inverse mappings.

Let $\gamma_i = \alpha$ if i is odd and $\gamma_i = \beta$ if i is even. Then $\operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$ has basis consisting of 1 and all products $\gamma_1\bar{r}(\mathcal{H}_1) \dots \gamma_m\bar{r}(\mathcal{H}_m)$ or $\gamma_2\bar{r}(\mathcal{H}_2) \dots \gamma_m\bar{r}(\mathcal{H}_{m+1})$ where $\mathcal{H}_i \in Y_{(n-1)}$ if i is odd and $\mathcal{H}_i = \{1\}^{j_i}$ if i is even. Then

$$\begin{aligned}\lambda\big(\gamma_1\bar{r}(\mathcal{H}_1) \dots \gamma_m\bar{r}(\mathcal{H}_m)\big) &= \bar{r}\big(\delta(\mathcal{H}_1)\bar{r}(\mathcal{H}_2) \dots\big) = \bar{r}\big(\delta(\mathcal{H}_1)\mathcal{H}_2 \dots\big) \\ &= \bar{r}\big(\big(\delta(\mathcal{H}_1)\mathcal{H}_2 \dots\big)^\vee\big)\end{aligned}$$

and $\delta(\mathcal{H}_1)\mathcal{H}_2 \dots \in \mathcal{T}$. Also, $\lambda(\gamma_2\bar{r}(\mathcal{H}_2) \dots \gamma_{m+1}\bar{r}(\mathcal{H}_{m+1}) = \bar{r}(\mathcal{H}_2\delta(\mathcal{H}_3) \dots) = \bar{r}(\mathcal{H}_2\delta(\mathcal{H}_3) \dots)^\vee)$ and $\mathcal{H}_2\delta(\mathcal{H}_3) \dots \in \mathcal{T}$. Every element of $\mathcal{T}$ arises in this way. Since $\Phi : \mathcal{T} \to Y(1)$ is a bijection, we see that λ maps a basis of $\operatorname{gr} Q_{n-1} * F[\bar{r}(1)]$ bijectively onto the basis $\bar{r}(Y(1))$ of $\operatorname{gr} Q_n(1)$, proving the lemma. □

4. Proof of Theorem 1

Let $\theta : S \times S \to S$ be defined by

$$\theta\big((B_1, \dots, B_l), (C_1, \dots, C_k)\big) = (B_1, \dots, B_l, C_1, \dots, C_k).$$

Lemma 4.1. *If $\mathcal{B} = (B_1, \dots, B_l)$, $\mathcal{C} = (C_1, \dots, C_k) \in Y$, $1 \notin B_l$ and $1 \in C_1$, then $\mathcal{BC} \in Y$.*

Proof. Since $\mathcal{B} \in Y$ we may write $\mathcal{B} = (A_1, j_1) \dots (A_s, j_s)$ where (2.1) is satisfied. Since $1 \notin B_l$ we have $1 \notin A_s$. Similarly since $\mathcal{C} \in Y$ we may write $\mathcal{C} = (D_1, m_1) \dots (D_t, m_t)$ where condition (2.1) holds. Since $1 \in C_1$ we have $1 \in D_1$. Then $\mathcal{BC} = (A_1, j_1) \dots (A_s, j_s)(D_1, m_1) \dots (D_t, m_t)$. Since (2.1) holds for $\mathcal{B}$ and $\mathcal{C}$, and since $D_1 \nsubseteq A_s$ (for $1 \in D_1$, $1 \notin A_s$), (2.1) is satisfied for $\mathcal{BC}$ and so $\mathcal{BC} \in Y$. □

Let $\mathcal{B} = (B_1, \dots, B_l) \in S$. Define $A(\mathcal{B}) = \{i \mid 1 \leqslant i \leqslant l-1, 1 \in B_i, 1 \notin B_{i+1}\}$ and $a(\mathcal{B}) = |A(\mathcal{B})|$. Set $S_{\{i\}} = \{\mathcal{B} \in S \mid a(\mathcal{B}) = i\}$. Then S is equal to the disjoint union $\bigcup_{i \geqslant 0} S_{\{i\}}$.

Lemma 4.2. $(SS(1)^\circ \cap S_{\{0\}}) \times (S(\hat{1})^\circ S \cap S_{\{i\}})$ *injects into* $S_{\{i+1\}}$.

Proof. Let $\mathcal{B} = (B_1, \ldots, B_l) \in S_{\{i+1\}}$. There are $l + 1$ pairs in $S \times S$ which θ maps to $\mathcal{B}$, namely $(B_1, \ldots, B_j) \times (B_{j+1}, \ldots, B_l)$ for $0 \leqslant j \leqslant l$. Now $(B_1, \ldots, B_j) \in S_{\{0\}}$ implies $j \leqslant \min A(\mathcal{B})$ while $(B_1, \ldots, B_j) \in SS(1)^\circ$ and $(B_{j+1}, \ldots, B_l) \in S(\hat{1})^\circ S$ implies $j \in A(\mathcal{B})$ and the lemma follows. □

Let $L(1) = S(1)^\circ \times S(\hat{1})^\circ$, and $L(i + 1) = L(1) \times L(i)$, $i \geqslant 1$.

Corollary 4.3. $\bigcup_{i \geqslant 0} S(\hat{1}) \times L(i) \times S(1)$ *injects into* S.

Proof. The ith term in the union maps into $S_{\{i\}}$, so it is enough to prove that this is an injection. Write this term as $(S(\hat{1}) \times S(1)^\circ) \times (S(\hat{1})^\circ \times L(i - 1) \times S(1))$ and observe that the result follows by the lemma and by induction on i. □

Let M denote the span of $Y(1)^\circ Y(\hat{1})^\circ \cap Y$ and let N denote the subalgebra of Q_n generated by M.

Lemma 4.4. *N is isomorphic to the free algebra generated by M and the map*

$$Q_n(\hat{1}) \otimes N \otimes Q_n(1) \to Q_n$$

induced by multiplication is an isomorphism of graded vector spaces.

Proof. Let $W = Y(1)^\circ Y(\hat{1})^\circ \cap Y$ and let $W^i = W \times \cdots \times W$ (i times). Then W, being linearly independent, is a basis for M. By Lemma 4.2, $\bigcup_{i \geqslant 0} W^i$ injects into S. Indeed, Lemma 4.1 shows that the image is in Y. Thus $\bigcup_{i \geqslant 0} W^i$ injects onto a basis for N, so N is isomorphic to the free algebra generated by M. Again by Lemma 4.1 we have that $\bigcup_{i \geqslant 0} Y(\hat{1}) \times W^i \times Y(1)$ maps into Y. Since any substring of an element of Y is again in Y, this map is onto. By Corollary 4.3 the map is an injection. This proves the final statement of the lemma. □

We now recall some well-known facts about Hilbert series (cf. [14, Section 3.3]).

Lemma 4.5. (a) *If W_1 and W_2 are graded vector spaces then*

$$H(W_1 \otimes W_2, t) = H(W_1, t)H(W_2, t).$$

(b) *If W is a graded vector space, then*

$$H\bigl(T(W), t\bigr) = \frac{1}{1 - H(W, t)}.$$

(c) *If* $A = \sum_{i \geqslant 0} A_i$ *and* $B = \sum_{i \geqslant 0} B_i$ *are graded algebras with* $A_0 = B_0 = F.1$, *then*

$$\frac{1}{H(A * B, t)} = \frac{1}{H(A, t)} + \frac{1}{H(B, t)} - 1.$$

Let $U(A : j)$ denote the span of all strings $(A_1 : j_1) \ldots (A_s : j_s)$ satisfying (2.1) such that $1 \in A_i$ for all i and $(A_s : j_s) = (A, j)$.

Lemma 4.6. $H(U(A : j), t) = t^j (1 - t)^{n-|A|} H(Q_n(1), t)$.

Proof. Since whenever the string $(A_1 : j_1) \ldots (A_{s-1} : j_{s-1})$ satisfies (2.1) then the string $(A_1 : j_1) \ldots (A_{s-1} : j_{s-1})(\{1, \ldots, n\}, j)$ also satisfies (2.1), we have

$$H\big(U\big(\{1, \ldots, n\}, j\big), t\big) = t^j H\big(Q_n(1), t\big).$$

We now proceed by downward induction on $|A|$, assuming the result is true whenever $|A| > l$. Let $|A| = l$. Then

$$H\big(U(A : j), t\big) = t^j H\big(Q_n(1), t\big) - t^j \sum_{C \supseteq A,\, |C| = |A| + m,\, m \geqslant 1} H\big(U(C : m), t\big).$$

By the induction assumption this is

$$t^j \left(1 - \sum_{C \supseteq A,\, |C| = |A| + m,\, m \geqslant 1} t^m (1 - t)^{n-|C|}\right) H\big(Q_n(1), t\big).$$

Let $C = D \cup A$ where $D \subseteq \{1, \ldots, n\} \setminus A$. Then the expression becomes

$$t^j \left(1 - \sum_{\emptyset \neq D \subseteq \{1, \ldots, n\} \setminus A} t^{|D|} (1 - t)^{n-|A|-|D|}\right) H\big(Q_n(1), t\big).$$

By the binomial theorem the quantity in parenthesis is $(1 - t)^{n-|A|}$, proving the result. □

Let $B \subseteq \{2, \ldots, n\}$ and let $Z(B)$ denote the span of all strings $(A_1 : j_1) \ldots (A_s : j_s)$ such that $1 \in A_1, \ldots, A_s$, $(A_1 : j_1) \ldots (A_s : j_s)$ satisfies (2.1), $|B| = |A_s| - j_s$, $A_s \supseteq B$.

Lemma 4.7. $H(Z(B), t) = t H(Q_n(1), t)$.

Proof. Write $A_s = B \cup E \cup \{1\}$ where $B \cap E = \emptyset$ and $E \subseteq \{2, \ldots, n\}$. Then

$$Z(B) = \sum_{E \subseteq \{2, \ldots, n\} \setminus B} U\big(B \cup E \cup \{1\}, |E| + 1\big)$$

and so

$$H(Z(B), t) = \sum_{E \subseteq \{2, \ldots, n\} \setminus B} t^{|E|+1} (1 - t)^{n-|B|-|E|-1} H\big(Q_n(1), t\big).$$

By the binomial theorem this is $t H(Q_n(1), t)$. □

Lemma 4.8. $H(M,t) = (H(Q_n(\hat{1}),t) - 1)(H(Q_n(1),t) - 1) - tH(Q_n(1),t) \times (H(Q_n(\hat{1}),t) - 1)$.

Proof. M is the span of $Y(1)^\circ Y(\hat{1})^\circ \cap Y$. The complement $\mathcal{Z}$ of $Y(1)^\circ Y(\hat{1})^\circ \cap Y$ in $Y(1)^\circ Y(\hat{1})^\circ$ is the set of all strings $(A_1 : j_1)\ldots(A_s : j_s)(B_1,\ldots,B_l)$ such that $(A_1 : j_1)\ldots(A_s : j_s) \in Y(1)^\circ$ satisfies (2.1), $|B_1| = |A_s| - j_s$, $A_s \supseteq B_1$, $(B_1,\ldots,B_l) \in Y(\hat{1})$. Let Z denote the span of $\mathcal{Z}$. The lemma follows from showing that

$$H(Z,t) = tH\big(Q_n(1),t\big)\big(H\big(Q_n(\hat{1}),t\big) - 1\big).$$

For $\emptyset \neq B \subseteq \{2,\ldots,n\}$ let $P(B)$ denote the span of all strings in $\mathcal{Z}$ such that $B_1 = B$ and $P_0(B)$ denote the span of all strings $(B_1,\ldots,B_l) \in Y(\hat{1})$ such that $B_1 = B$. Then $Z = \sum_{\emptyset \neq B \subseteq \{2,\ldots,n\}} P(B)$ and $H(P(B),t) = H(Z(B),t)H(P_0(B),t)$. By Lemma 4.7, this equals to $tH(Q_n(1),t)H(P_0(B),t)$. Thus

$$\begin{aligned} H(Z,t) &= \sum_{\emptyset \neq B \subseteq \{2,\ldots,n\}} H\big(P(B),t\big) \\ &= \sum_{\emptyset \neq B \subseteq \{2,\ldots,n\}} tH\big(Q_n(1),t\big)H\big(P_0(B),t\big) \\ &= tH\big(Q_n(1),t\big) \sum_{\emptyset \neq B \subseteq \{2,\ldots,n\}} H\big(P_0(B),t\big). \end{aligned}$$

But $\sum_{\emptyset \neq B \subseteq \{2,\ldots,n\}} H(P_0(B),t) = H(Q_n(\hat{1}),t) - 1$ and the lemma is proved. □

Corollary 4.9. $\dfrac{1}{H(Q_n,t)} = (2-t)\dfrac{1}{H(Q_{n-1},t)} - 1.$

Proof. $H(Q_n,t) = H(Q_n(\hat{1}),t)H(N,t)H(Q_n(1),t)$ by Lemma 4.4. For brevity we write $H(Q_n(\hat{1}),t) = a$ and $H(Q_n(1),t) = b$. Then

$$H(N,t) = \frac{1}{1 - H(M,t)} = \frac{1}{(1-t)b + a + (t-1)ab},$$

and so

$$\frac{1}{H(Q_n,t)} = \frac{(1-t)b + a + (t-1)ab}{ab} = \frac{1-t}{a} + \frac{1}{b} + t - 1.$$

Since

$$\frac{1}{b} = \frac{1}{a} - t,$$

(by Lemmas 3.5 and 4.5(c)) this gives

$$\frac{1}{H(Q_n,t)} = \frac{1-t}{a} + \frac{1}{a} - 1 = \frac{2-t}{a} - 1.$$

Now $a = H(Q_n(\hat{1}),t) = H(Q_{n-1},t)$, so the corollary follows. □

Theorem 1 now follows from Corollary 4.9 and the fact that $Q_0 = F$.

5. Generalities about the dual algebra $Q_n^!$

Let V^* denote the dual space of V. Thus V^* has basis

$$\{v^*(A) \mid \emptyset \neq A \subseteq \{1, \dots, n\}\} \quad \text{where } \langle v(A), v^*(B)\rangle = \delta_{A,B}.$$

Note that $T(V^*) = \sum_{i \geqslant 0} T(V^*)_i$ is a graded algebra where $T(V^*)_i = \operatorname{span}\{v^*(A_1) \dots v^*(A_i) \mid \emptyset \neq A_1, \dots, A_i \subseteq \{1, \dots, n\}\}$. Also, $T(V^*)$ has a decreasing filtration

$$T(V^*) = T(V^*)_{(0)} \supset T(V^*)_{(1)} \supset \cdots \supset T(V^*)_{(j)} \supset \cdots$$

where

$$T(V^*)_{(j)} = \operatorname{span}\{v^*(A_1) \dots v^*(A_i) \mid |A_1| + \cdots + |A_i| \geqslant j\}.$$

In fact

$$T(V^*)_{(j+1)} = \left(T(V)_{(j)}\right)^{\perp} \quad \text{for } j \geqslant 0.$$

Define $Q_n^! = T(V^*)/\langle \mathcal{Q}^{\perp}\rangle$.

We may explicitly describe $\mathcal{Q}^{\perp}$ and thus give a presentation of $Q_n^!$. To do this define the following subsets of $T(V^*)_2$:

$$S_1 = \{v^*(A)v^*(B) \mid B \not\subseteq A \text{ or } |B| \neq |A|,\ |A| - 1\},$$

$$S_2 = \left\{v^*(C)\left(\sum_{i \in C} v^*(C \setminus i)\right) + v^*(C)^2 \,\middle|\, |C| \geqslant 2\right\},$$

$$S_3 = \left\{\left(\sum_{i \notin C} v^*(C \cup i)v^*(C)\right) + v^*(C)^2 \,\middle|\, C \neq \{1, \dots, n\}\right\},$$

$$S_4 = \{v^*(\{1, \dots, n\})^2\}.$$

Theorem 5.1. *$S_1 \cup S_2 \cup S_3 \cup S_4$ spans $\mathcal{Q}^{\perp}$. Therefore, $Q_n^!$ is presented by generators $\{v^*(A) \mid \emptyset \neq A \subseteq \{1, \dots, n\}\}$ and relations $S_1 \cup S_2 \cup S_3 \cup S_4$.*

Before beginning the proof of this theorem, we present some examples and develop some notation. Set

$$s(A) = \left(v^*(A) + \langle \mathcal{Q}^{\perp}\rangle\right) / \langle \mathcal{Q}^{\perp}\rangle \in Q_n^!.$$

Write $s(i)$ for $s(\{i\})$, $s(ij)$ for $s(\{i, j\})$, etc.

Example. (a) $Q_2^!$ is 5-dimensional with basis

$$\{1,\ s(1),\ s(2),\ s(12),\ s(12)s(1)\}.$$

(b) $Q_3^!$ is 14-dimensional with basis

$$\big\{1,\ s(1),\ s(2),\ s(3),\ s(12),\ s(13),\ s(23),\ s(123),\ s(123)s(12),\\ s(123)s(13),\ s(12)s(1),\ s(13)s(1),\ s(23)s(2),\ s(123)s(12)s(1)\big\}.$$

These assertions follow from Proposition 6.4.

Note that $Q_n^! = \sum_{i\geqslant 0} Q_{n,i}^!$ is graded where

$$Q_{n,i}^! = \big(T(V^*)_i + \langle \mathcal{Q}^\perp\rangle\big)/\langle \mathcal{Q}^\perp\rangle$$

and that $Q_n^!$ has a decreasing filtration

$$Q_n^! = Q_{n,(0)}^! \supseteq Q_{n,(1)}^! \supseteq \cdots \supseteq Q_{n,(j)}^! \supseteq \cdots$$

where

$$Q_{n,(j)}^! = \big(T(V^*)_{(j)} + \langle \mathcal{Q}^\perp\rangle\big)/\langle \mathcal{Q}^\perp\rangle.$$

Clearly

$$Q_{n,(j)}^! = \sum_{i\geqslant 0} Q_{n,i}^! \cap Q_{n,(j)}^! \quad \text{and} \quad Q_{n,i}^! \cap Q_{n,(j)}^! = (0) \quad \text{if } j > ni.$$

Let $\operatorname{gr} Q_n^! = \sum_{j=0}^{\infty} Q_{n,(j)}^!/Q_{n,(j+1)}^!$, the associated graded algebra of $Q_n^!$. Denote $s(A) + Q_{n,(|A|+1)}^! \in \operatorname{gr} Q_n^!$ by $\bar{s}(A)$. Then $\{\bar{s}(A) \mid \emptyset \neq A \subseteq \{1,\dots,n\}\}$ generates $\operatorname{gr} Q_n^!$.

Proof of Theorem 5.1. We first show that each S_h, $1 \leqslant h \leqslant 4$, is contained in $\mathcal{Q}^\perp$, i.e., that $\langle \mathcal{V}(A\colon \{c,d\}), u_h\rangle = 0$ whenever $c < d, c, d \in A \subseteq \{1,\dots,n\}$ and $u_h \in S_h$. For $h = 1$ or 4 this is clear.

If $h = 2$, we note that

$$\left\langle \mathcal{V}\big(A:\{c,d\}\big), v^*(C)\left(\sum_{i\in C} v^*(C\setminus i)\right) + v^*(C)^2\right\rangle = 0$$

unless $A = C$, $A\setminus c = C$, or $A \setminus d = C$. In the first case,

$$\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ v^*(C)\left(\sum_{i\in C} v^*(C\setminus i)\right) + v^*(C)^2\right\rangle \\ = \left\langle v(A)v(A\setminus d) - v(A)v(A\setminus c),\ v^*(A)\left(\sum_{i\in A} v^*(A\setminus i)\right)\right\rangle = 0.$$

In the second case,

$$\begin{aligned}
&\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ v^*(C)\left(\sum_{i\in C} v^*(C\setminus i)\right) + v^*(C)^2 \right\rangle \\
&\quad = \left\langle -v(A\setminus c)v(A\setminus c\setminus d),\ v^*(A\setminus c)\left(\sum_{i\in A\setminus c} v^*(A\setminus c\setminus i)\right)\right\rangle \\
&\qquad + \left\langle v(A\setminus c)^2,\ v^*(A\setminus c)^2\right\rangle \\
&\quad = -1+1=0.
\end{aligned}$$

In the third case,

$$\begin{aligned}
&\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ v^*(C)\left(\sum_{i\in C} v^*(C\setminus i)\right) + v^*(C)^2 \right\rangle \\
&\quad = \left\langle v(A\setminus d)v(A\setminus c\setminus d),\ v^*(A\setminus d)\left(\sum_{i\in A\setminus d} v^*(A\setminus d\setminus i)\right)\right\rangle \\
&\qquad + \left\langle -v(A\setminus d)^2,\ v^*(A\setminus d)^2\right\rangle \\
&\quad = 1-1=0.
\end{aligned}$$

If $h=3$, we note that

$$\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ \left(\sum_{i\notin C} v^*(C\cup i)v^*(C)\right) + v^*(C)^2 \right\rangle = 0$$

unless $A\setminus c = C$, $A\setminus d = C$, or $A\setminus c\setminus d = C$. In the first case,

$$\begin{aligned}
&\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ \left(\sum_{i\notin C} v^*(C\cup i)v^*(C)\right) + v^*(C)^2 \right\rangle \\
&\quad = \left\langle -v(A)v(A\setminus c),\ \sum_{i\notin A\setminus c} v^*(A\cup i\setminus c)v^*(A\setminus c)\right\rangle \\
&\qquad + \left\langle v(A\setminus c)^2,\ v^*(A\setminus c)^2\right\rangle \\
&\quad = -1+1=0.
\end{aligned}$$

In the second case,

$$\begin{aligned}
&\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ \left(\sum_{i\notin C} v^*(C\cup i)v^*(C)\right) + v^*(C)^2 \right\rangle \\
&\quad = \left\langle v(A)v(A\setminus d),\ \sum_{i\notin A\setminus d} v^*(A\cup i\setminus d)v^*(A\setminus d)\right\rangle \\
&\qquad + \left\langle -v(A\setminus d)^2,\ v^*(A\setminus d)^2\right\rangle \\
&\quad = 1-1=0.
\end{aligned}$$

In the third case,

$$\left\langle \mathcal{V}\big(A:\{c,d\}\big),\ \left(\sum_{i\notin C} v^*(C\cup i)v^*(C)\right) + v^*(C)^2\right\rangle$$
$$= \Big\langle v(A\setminus d)v(A\setminus c\setminus d) - v(A\setminus c)v(A\setminus c\setminus d),$$
$$\sum_{i\notin A\setminus c\setminus d} v^*(A\cup i\setminus c\setminus d)v^*(A\setminus c\setminus d)\Big\rangle = 0.$$

We will now use downward induction on l to show that $(S_1\cup S_2\cup S_3\cup S_4)\cap T(V^*)_{(l)}$ spans $\mathcal{Q}^\perp\cap T(V^*)_{(l)}$ for all $l\geqslant 0$. Note that, since $T(V^*)=T(V^*)_{(0)}$, this will complete the proof of the lemma. Now $\mathcal{Q}^\perp$ is contained in $T(V^*)_2$ and $T(V^*)_2\cap T(V^*)_{(2n+1)}=(0)$, so the result holds for $l=2n+1$. Assume the result holds whenever $l>m$ and let $u\in\mathcal{Q}^\perp\cap T(V^*)_{(m)}$. Suppose that m is even. Then by subtracting an element in the span of S_1 we may assume that $u\in\sum_{|C|=m/2} a_C v^*(C)^2 + T(V^*)_{(m+1)}$ for some scalars a_C. Then by subtracting an element in the span of $S_3\cup S_4$ we may assume that $u\in T(V^*)_{(m+1)}$. Hence the induction assumption gives our result in this case. Now suppose that m is odd. Then by subtracting an element in the span of S_1 we may assume that

$$u\in \sum_{|C|=(m+1)/2,\, i\in C} b_{C,i}v^*(C)v^*(C\setminus i) + T(V^*)_{(m+1)}$$

for some scalars $b_{C,i}$. Since $0=\langle\mathcal{V}(C:\{c,d\}),u\rangle$ for all C with $|C|=(m+1)/2$ we see that $b_{C,c}=b_{C,d}$ for all $c,d\in C$. Then by subtracting an element in the span of S_2 we may assume that $u\in T(V^*)_{(m+1)}$. Hence the induction assumption gives our result in this case and the proof of the lemma is complete. □

6. A basis for $Q_n^!$

Let $B=\{b_1,\ldots,b_k\}\subseteq A\subseteq\{1,\ldots,n\}$ with $b_1>\cdots>b_k$. Define $\mathcal{S}(A:B)\in Q_n^!$ by

$$\mathcal{S}(A:B) = s(A)s(A\setminus b_1)\ldots s(A\setminus b_1\ldots\setminus b_k).$$

Let $\min A$ denote the smallest element of A. Define

$$\mathcal{S}=\big\{\mathcal{S}(A:B)\ \big|\ B\subseteq A\subseteq\{1,\ldots,n\},\ \min A\notin B\big\},$$
$$\overline{\mathcal{S}}(A:B)=\mathcal{S}(A:B)+Q^!_{n,(1+(|B|+1)(2|A|-|B|)/2)},$$

and

$$\overline{\mathcal{S}}=\big\{\overline{\mathcal{S}}(A:B)\ \big|\ B\subseteq A\subseteq\{1,\ldots,n\},\ \min A\notin B\big\}.$$

Lemma 6.1. $\overline{\mathcal{S}}\cup\{\bar{s}(\emptyset)\}$ *spans* $\operatorname{gr} Q_n^!$ *and* $\mathcal{S}\cup\{s(\emptyset)\}$ *spans* $Q_n^!$.

Proof. It is sufficient to show the first assertion. We know that $\{\bar{s}(A) \mid \emptyset \neq A \subseteq \{1,\dots,n\}\}$ generates $\operatorname{gr} Q_n^!$. Since the sets S_1, S_3, and S_4 are contained in $\mathcal{Q}^\perp$, we see that $\bar{s}(A)\bar{s}(B) = 0$ unless $B \subset A$ and $|B| = |A| - 1$. Furthermore, since the set S_2 is contained in $\mathcal{Q}^\perp$, we have $\bar{s}(A)(\sum_{i\in A} \bar{s}(A \setminus i)) = 0$ for all $A \subseteq \{1,\dots,n\}$, $|A| \geqslant 2$. Then if $i, j \in A \subseteq \{1,\dots,n\}$ we have

$$\begin{aligned}\bar{s}(A)\bar{s}(A\setminus i)\bar{s}(A\setminus i\setminus j) &= -\bar{s}(A)\Bigg(\sum_{l\in A,\, l\neq i} \bar{s}(A\setminus l)\Bigg)\bar{s}(A\setminus i\setminus j)\\ &= -\bar{s}(A)\bar{s}(A\setminus j)\bar{s}(A\setminus i\setminus j).\end{aligned}$$

The lemma is then immediate. □

Lemma 6.2. *Let $B = \{b_1,\dots,b_k\} \subseteq A \subseteq \{1,\dots,n\}$ with $b_1 > \cdots > b_k$ and $k > 2$. Then, for $0 \leqslant m \leqslant k-3$,*

$$\Bigg\langle \sum_{\sigma\in \mathrm{Sym}(B)} \operatorname{sgn}(\sigma)\sigma\big\{v(A\setminus b_1)\dots v(A\setminus b_1\dots\setminus b_{k-1})\big\},\\ V^{*m}\mathcal{Q}^\perp V^{*k-m-3}\Bigg\rangle = 0.$$

Proof. If $k = 3$ then

$$\begin{aligned}&\sum_{\sigma\in \mathrm{Sym}(B)} \operatorname{sgn}(\sigma)\sigma\big\{v(A\setminus b_1)\dots v(A\setminus b_1\dots\setminus b_{k-1})\big\}\\ &\quad= \mathcal{V}\big(A:\{b_1,b_2\}\big) + \mathcal{V}\big(A:\{b_2,b_3\}\big) + \mathcal{V}\big(A:\{b_3,b_1\}\big) \in \mathcal{Q},\end{aligned}$$

so the result holds. Now assume that $k > 3$ and that the result holds for $k-1$. Then it is sufficient to show that

$$\Bigg\langle \sum_{\sigma\in \mathrm{Sym}(B)} \operatorname{sgn}(\sigma)\sigma\big\{v(A\setminus b_1)\dots v(A\setminus b_1\dots\setminus b_{k-1})\big\},\\ v^*(A\setminus b_1)V^{*m-1}\mathcal{Q}^\perp V^{*k-m-3}\Bigg\rangle = 0$$

whenever $m > 0$ and that

$$\Bigg\langle \sum_{\sigma\in \mathrm{Sym}(B)} \operatorname{sgn}(\sigma)\sigma\big\{v(A\setminus b_1)\dots v(A\setminus b_1\dots\setminus b_{k-1})\big\},\\ \mathcal{Q}^\perp V^{*k-4}v^*(A\setminus b_1\setminus\dots\setminus b_{k-1})\Bigg\rangle = 0.$$

Both of these are immediate from the induction assumption. □

Lemma 6.3. *Let $B \subseteq A \subseteq \{1,\dots,n\}$ and $|B| = k \geqslant 2$. Then*

$$\mathcal{V}(A:B) \in \bigcap_{m=0}^{k-2} V^m \mathcal{Q} V^{k-m-2}.$$

Proof. It is enough to show that $\langle \mathcal{V}(A:B), V^{*m}\mathcal{Q}^\perp V^{*k-m-2}\rangle = 0$ for all m, $0 \leqslant m \leqslant k-2$. This is immediate if $k=2$. We will proceed by induction on k. Thus we assume $k>2$ and that the assertion is true for $k-1$. Now if $m>0$ then $V^{*m}\mathcal{Q}^\perp V^{*k-m-2} = \sum_{C\subseteq\{1,\dots,n\}} v^*(C)V^{*m-1}\mathcal{Q}^\perp V^{*k-m-2}$. Now $v^*(C)V^{*m-1}\mathcal{Q}^\perp V^{*k-m-2}$ is orthogonal to $\mathcal{V}(A:B)$ unless $A \supseteq C \supseteq A\setminus B$ and $|C| = |A|$ or $|A|-1$. But if $|C| = |A|$ then $v^*(C)V^{*m-1}\mathcal{Q}^\perp V^{*k-m-2}$ is orthogonal to $\mathcal{V}(A:B)$ by Lemma 6.2 and if $|C| = |A|-1$, the induction assumption yields the same result. Thus, the assertion holds if $m>0$. For $m=0$ we must consider $\mathcal{Q}^\perp V^{*k-2} = \sum_{C\subseteq\{1,\dots,n\}} \mathcal{Q}^\perp V^{*k-3}v^*(C)$. Now, $\mathcal{Q}^\perp V^{*k-3}v^*(C)$ is orthogonal to $\mathcal{V}(A:B)$ unless $A\supseteq C\supseteq A\setminus B$ and $|C| = |A|-|B|$ or $|A|-|B|+1$. If $|C| = |A|-|B|$ then $\mathcal{Q}^\perp V^{*k-3}v^*(C)$ is orthogonal to $\mathcal{V}(A:B)$ by Lemma 6.2 and if $|C| = |A|-|B|+1$ the induction assumption yields the same result. This completes the proof of the lemma. □

Let $\mathcal{V}$ denote the span of $\{\mathcal{V}(A:B) \mid B\subseteq A\subseteq\{1,\dots,n\}\}$. The lemma shows that $\mathcal{V}$ is orthogonal to $\langle \mathcal{Q}^\perp\rangle$ and so, the pairing of $T(V)$ and $T(V^*)$ induces a pairing of $\mathcal{V}$ and $Q_n^!$.

Proposition 6.4. *$\mathcal{S}$ is a basis for $Q_n^!$.*

Proof. Suppose $\min A \notin B \subseteq A \subseteq \{1,\dots,n\}$. Then

$$\bigl\langle \mathcal{V}\bigl(A : B\cup\{\min A\}\bigr),\ \mathcal{S}(A:B)\bigr\rangle = 1$$

and

$$\bigl\langle \mathcal{V}(C,D), \mathcal{S}(A:B)\bigr\rangle = 0$$

if $|C| < |A|$ or if $|C| = |A|$ and $C \neq A$ or if $\min A \in D$ and $D \neq B$. It is then easy to see that $\mathcal{S}$ is linearly independent. In view of Lemma 6.3, this proves the proposition. □

Corollary 6.5. $\operatorname{gr} Q_n^!$ *is presented by generators $\{v^*(A) \mid \emptyset \neq A \subseteq \{1,\dots,n\}\}$ and relations $\overline{S_1}\cup\overline{S_2}\cup\overline{S_3}$ where*

$$\overline{S_1} = \bigl\{v^*(A)v^*(B) \mid B \not\subseteq A \text{ or } |B| \neq |A|,\ |A|-1\bigr\},$$

$$\overline{S_2} = \Bigl\{v^*(C)\sum_{i\in C} v^*(C\setminus i)\ \Big|\ |C| \geqslant 2\Bigr\},$$

$$\overline{S_3} = \bigl\{v^*(C)^2 \mid \emptyset \neq C \subseteq \{1,\dots,n\}\bigr\}.$$

7. Proof of Theorem 2

Clearly if $i > 0$ then $\mathcal{S} \cap Q^!_{n,i}$ is a basis for $Q^!_{n,i}$. Thus, for $i > 0$, $\dim Q^!_{n,i}$ is equal to $|\mathcal{S} \cap Q^!_{n,i}|$. This is the same as $|\{\mathcal{S}(A:B) \in \mathcal{S} \mid |B| = i-1\}|$. Now

$$\Bigl|\bigl\{\mathcal{S}(A:B) \in \mathcal{S} \bigm| |B| = i-1 \text{ and } |A| = u\bigr\}\Bigr| = \binom{n}{u}\binom{u-1}{i-1}.$$

Thus

$$\begin{aligned}
H\bigl(Q^!_n, t\bigr) &= 1 + \sum_{i>0}\left(\sum_{u=i}^{n}\binom{n}{u}\binom{u-1}{i-1}\right)t^i \\
&= 1 + t\sum_{i>0}\left(\sum_{u=i}^{n}\binom{n}{u}\binom{u-1}{i-1}\right)t^{i-1} \\
&= 1 + t\sum_{v=0}^{n-1}\left(\sum_{u=v+1}^{n}\binom{n}{u}\binom{u-1}{v}\right)t^v \\
&= 1 + t\sum_{u=1}^{n}\sum_{v=0}^{u-1}\binom{n}{u}\binom{u-1}{v}t^v \\
&= 1 + t\sum_{u=1}^{n}\binom{n}{u}\sum_{v=0}^{u-1}\binom{u-1}{v}t^v \\
&= 1 + t\sum_{u=1}^{n}\binom{n}{u}(t+1)^{u-1} \\
&= 1 + \frac{t}{t+1}\sum_{u=1}^{n}\binom{n}{u}(t+1)^u \\
&= 1 + \frac{t}{t+1}\bigl((2+t)^n - 1\bigr) = \frac{1}{t+1}\bigl(t+1+t(2+t)^n - t\bigr) \\
&= \frac{1}{t+1}\bigl(1 + t(2+t)^n\bigr).
\end{aligned}$$

This completes the proof of Theorem 2.

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