LUSTERNIK–SCHNIRELMANN CATEGORY OF 3-MANIFOLDS

J. C. GÓMEZ-LARRAÑAGA and F. GONZALEZ-ACUÑA

(Received 1 October 1990; in revised form 4 October 1991)

§1. INTRODUCTION

The Lusternik–Schnirelmann category \( \text{cat} X \) of a space \( X \) is the smallest number of sets, open and contractible in \( X \), needed to cover \( X \). We prove in this paper (Corollary 4.2) that, for a closed 3-manifold \( M^3 \), \( \text{cat} M^3 \) depends only on \( \pi_1 M^3 \): it is 2 if \( \pi_1 M^3 \) is trivial, 3 if it is free nontrivial, and 4 if it is not free. Recall that a closed 3-manifold \( M^3 \) has a free fundamental group if and only if each prime summand of \( M^3 \) is a homotopy sphere or an \( S^2 \)-bundle over \( S^1 \) [13; chapter 5]. Thus, question 12 in [26] has an affirmative answer.

The smallest number of open cells needed to cover \( M^3 \) is denoted by \( C(M^3) \). This invariant has been calculated by Hempel and McMillan [14]. Endowing \( M^3 \) with a differentiable structure (and there is essentially only one), if \( f \) is a smooth function on \( M^3 \), the number of critical points of \( f \) is called \( \mu_M(f) \). The minimal value of \( \mu_M(f) \) for a fixed \( M^3 \), over all \( f \), is denoted by \( F(M^3) \). This invariant has been calculated by Takens [33]. From their results it follows that \( C(M^3) = F(M^3) \). We will show that \( \text{cat} M^3 = C(M^3) \) if and only if \( M^3 \) contains no fake cells or \( \pi_1 M^3 \) is not free and so, modulo the Poincaré conjecture, the three invariants \( \text{cat} \), \( C \) and \( F \) coincide on closed 3-manifolds.

A subset \( U \) of a space \( X \) is \( \pi_1 \)-contractible if every loop in \( U \) is contractible in \( X \). We denote by \( \text{cat}_* X \) the smallest number of open \( \pi_1 \)-contractible sets needed to cover \( X \). This invariant was defined by Fox [7], who denoted it by \( h_1 \text{cat} X \), and has been studied, for example, in [5] and [9]. For a closed 3-manifold \( M^3 \) we prove that \( \text{cat}_* M^3 \), again, depends only on \( \pi_1 M^3 \): it is 1 if \( \pi_1 M^3 \) is trivial, 2 if it is free nontrivial, and 4 if it is not free (Corollary 4.2).

To prove our results we use a theorem of Ganea–Eilenberg (see also Proposition 2.1) and the natural homomorphism \( H_3(M^3; A) \to H_3(\pi_1 M^3; A) \). The homology of (possibly nonorientable) 3-manifold groups is calculated in §3.

With similar methods we prove that if \( M^n \) is properly covered by a homotopy \( n \)-sphere then \( \text{cat} M^n = \text{cat} M^n = n + 1 \), thus giving a new proof of a theorem of Krasnoselski ([18]). This proof, as well as that in [23], is algebraic-topological as requested by James in [16]. We have learned that S. Husseini also has an unpublished proof of Krasnoselski's theorem.

In §2 we give an \( H_1 \) version and a \( \pi_1 \) version of a theorem of Eilenberg and Ganea ([5, Prop. 3], [9, Prop. 1.18]). Our proof of the \( \pi_1 \) version seems to us more direct than Ganea's proof. Also we omit the condition of semilocal 1-connectedness. The \( H_1 \) version will be used in a subsequent paper. We thank Mónica Clapp for pointing out the existence of [9].

In §3 we study the homomorphism \( H_3(M^3; A) \to H_3(\pi_1 M^3; A) \). If \( M^3 \) is irreducible with infinite fundamental group, the classifying space of \( \pi_1 M^3 \) is obtained by attaching to \( M^3 \) copies of \( \mathbb{P}^\infty \) along a maximal family of two sided \( \mathbb{P}^2 \)'s no two of which cobound 791
a homotopy $P^i \times I$. This allows one to calculate the homology of $\pi_1 M^4$ and the image of $H_3(M^3; A) \to H_3(\pi_1 M^3; A)$.

In §4 we prove the main theorem (Theorem 4.1), and calculate $cat M^3$ and $cat_{\pi_1} M^3$ for any closed 3-manifold $M^3$ (Corollary 4.2). We also relate $cat(M)$ to two other invariants $C(M)$ and $N_0(M)$, the smallest number of balls and the smallest numbers of charts needed to cover $M$. It is also pointed out that if $M^3$ is obtained by non trivial surgery on a non trivial knot $k$, then $C(M^3) = 4$ and, if $\pi_1 M^3 \neq 1$, $cat M^3 = 4$. We prove that, if $M^3$ is a closed 3-manifold, $cat(M^3, point) = cat(M^3) - 1$; this answers affirmatively, for 3-manifolds, question 10 in [26] (see also Conjecture 7.2 in [27]).

In §5 we apply our methods to prove that if $M^n$ is properly covered by $S^n$ then $cat M^n = cat_{\pi_1} M^n = n + 1$.

§2. PRELIMINARIES

The Lusternik–Schnirelmann category $cat X$ of a topological space is the least integer $n$ such that $X$ can be covered by $n$ open sets $U_1, \ldots, U_n$ such that each $U_i$ is contractible in $X$. If no such integer exists then $cat X = \infty$.

More generally, let $F: \text{Top} \to \mathcal{C}$ be a functor on the category of topological spaces. A subset $U$ of a space $X$ is $F$-contractible (in $X$) if $F(i)$ is a constant morphism ([15, Definition 8.2]) where $i: U \to X$ is the inclusion. We write $cut_F X = n$ if $n$ is the least integer such that $X$ can be covered by $n$ open subsets $U_1, \ldots, U_n$ such that each $U_i$ is $F$-contractible. Again, if no such integer exists then $cut_F X = \infty$.

If $F: \text{Top} \to h\text{Top}$ is the natural functor to the homotopy category of topological spaces then $cat_F$ is the Lusternik–Schnirelmann category $cat$. We are especially interested in the functor $\pi_1: \text{Top} \to \text{Grp}$, to the category of groups, defined on objects by $\pi_1(X) = \pi_1(X, x)$; if $f: X \to Y$ is a map, then $\pi_1(f): \pi_1(X, x) \to \pi_1(Y, y)$ restricted to $\pi_1(X, x)$ is the homomorphism $f_*: \pi_1(X, x) \to \pi_1(Y, f(x))$ induced by $f$ on fundamental groups. One can see that a subset $U$ of $X$ is $\pi_1$-contractible if and only if every loop in $U$ is contractible in $X$. We are also interested in the functor $H_1: \text{Top} \to \text{Ab}$ where $H_1(X)$ is the first singular homology group of $X$.

Proposition 2.1 (ii), with the additional hypothesis of semilocal 1-connectedness of $X$, is due to Ganea and Eilenberg ([5, Prop. 3], [9, Prop. 1.18]). Our results are based on it.

A paracompact space is a Hausdorff space such that every open cover has a locally finite refinement [17, page 156].

**Proposition 2.1.** Let $X$ be a paracompact, locally pathwise connected space and let $n$ be a natural number.

(i) In order that $cut_{H_1} X \leq n$ it is necessary and sufficient that there exist a complex $L$ of dimension less than $n$ and a map $f: X \to L$ such that $f_*: H_1 X \to H_1 L$ is an isomorphism.

(ii) If $X$ is connected, in order that $cut_{\pi_1} X \leq n$ it is necessary and sufficient that there exist a connected complex $L$ of dimension less than $n$ and a map $f: X \to L$ such that $f_*: \pi_1 X \to \pi_1 L$ is an isomorphism.

**Proof.** First, we prove the sufficiency of the conditions in (i). Suppose $f: X \to L$ induces an isomorphism of first homology groups and $L$ is a complex of dimension less than $n$. Then $cut_{H_1} L \leq n$, that is, $L$ can be covered with open sets $A_1, A_2, \ldots, A_n$ that are $H_1$-contractible in $L$. Then $f^{-1}(A_1), \ldots, f^{-1}(A_n)$ are $H_1$-contractible in $X$ and so $cut_{H_1} X \leq n$. 

To prove the necessity in (i) let \( \{ U_1, \ldots, U_n \} \) be an open cover of \( X \) where \( U_i \) is \( H_1 \)-contractible in \( X \) \((i = 1, \ldots, n)\). Let \( \{ \rho_i \}_{1 \leq i \leq n} \) be a partition of unity such that support \( \rho_i \subset U_i \) \((i = 1, \ldots, n)\).

For \( S \) nonempty contained in \( \{ 1, \ldots, n \} \) define
\[
W_S = \{ x \in X | \rho_i(x) > \rho_j(x) \text{ and } \rho_k(x) > 0 \text{ for } i \in S \text{ and } j \notin S \}.
\]

Let \( v = \{ V_j \}_{j \in J} \) be the family of components of the sets \( W_S \). Let \( L \) be the nerve of \( v \). A point of \( L \) will be denoted by a convex linear combination of the vertices of \( L \). Notice that \( W_S \subset U_i \) if \( i \in S \). Now, if \( W_S \cap W_{S'} \neq \emptyset \) then \( S \cap S' \) is either \( S \) or \( S' \) and \( W_S \cup W_{S'} \subset U_i \) for \( i \in S \cap S' \). Hence, if \( V_{j_i} \cap V_{j_S} \neq \emptyset \), then \( V_{j_i} \cup V_{j_S} \) is contained in some \( U_i \). Thus \( V_{j_i} \cup V_{j_S} \) is \( H_1 \)-contractible for \( j_i, j_S \in J \). Also, if \( V_{j_i} \cap V_{j_S} \neq \emptyset \) and \( V_{j_i} \neq V_{j_S} \), then \( |S_1| \neq |S_2| \), where \( V_{j_i} \subset W_{S_i}, i = 1, 2 \). Hence, no point of \( X \) belongs to \( n \) different members of \( v \) and so, \( \dim L \leq n - 1 \).

If \( x \in W_S \) we denote by \( W_S(S) \) the vertex corresponding to the component of \( W_S \) containing \( x \). If \( S \) is not empty and contained in \( \{ 1, \ldots, n \} \) define \( p_S : X \rightarrow R \) by
\[
p_S(x) = \max \left\{ \min_{i \in S} \rho_i(x) - \max_{j \notin S} \rho_j(x), 0 \right\}
\]
where \( \max_{j \notin S} \rho_j(S) \) is taken to be 0 when \( S = \{ 1, \ldots, n \} \). The mapping \( p_S \) is positive on \( W_S \) and zero on \( X - W_S \). Let
\[
\Phi_S(x) = \sum_{\emptyset \neq T \subset \{ 1, \ldots, n \}} p_T(x) p_S(x).
\]
Now define the map \( f : X \rightarrow L \) by \( f(x) = \sum_{S} \Phi_S(x) W_S(S) \) where \( S \) runs over the subsets of \( \{ 1, \ldots, n \} \) such that \( x \in W_S \). Notice that, for every \( j \in J \), \( f^{-1}(st(V_j)) = V_j \) where \( V_j \) is the vertex associated to \( V_j \).

We now prove that if \( X' \) is a component of \( X \) and \( L' \) is the component of \( L \) containing \( f(X') \) then \( f|X' : X' \rightarrow L' \) induces an epimorphism of fundamental groups. This implies that \( f_* : H_1(X) \rightarrow H_1(L) \) is an epimorphism.

Take a point \( * \) in \( X' \) and its image under \( f \), also denoted by \( * \), as base points for \( \pi_1 X' \) and \( \pi_1 L' \). Let \( \alpha \) be a loop of \( L' \) based on \( * \). It is clear that \( \alpha \) is a product \( \alpha_1 \cdots \alpha_m \) of paths such that, for \( i = 1, \ldots, m \), the path \( \alpha_i \) is contained in \( st(V_{j_i}) \) for some index \( j_i \). Then \( V_{j_i} \cap V_{j_{i+1}} \neq \emptyset \) for \( i = 1, \ldots, m - 1 \). Notice that the base point of \( X' \) is in \( V_{j_1} \). For \( i = 1, \ldots, m - 1 \) take a point \( x_i \in V_{j_i} \cap V_{j_{i+1}} \). Also define \( x_0 = x_m = * \).

Let \( \beta \) be a path in \( V_{j_i} \) from \( x_{i-1} \) to \( x_i \) \((i = 1, \ldots, m)\) and let \( \beta = \beta_1 \beta_2 \cdots \beta_m \). We claim that \( \beta \) and \( x \) represent the same element of \( \pi_1(X', *) \). Notice that \( f(\beta_i) \) is a path in \( st(V_{j_i}) \).

For \( i = 1, \ldots, m - 1 \) let \( \gamma_i \) be a path in \( st(V_{j_i}) \cap st(V_{j_{i+1}}) \) from the terminal point of \( f\beta_i \) to the terminal point of \( x_i \). Also let \( \gamma_0 \) and \( \gamma_m \) be the constant map with image \( * \). Since \( \gamma_i^{-1} f\beta_i \gamma_i^{-1} \) is a path in \( st(V_{j_i}) \), it is nullhomotopic and so \( \gamma_i^{-1} f\beta_i \gamma_i^{-1} \sim x_i \).

Therefore
\[
f\beta = \prod_{i=1}^{m} f\beta_i \sim \prod_{i=1}^{m} \gamma_i^{-1} f\beta_i \gamma_i \sim \prod x_i = \alpha.
\]
Hence \( [\alpha] = f_*[\beta] \) in \( \pi_1 X' \). This proves that \( (f|X')_* : \pi_1 X' \rightarrow \pi_1 L' \) is surjective.

We now prove that \( f_* \) is a monomorphism. Suppose that \( f_*([x]) \in H_1(L) \) is trivial, where \( [x] \in H_1(X) \) is represented by a map \( x \) from an oriented 1-sphere \( S \) into \( X \). Then \( x \) can be extended to a map \( \beta : F^2 \rightarrow L \) where \( F^2 \) is a compact oriented 2-manifold, with \( \partial F^2 = S \) and the orientation of \( S \) is induced from that of \( F^2 \). Let \( K \) be a triangulation of \( F^2 \) such that the
image, under \( \beta \), of any 2-simplex \( \tau \) of \( K \) is contained in the star of some vertex of \( L \). For every such \( \tau \) choose an index \( j(\tau) \) such that \( \hat{\beta}(\tau) \subset \text{st}(v_{j(\tau)}) \). If \( \sigma \) is a 1-simplex of \( K \) not contained in \( \partial F^2 \), let \( E_\sigma \) be the closed star of the baricenter of \( \sigma \) in \( K' \), the second barycentric subdivision of \( K \). Also, if \( \tau \) is a 2-simplex of \( K \), let \( E_\tau \) be a 2-disk in the interior of \( \tau \) that does not intersect any \( E_\sigma \). We will define an extension \( \hat{\beta}: F^2 - \bigcup_\sigma E_\sigma - \bigcup_\tau E_\tau \to X \) of \( \alpha \) such that \( \hat{\beta}|_{\partial E_\sigma} \) and \( \hat{\beta}|_{\partial E_\tau} \) are nullhomologous for any \( \sigma \) and any \( \tau \). If \( u \) is a vertex of \( K \) not lying on \( \partial F \) and \( \tau_1, \ldots, \tau_n \) are the 2-simplexes of \( K \) containing \( u \) define \( \hat{\beta}(u) \) to be a point of the nonempty set \( V_{\tau_1} \cap \ldots \cap V_{\tau_n} \). Also, if \( u \) is any vertex of \( K \) and \( x \) lies in a 1-simplex of \( K' \) containing \( u \), contained in \( K \) and not contained in \( \partial F \), we define \( \hat{\beta}(x) = \hat{\beta}(u) \). Now if \( \sigma \) is a 1-simplex of \( K \) not contained in \( \partial F \) we extend \( \hat{\beta} \), already defined in \( \partial E_\sigma \cap \sigma \), to \( \partial E_\sigma \) in such a way that \( \hat{\beta}(\tau \cap \partial E_\sigma) \subset V_{\tau} \) for any 2-simplex \( \tau \) of \( K \). This is possible since \( V_{\tau} \) is path-connected. Finally, if \( \tau \) is a 2-simplex of \( K \) then \( \tau - \cup E_\sigma - E_\tau \) is an annulus \( A \) and \( \hat{\beta}(A - \partial E_\tau) \subset \text{st}(v_{j(\tau)}) \) for any 2-simplex \( \tau \) of \( K \). This is possible since \( V_{\tau} \) is path-connected. If \( x \) lies in \( \partial F \) we extend \( \hat{\beta} \), already defined in \( \partial E_\sigma \cap \sigma \), to \( \partial E_\sigma \) in such a way that \( \hat{\beta}(\partial E_\sigma) \subset V_{\tau} \) for any 2-simplex \( \tau \), and \( \hat{\beta}(\partial E_\tau) \subset V_{\tau_1} \cup V_{\tau_2} \) where \( \tau_1 \cap \tau_2 = \sigma \). Since \( V_{\tau_1} \cap V_{\tau_2} \) are \( H_1 \)-contractible, \( \hat{\beta}([\tau \cap \partial E_\sigma] \cup ([\tau \cap \partial E_\tau]) \) is homologically trivial and, therefore, \( \alpha \) is homologically trivial. Hence \( f_* \) is a monomorphism. This completes the proof of (i).

The proof of the sufficiency of the condition in (ii) is analogous to that of (i).

To prove necessity in (ii) let \( \{ U_1, \ldots, U_n \} \) be an open cover of \( X \) such that \( U_i \) is \( \pi_1 \)-contractible in \( X \) (i = 1, \ldots, n). Let \( \{ \rho_i \}_{1 \leq i \leq n} \) be a partition of unity such that \( \rho_i \subset U_i \) (i = 1, \ldots, n). Define \( v = \{ V_{\rho_i} \}_{i \in J} \) and \( f: X \to L \) exactly as in the proof of (i). Then \( \dim L < n \), \( L \) is connected and \( V_{\rho_1} \cap V_{\rho_2} \) is \( \pi_1 \)-contractible in \( X \) if \( j_1, j_2 \in J \). As shown in the proof of (i), \( f_*: \pi_1 X \to \pi_1 L \) is surjective.

To prove that \( f_*: \pi_1 X \to \pi_1 L \) is a monomorphism suppose that \( f_*([\alpha]) \) is trivial, where \( [\alpha] \in \pi_1(X) \) is represented by a map \( \alpha \) from the 1-sphere \( S \) into \( X \). Then \( f_* \) can be extended to a map \( \beta \) from a 2-disk \( F^2 \) into \( L \). Proceed as in the proof that \( f_*: H_1 X \to H_1 L \) is a monomorphism in (i) to extend \( \alpha \) to a map \( \hat{\beta}: F^2 - \bigcup_\sigma E_\sigma - \bigcup_{\tau} E_\tau \to X \) where the \( E_\sigma \) and \( E_\tau \) are defined as in the proof of (i). Now \( \hat{\beta}(\partial E_\sigma) \subset V_{\tau} \) for any \( \tau \) and \( \hat{\beta}(\partial E_\tau) \subset V_{\tau_1} \cap V_{\tau_2} \) where \( \tau_1 \cap \tau_2 = \sigma \). Since \( V_{\tau_1} \cap V_{\tau_2} \) are \( H_1 \)-contractible in \( X \), \( \hat{\beta} \) can be extended to a map \( \hat{\beta}: F^2 \to X \). Since \( \hat{\beta}(\partial F^2) = \alpha, [\alpha] \) is trivial. Hence \( f_*: \pi_1 X \to \pi_1 L \) is a monomorphism. \( \square \)

§ 3. HOMOLOGY OF 3-MANIFOLD GROUPS

Another ingredient in the proof of our results is the study of the natural homomorphism \( H_3(M^3; A) \to H_3(S^2 M^3; A) \) which we undertake in this section. As a by-product we calculate the homology of closed 3-manifold groups (see the table at the end of the section).

If \( M \) is a 3-manifold we denote by \( \tilde{M} \) the manifold obtained from \( M \) by capping off each 2-sphere component of \( \partial M \) with a 3-disk.

**Lemma 3.1.** Let \( M^3 \) be a prime closed 3-manifold. Let \( A \) be \( \mathbb{Z} \) if \( M^3 \) is orientable and \( \mathbb{Z}_2 \) if \( M^3 \) is nonorientable. Let \( g: M^3 \to BG \) be the natural map from \( M^3 \) to the classifying space of the fundamental group \( G \) of \( M^3 \). If \( g_*: H_3(M^3; A) \to H_3(BG; A) \) is zero, then \( G \) is trivial or infinite cyclic.
Proof. Suppose $M^3$ is orientable. If $G$ were infinite noncyclic then we could take $g$ to be
the identity and $g_*$ would be nonzero. If $G$ were nontrivial of finite order $d$, then $g_*$ would be
a surjection onto a cyclic group of order $d$. (Compare [32; §2] or [4; chapter XIII]). Hence $G$ is either $\mathbb{Z}$ or 1.

Next, suppose that $M^3$ is nonorientable (and therefore $G$ is infinite). To complete the
proof we will show that, if $G \cong \mathbb{Z}$ (that is, if $M^3 \cong S^1 \times S^2$), then

$$g_*: H_3(M^3; \mathbb{Z}_2) \to H_3(BG; \mathbb{Z}_2)$$

is nonzero.

If $M^3 \cong \mathbb{P}^2 \times S^1$, then $BG = \mathbb{P}^\infty \times S^1$, $g = (j \times \text{id})h$ where $j: \mathbb{P}^2 \to \mathbb{P}^\infty$ is the natural
inclusion, $h: M^3 \to \mathbb{P}^3 \times S^1$ is a homotopy equivalence and therefore $g_* \neq 0$.

Henceforth we will assume that $M^3 \not\cong \mathbb{P}^2 \times S^1$. Notice that $M^3$ is irreducible. If $M^3$ contains
no two-sided projective plane, then by the projective plane theorem ([6], [13, Theorem 4.12]), $\pi_2 M^3 = 0$, we can take $g$ to be the identity and so $g_* \neq 0$. If $M^3$ contains
a two-sided projective plane, let $\{P_1^\infty, \ldots, P_k^\infty\}$ be a maximal collection of pairwise disjoint,
two-sided $\mathbb{P}^2$s in $M^3$ such that no pair cobounds a homotopy $\mathbb{P}^2 \times I$ [13, Lemma 13.2]. Let
$N^3$ be $M^3$ cut along $\bigcup_{i=1}^k P_i^\infty$. Notice that no component of $N$ is a homotopy $\mathbb{P}^2 \times I$ (here we
use the fact that $M \cong \mathbb{P}^2 \times S^1$). Hence, by a theorem of Epstein ([6], [13, Theorem 9.6])
every component of $N^3$ has infinite fundamental group. Let $\tilde{N}^3$ be the orientable double
covering of $N$.

By the proof of [31, Lemma 2.1], $\tilde{N}$ is aspherical (see [31, Proposition 2.2 (c)]).

Now choose $k$ copies $P_1^\infty, \ldots, P_k^\infty$ of $\mathbb{P}^\infty$ and identify a standard copy of $\mathbb{P}^\infty \times I$ in
$P_i^\infty \times I$ with a regular neighbourhood of $P_i^\infty$ in $M^3$ to obtain a complex
$X = M \cup P_1^\infty \times I \cup \ldots \cup P_k^\infty \times I$. We will show that $X$ can be taken as $BG$. First the
inclusion of $M$ in $X$ induces an isomorphism of fundamental groups since $\pi_1 P_i^\infty \to \pi_1 P_i^\infty$ is
an isomorphism ($i = 1, \ldots, k$). Let $C_1, \ldots, C_j$ be the components of $X - \bigcup_{i=1}^k P_i^\infty \times I$. Each $C_j$
is obtained from a component $W$ of $N$ by attaching to $W$ copies of $\mathbb{P}^\infty$ one for each component of $\partial W$; its orientable double cover $\tilde{C}_j$ is obtained by attaching to $\tilde{W}$ copies of $S^\infty$
along each (spherical) component of $\partial \tilde{W}$. Hence $\tilde{C}_j \cong \tilde{W}$ and so $C_j$ is aspherical. Notice also
that the inclusion of a copy of $\mathbb{P}^\infty$ (in $\text{Fr} C_j$) in $C_j$ induces a monomorphism of fundamental
groups. We therefore have a graph of aspherical spaces, as described in [29, pp. 155–156],
whose total space is $X$ and by [29, Proposition 3.6(ii)] $X$ is aspherical. Hence we can take
$X$ as $BG$ and the inclusion of $M$ in $X$ as $g$. The Mayer–Vietoris sequence

$$0 \to H_3(M; \mathbb{Z}_2) \oplus H_3 \left( \bigcup_{i=1}^k P_i^\infty \times I \right) \xrightarrow{(j_*, \ldots, j_*)} H_3(X; \mathbb{Z}_2)$$

show that $g_* \neq 0$. \(\square\)

Remark. The construction of the space $X = B\pi_1 M$ in the proof of Lemma 3.1 allows one to complete the description of the homology of a (closed) 3-manifold group. Such a group can be expressed uniquely as the free product of indecomposable 3-manifold groups
and $H_n(\pi_i G_i) = \bigoplus H_n(G_i)$ for $n > 0$. Also $H_*(\mathbb{Z})$ is well known. We therefore describe the
integral homology group $H_*(\pi_1 M^3)$ only when $M^3$ is irreducible.
J. C. Gómez-Larrañaga and F. González-Acuña

$H_n(\pi_1 M^3)$ for $M^3$ closed irreducible

<table>
<thead>
<tr>
<th>$M^3$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n$ even</th>
<th>$n = 1 \mod 4$</th>
<th>$n = 3 \mod 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>with finite</td>
<td>0</td>
<td>$\mathbb{Z}_{\pi_1 M^3}$</td>
<td>0</td>
<td>$H_1(\pi_1 M^3)$</td>
<td>$\mathbb{Z}_{\pi_1 M^3}$</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>orientable</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>with infinite</td>
<td>$\pi_1 \approx \mathbb{Z} \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nonorientable</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2^{-1}$</td>
<td>$\mathbb{Z}_1$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>and with</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_1 \approx \mathbb{Z} \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$r = \dim_{\mathbb{Q}} H_3(\pi_1 M^3; \mathbb{Q})$

$k =$ number of conjugacy classes of elements of order 2 in $\pi_1 M^3$

In connection with the last line of the table one can show that, if $M^3$ is closed nonorientable and irreducible, then the number of conjugacy classes of order 2 in $\pi_1 M^3$ equals the cardinality of a maximal family $\{P_1^2, \ldots, P_k^2\}$ of two-sided projective planes in $M^3$ such that no pair cobounds a homotopy $P^2 \times I$. To see this one uses [31, Theorem 4.1], the first remark after [29, Theorem 3.7], and [13, Theorem 9.8].

§4. CATEGORIES OF 3-MANIFOLDS

Theorem 4.1 is our main result. It allows us to compute the category of any closed 3-manifold (see Corollary 4.2).

**Theorem 4.1.** Let $M^3$ be a closed 3-manifold. The following statements are equivalent:

(i) $\text{cat} M^3 \leq 3$

(ii) $\text{cat}_{\pi_1} M^3 \leq 3$

(iii) $\pi_1 M^3$ is free

**Proof.** (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) By Proposition 2.1 there is a connected 2-complex $L^2$ and a map $f: M^3 \rightarrow L^2$ inducing isomorphism of fundamental groups. Write $G = \pi_1 M^3$. Let $g$ be the natural map from $M^3$ to $BG$, the classifying space of $G$. Let $h: L^2 \rightarrow BG$ be a map such that $hf$ and $g$ induce the same homomorphism of fundamental groups. Then $hf$ and $g$ are homotopic and so, for any group of coefficients $A$, the diagram

$$
\begin{array}{ccc}
H_3(M^3; A) & \xrightarrow{g_*} & H_3(BG; A) \\
\downarrow & & \downarrow \\
H_3(L^2; A) & \xrightarrow{h_*} & H_3(BG; A)
\end{array}
$$

commutes and $H_3(L^2; A) = 0$. Hence $g_* = 0$.

Now let $M_1$ be any prime summand of $M$ and let $g_1: M_1 \rightarrow BG_1$ be the natural map where $G_1 = \pi_1 M_1$. We have a commutative diagram

$$
\begin{array}{ccc}
H_3(M; A) & \xrightarrow{g_*} & H_3(BG; A) \\
\downarrow & & \downarrow \\
H_3(M_1; A) & \xrightarrow{g_1*} & H_3(BG_1; A)
\end{array}
$$
where the left vertical homomorphism is induced by a collapsing map $c$.

Assume that $M$ is orientable and take $A = \mathbb{Z}$. Then $c_\ast$ is surjective and, therefore $g_1 = 0$. Hence, by Lemma 3.1, $\pi_1 M$ is trivial or infinite cyclic. This proves (iii) in case $M$ is orientable.

Assume now that $M$ is nonorientable, $A = \mathbb{Z}_2$ and $M_1$ is any nonorientable prime summand of $M$. Then, again, $c_\ast$ is surjective, $g_1 = 0$ and, by Lemma 4.1, $\pi_1 M_1 \approx \mathbb{Z}$, that is, $M_1 \approx S^1 \times S^1$. Therefore $M \approx S^1 \times S^2 \# N^3$ where $N^3$ is orientable. Let $\rho: \tilde{M} \to M$ be the orientable double covering and let $\{U_1, U_2, U_3\}$ be an open cover of $M$ with $U_i$ $\pi_1$-contractible ($i = 1, 2, 3$). Then $\{\rho^{-1}(U_1), \rho^{-1}(U_2), \rho^{-1}(U_3)\}$ is an open cover of $\tilde{M}$ and $\rho^{-1}(U_i)$ is $\pi_1$-contractible ($i = 1, 2, 3$). Hence $\text{cat}_{\pi_1} \tilde{M} \leq 3$ and, since we already know that (ii) $\Rightarrow$ (iii) in the orientable case, $\pi_1 M$ is free. Since $\tilde{M} \approx N^3 \# S^1 \times S^2 \# (-N^3)$, $\pi_1 N^3$, and therefore $\pi_1 M$, is free.

(iii) $\Rightarrow$ (i) If $\pi_1 M$ is free then $M$ is homotopy equivalent to a connected sum of copies of $S^1 \times S^2$ or $S^1 \times S^2$ ([13, Chapter V]). Since this connected sum can be covered with three open balls ([10]), it follows that $\text{cat} M \leq 3$.

Remark. One may conjecture that, for a closed $n$-manifold $M^n$ with $n \geq 3$, $\text{cat}(M^n) \leq 3$ implies that $\pi_1(M^n)$ is free.

Theorem 4.1 enables us to calculate $\text{cat} M^3$ and $\text{cat}_{\pi_1} M^3$ for any closed 3-manifold.

**Corollary 4.2.** Let $M^3$ be a closed 3-manifold. Then

$$
\text{cat} M^3 = \begin{cases} 
2 & \text{if } \pi_1 M^3 = 1 \\
3 & \text{if } \pi_1 M^3 \text{ is free nontrivial} \\
4 & \text{if } \pi_1 M^3 \text{ is not free}
\end{cases}
$$

$$
\text{cat}_{\pi_1} M^3 = \begin{cases} 
1 & \text{if } \pi_1 M^3 = 1 \\
2 & \text{if } \pi_1 M^3 \text{ is free nontrivial} \\
4 & \text{if } \pi_1 M^3 \text{ is not free}.
\end{cases}
$$

**Proof.** Since $\text{cat}_{\pi_1} M^3 \leq \text{cat} M^3 \leq 4$, it follows from Theorem 4.1 that, if $\pi_1 M^3$ is not free, then $\text{cat}_{\pi_1} M^3 = \text{cat} M^3 = 4$.

If $\pi_1 M^3 = 1$ then clearly $\text{cat}_{\pi_1} M^3 = 1$ and, since $M^3$ minus a 3-disk is a homotopy 3-disk and $M^3$ is not contractible, $\text{cat} M^3 = 2$.

Suppose $\pi_1 M^3$ is free of rank $r > 0$. Then

$$
M^3 \approx \Sigma^3 \# \left( \bigcup_{i=1}^{m} S^2 \times S^1 \right) \# \left( \bigcup_{j=1}^{n} S^2 \times S^1 \right)
$$

($m > 0$, $n > 0$, $\Sigma^3$ a homotopy sphere) and therefore we can find $r$ disjoint spheres $S^2_1, \ldots, S^2_r$ in $M$ such that $M^3 \setminus \bigcup_{k=1}^{r} S^2_k$ is 1-connected. Since every component of a regular neighborhood of $\bigcup_{k=1}^{r} S^2_k$ is also 1-connected, $\text{cat}_{\pi_1} M^3 \leq 2$. As $\pi_1 M^3 \neq 1$, $\text{cat}_{\pi_1} M^3 = 2$.

(See also [7, Theorem 23.1]). Also $M^3$ can be covered with three homotopy 3-cells and $\text{cat} M^3 > 2$ since $M^3$ is not a homotopy sphere (see [16, page 336]). Hence $\text{cat} M^3 = 3$.

Remarks. As a consequence of the corollary one can see that if $M^3$ is a closed PL 3-manifold with $\text{cat} M^3 = k$ then $M^3$ can be covered with $k$ subpolyhedra contractible in
themselves. Thus, with the terminology of [26] and [27], the category, the geometric category and the strong category of $M^3$ are the same.

Denote by $(k; r), r \in \mathbb{Q}$, the manifold obtained by $r$-surgery on the knot $k$ of $S^3$. Then it follows from Theorem 4.1 and Gabai's theorem [8], that $cat(k; r) = 4 \forall r \in \mathbb{Q}$ if and only if $k$ has property P. Thus the conjecture that every nontrivial knot has property P is equivalent to the conjecture that $cat(k; r) = 4$ if $k$ is nontrivial and $r \in \mathbb{Q}$.

Two invariants of a topological manifold $M^n$ related to $cat(M^n)$ are: the smallest number $C(M^n)$ of open balls needed to cover $M^n$, and the minimal number $N_0(M^n)$ of charts (spaces homeomorphic to open sets of $\mathbb{R}^n$) needed to cover $M^n$. (Another invariant $F(M^3)$ of $M^3$, which turns out to be equal to $C(M^3)$, is defined in the introduction.) Clearly $cat(M^n) \leq C(M^n)$ and $N_0(M^n) \leq C(M^n)$. One has $C(M^n) \leq n + 1$ by [20] or [28] and frequently (perhaps always) $cat(M^n) = C(M^n)$ (see [30], [25], question 9 in [26], and Conjecture 7.1 in [27]). It was proven in [11] that for a closed 3-manifold $M^3$, $N_0(M^3)$ is two if the Bockstein of the first Stiefel–Whitney class of $M^3, \beta \omega_1(M^3) \in H^2(M^3; \mathbb{Z}_2)$, is zero and three if it is not. Hempel and Macmillan proved that $C(M^3) \leq 3$ if and only if $\pi_1(M^3)$ is free and $M^3$ contains no fake cells [14]. This is analogous to our Theorem 4.1. By these two results $cat(M^3) = C(M^3)$ if and only if the Poincaré conjecture is true; in fact $catM^3 = C(M^3)$ if and only if $M^3$ contains no fake cells or $\pi_1(M^3)$ is free. The values of the pair $(N_0(M^3), cat(M^3))$ as $M^3$ runs over the class of closed 3-manifolds are $(2,2), (2,3), (2,4)$ and $(3,4)$; in particular $N_0(M^3) < cat(M^3)$ if $M^3$ is not a homotopy sphere.

Considering, again, the manifolds obtained by surgery on knots, it follows from [14], [8] and [12] that $C(k; r) = 4$ if $k$ is a nontrivial knot and $r \in \mathbb{Q}$.

It has been conjectured ([26] Question 10, and [27], §7) that if $M^n$ is a closed manifold then the category of $M^n$ minus a point equals $cat(M^n) - 1$. We now prove this conjecture for the case $n = 3$.

**Corollary 4.3.** Let $M^3$ be a closed 3-manifold and let $p \in M^3$. Then $\text{cat}(M^3 - p) = \text{cat}(M^3) - 1$.

**Proof.** Suppose first that $\pi_1 M^3$ is not free. Then $\text{cat}M^3 = 4$ and, since $M^3 - p$ is homotopy equivalent to a 2-complex and $\pi_1(M^3 - p)$ is not free, $\text{cat}(M^3 - p) = 3$ ([2]).

Next assume that $\pi_1 M^3$ is free and non trivial. Then $\text{cat}M^3 = 3$ and, since $M^3$ is the connected sum of a homotopy 3-sphere and $S^2$-bundles over $S^1$, $M^3 - p$ can be covered with 2 homotopy cells and so $\text{cat}(M^3 - p) = 2$.

Finally, if $\pi_1 M^3$ is trivial then $\text{cat}(M^3) = 2$ and $\text{cat}(M^3 - p) = 1$. \hfill $\Box$

**Remark.** In contrast, it is possible to prove that, if $M^3 \neq S^3, N_0(M^3 - p) = N_0(M^3)$.

§8. CATEGORIES OF QUOTIENTS OF SPHERES

Our methods can be used to prove a theorem of Krasnoselski ([18]) stating that a manifold properly covered by the $n$-sphere has category $n + 1$. We give a proof in this section.

**Theorem 5.1.** Let $X$ be a connected space of the homotopy type of a CW-complex. Let $n$ be a natural number such that $\pi_i(X) = 0$ for $1 < i < n$ and $\pi_n X \to H_n X$ is not surjective. Then $\text{cats}_n X \geq n + 1$. 


Proof. We may assume $X$ is a connected $CW$-complex. Let $G = \pi_1 X$. Kill $\pi_n X$ by attaching $(n + 1)$-cells to $X$ by maps $\varphi_i: \partial D^{n+1}_i \rightarrow X$ such that $\{[\varphi_i]\}$ generate $\pi_n X$. Then, as usual, we construct the classifying space $BG$, killing successively $\pi_{n+1}, \pi_{n+2}, \ldots$ by attaching cells of dimensions $n+2, n+3, \ldots$. The group $H_n BG$ is isomorphic to the cokernel of the Hurewicz homomorphism $\pi_n X \rightarrow H_n X$, and so $H_n BG \neq 0$. Moreover the inclusion induced homomorphism $g_*: H_n X \rightarrow H_n BG$ is surjective and, therefore, nonzero.

Now suppose $\text{cat}_n(X) \leq n$. Recall that a $CW$-complex is paracompact and locally connected [21, Theorem 11.4.2 and Corollary 11.6.7]. Then by [5, Proposition 3] (see also Prop. 2.1) there is a connected $(n-1)$-complex $L^{n-1}$ and a map $f: X \rightarrow L^{n-1}$ inducing isomorphism of fundamental groups. Let $h: L^{n-1} \rightarrow BG$ be a map such that $hf$ and $g$ induce the same homomorphisms of fundamental groups. Then $hf$ and $g$ are homotopic and so, the diagram

$$
\begin{array}{ccc}
H_n(X) & \xrightarrow{a_*} & H_n(BG) \\
\downarrow f_* & & \downarrow g_*
\end{array}
$$

commutes, which is impossible because $H_n(L^{n-1}) = 0$ and $g_*$ is non-zero. Hence $\text{cat}_n(X) \geq n + 1$. \hfill $\Box$

Krasnoselki's theorem, stating that $\text{cat}(S^n/G) = n + 1$ for a free action of a finite nontrivial group $G$ in the sphere $S^n$, is a consequence of Theorem 5.1:

Corollary 5.2. If $G$ is a finite nontrivial group acting freely on a homotopy sphere $\Sigma^n$, then $\text{cat}_n(\Sigma^n/G) = \text{cat}(\Sigma^n/G) = n + 1$.

Proof. We have $\text{cat}_n(\Sigma^n/G) \leq \text{cat}(\Sigma^n/G) \leq n + 1$ since $\Sigma^n/G$ is an $n$-manifold. Also $\Sigma^n/G$ has the homotopy of a $CW$-complex ([24] or [21, Corollary IV.5.7]). If the action of $G$ preserves the orientation then $\pi_i(\Sigma^n/G) = 0$ for $1 < i < n$, the image of a generator of $\pi_n(\Sigma^n/G)$ in $H_n(\Sigma^n/G)$ is $|G|$ times a generator of the infinite cyclic group and so, by Theorem 5.1, $n + 1 \leq \text{cat}_n(\Sigma^n/G)$.

If the action of $G$ does not preserve the orientation then $\Sigma^n/G$ is homotopy equivalent to an even dimensional projective space [19, IV.3.1] and it is easy to show (applying $H_n(\cdot; \mathbb{Z}_2)$) that the natural map $g: \Sigma^n/G \rightarrow BG$ cannot be factored through an $(n-1)$-complex and so $n + 1 \leq \text{cat}_n(\Sigma^n/G)$ again. \hfill $\Box$

Krasnoselski's proof of his result is complicated. James ([16, p. 334]) asked for a proof more in the spirit of algebraic topology. Marzantowicz ([23]) provided one such proof and we have given another one above. We now give a brief proof in case $G$ has even order. Let $Z_2$ be a subgroup of order 2 of $G$. Then $\Sigma^n/G$ is covered by $\Sigma^n/Z_2$ which is homotopy equivalent to $P^n$ [19, IV.3.1]. A well known cup product argument then shows that $\text{cat}(\Sigma^n/Z_2) = n + 1$, and so, using the homotopy lifting property, we have $n + 1 \geq \text{cat}(\Sigma^n/G) \geq \text{cat}(\Sigma^n/Z_2) = n + 1$. This last proof also works in case $G$ has odd order using the fact that $\Sigma^n/Z_p$ (p a prime divisor of $|G|$) is homotopy equivalent to a lens space $L$ ([3, Lemma 1]) and the less elementary fact that $\text{cat}(L) = n + 1$ ([22], [1]).

REFERENCES


25. L. Montejano: A quick proof of Singhof's cat\( (M \times S^1) = \text{cat}(M) + 1 \) theorem, *Manuscripta Math.* 42 (1983), 49–52.


