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Extension dimensional approximation theorem

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Abstract

It is known that if an upper semicontinuous multivalued mapping $F:X \to Y$, defined on an n-dimensional compactum X, has UV^{n-1}-point images, then every neighbourhood of the graph of F (in the product $X \times Y$) contains the graph of a single-valued continuous mapping $f:X \to Y$. Similar result is known to be true when X is a compact C-space and images of F have trivial shape. We extend and unify both of these results in terms of extension theory.

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1. Introduction

Single-valued approximations of multivalued maps are proved to be very useful in geometric topology, fixed point theory, control theory and others (see a survey [7]). We consider the problem of single-valued continuous graph-approximation of upper semicontinuous (u.s.c.) multivalued mappings. We say that a multivalued mapping $F:X \to Y$ admits graph-approximations if every neighborhood of the graph of F (in the product $X \times Y$) contains the graph of a single-valued continuous mapping $f:X \to Y$.

Essentially there are three types of results concerning our problem. First assumes that multivalued mappings $F:X \to Y$ have UV^{n-1} point-images and $\dim X \leqslant n$ (see [9,10, 8]). The second type of results deal with UV^{∞}-valued mappings defined on C-spaces [1]. Finally results of the third type consider UV^{∞}-valued mappings defined on *ANR*-spaces [6,8].

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In this paper we prove an approximation theorem which generalizes and unifies the known results of the first and second types. Unification is achieved by exploiting recently created [4,5] theory of extension dimension and associated to it concepts of homotopy and shape [2]. Precise definitions will be given below in Section 2. Here we only provide some of the notation related to the extension dimension.

Let L be a CW-complex. A space X is said to have *extension dimension* $\leqslant [L]$ (notation: e-dim $X \leqslant [L]$) if any mapping of its closed subspace $A \subset X$ into L admits an extension to the whole space X. [1] It is known that $\dim X \leqslant n$ is equivalent to e-dim $X \leqslant [S^n]$ and that $\dim_G X \leqslant n$ is equivalent to e-dim $X \leqslant [K(G,n)]$ ($K(G,n)$ stands for the corresponding Eilenberg–MacLane complex). One can develop homotopy and shape theories specifically designed to work for at most $[L]$-dimensional spaces. Compacta of trivial $[L]$-shape are precisely $UV^{[L]}$-compacta [2].

Now we are ready to formulate our main result.

Theorem. *Let L be a countable CW-complex and $F : X \to Y$ be an u.s.c. $UV^{[L]}$-valued mapping of a paracompact space X to a completely metrizable space Y. If X is C-space of extension dimension* e-dim $X \leqslant [L]$, *then every neighborhood of the graph of F contains the graph of a single-valued continuous mapping $f : X \to Y$.*

Note that if L is the sphere S^n, we obtain an approximation theorem for UV^{n-1}-valued mappings of n-dimensional space. And if L is a point (or any other contractible complex), we obtain a theorem of Ancel on approximations of UV^{∞}-valued mappings of C-space [1].

What do we need to construct a mapping from a space X? Suppose that we can construct and, moreover, extend a mapping from X locally. Then one can try to obtain a fine cover of X and to construct a global mapping by induction, extending it successively over "skeleta" of this cover. The problem is to control this process when the cover has infinite order. Property C gives us a possibility of such a control.

Let us explain this with a bit more detail. A topological space X has *property* C if for each sequence $\{u^i \mid i \geqslant 1\}$ of open covers of X, there is an open cover Σ of X of the form $\bigcup_{i=1}^{\infty} \sigma_i$ such that for each $i \geqslant 1$, σ_i is a pairwise disjoint collection which refines u^i. If the space X is paracompact, we can choose the cover Σ to be locally finite. The cover Σ has very important property that every "simplex" $\{s_0, \ldots, s_n\}$ of this cover (i.e., the set of elements $\{s_0, \ldots, s_n\}$ such that $s_0 \cap \cdots \cap s_n \neq \emptyset$) has a natural order on its vertices. Indeed, for any element $s \in \Sigma$ denote by $\sigma(s)$ the integer such that $s \in \sigma_{\sigma(s)}$. Since $s_i \cap s_j \neq \emptyset$, then $\sigma(s_i) \neq \sigma(s_j)$ and we can order elements $s_0, s_1, \ldots, s_n$ according to the order of numbers $\sigma(s_0), \sigma(s_1), \ldots, \sigma(s_n)$.

We take a cover Σ of X which refines our fine cover so that every simplex $\langle \sigma_0, \ldots, \sigma_n \rangle$ of the nerve $N(\Sigma)$ has a natural order on its vertices. Then every simplex has a *basic* vertex (merely the smallest one). For every vertex $\langle \sigma \rangle$ of $N(\Sigma)$ (i.e., for every element σ of the cover Σ) we fix a "rule" of extension of mappings defined on subsets of σ. Then

[1] Everywhere below $[L]$ denotes the class of complexes generated by L with respect to the above extension property, see [4,5,2] for details.

the process goes on by induction on dimension of “skeleta” as follows: for a “simplex” the extension of mapping from the “boundary” to the “interior” is induced by the rule of the basic vertex of this simplex. Obviously the mapping on a simplex depends only on the basic vertex of this simplex and does not depend on the dimension of the simplex. This provides the needed control.

2. Preliminaries

Let us recall some definitions and introduce our notations. We denote by $\operatorname{Int} A$ and $\overline{A}$ the interior and closure of the set A, respectively. For a cover ω of a space X and for a subset $A \subseteq X$ let $\operatorname{St}(A, \omega)$ denote the star of the set A with respect to ω.

The *graph* of a multivalued mapping $F: X \to Y$ is the subset $\Gamma_F = \{(x, y) \in X \times Y\colon\ y \in F(x)\}$ of the product $X \times Y$. A multivalued mapping $F: X \to Y$ is called *upper semicontinuous* (notation: u.s.c.) if for any open set $U \subset Y$ the set $\{x \in X\colon\ F(x) \subset U\}$ is open in X.

Let L be a CW-complex. A pair of spaces $V \subset U$ is said to be $[L]$-*connected* if for every paracompact space X of extension dimension $\text{e-dim}\, X \leqslant [L]$ and for every closed subspace $A \subset X$ any mapping of A into V can be extended to a mapping of X into U. A compact subspace $K \subset Z$ is called $UV^{[L]}$-*compactum in* Z if any neighborhood U of K contains a neighborhood V of K such that the pair $V \subset U$ is L-connected. A compact-valued mapping $F: X \to Y$ is called $UV^{[L]}$-*valued* if for any point $x \in X$ the set $F(x)$ is $UV^{[L]}$-compactum in Y. A mapping $f: Y \to X$ is said to be $[L]$-*soft* if for any paracompact space Z with $\text{e-dim}\, Z \leqslant [L]$, its closed subspace $A \subset Z$ and any mappings $g: Z \to X$ and $\widetilde{g}_A: A \to Y$ such that $f \circ \widetilde{g}_A = g|_A$ there exists a mapping $\widetilde{g}: Z \to Y$ such that $f \circ \widetilde{g} = g$. Finally let $AE([L])$ denote the class of spaces with $[L]$-soft constant mappings.

Now we introduce the notion of $[L]$-extension which will represent a “rule” for extending mappings in the proof of our theorem. Let $V \subset U$ be a pair of spaces. An $[L]$-*extension* of the space V with respect to U is a pair $V' \subset W$ of spaces and a mapping $e: W \to U$ such that:

(1) $W \in AE([L])$;

(2) $e|_{V'}$ is $[L]$-soft mapping onto V.

The following is a key property of $[L]$-extensions needed in the proof (Section 3) of our theorem. Let a pair $V' \subset W$ of spaces and a mapping $e: W \to U$ represent an $[L]$-extension of the pair $V \subset U$.

$[L]$**-extension property.** *Let $A \subset B$ be a pair of closed subspaces of paracompact space X of extension dimension* $\text{e-dim}\, X \leqslant [L]$. *Suppose that we have mappings $f: B \to U$ and $g: A \to W$ such that $e \circ g = f|_A$, $f(\overline{B \setminus A}) \subset V$ and $g(A \cap \overline{B \setminus A}) \subset V'$. Then there exists a mapping $g': X \to W$ such that $e \circ g'|_B = f$.*

We construct g' in two steps. First, we use $[L]$-softness of e over V to extend g to a mapping $\widetilde{g}: B \to W$ such that $e \circ \widetilde{g} = f$ (we apply $[L]$-softness to the $[L]$-dimensional pair $A \cap \overline{B \setminus A} \subset \overline{B \setminus A}$). Finally we can extend $\widetilde{g}$ to the space X since W is $AE([L])$.

Lemma. *Let $V \subset U$ be $[L]$-connected pair. If V is completely metrizable space, then V admits an $[L]$-extension with respect to U.*

Proof. There exists a completely metrizable space V' with $\text{e-dim}\, V' \leqslant [L]$ and an $[L]$-soft mapping $e_V : V' \to V$ [3]. Consider an $AE([L])$-space W of dimension $\text{e-dim}\, W \leqslant [L]$ containing V' as a closed subspace [3]. Since the pair $V \subset U$ is $[L]$-connected, we can extend the mapping e_V to a mapping $e : W \to U$. □

3. Proof of the theorem

For a given $UV^{[L]}$-valued mapping $F : X \to Y$ we fix an arbitrary neighborhood $\mathcal{U} \subset X \times Y$ of its graph Γ_F. The proof of our theorem consists of the following two steps.

3.1. Construction of families of rectangles

For every integer $i \geqslant 0$ we construct families of open rectangles $\{u^i_\lambda \times U^i_\lambda\}_{\lambda \in \Lambda_i}$ and closed rectangles $\{v^i_\mu \times V^i_\mu\}_{\mu \in M_i}$ in the product $X \times Y$ such that:

(1) $u^0_\lambda \times U^0_\lambda \subset \mathcal{U}$ for every $\lambda \in \Lambda_0$;

(2) $u^i = \{u^i_\lambda\}_{\lambda \in \Lambda_i}$ and $v^i = \{v^i_\mu\}_{\mu \in M_i}$ are coverings of X (in fact, $\{\operatorname{Int} v^i_\mu\}_{\mu \in M_i}$ are coverings of X);

(3) $F(u^i_\lambda) \subset U^i_\lambda$ and $F(v^i_\mu) \subset \operatorname{Int} V^i_\mu$ for every $i \geqslant 0$, $\mu \in M_i$ and $\lambda \in \Lambda_i$;

(4) for every $i \geqslant 0$ and every $\mu \in M_i$ there exists $\lambda \in \Lambda_i$ such that $V^i_\mu \subset U^i_\lambda$, $v^i_\mu \subset u^i_\lambda$, and the pair $V^i_\mu \subset U^i_\lambda$ is $[L]$-connected;

  Choice 1. For given $i \geqslant 0$ and $\mu \in M_i$ we fix such a $\lambda = \lambda(\mu)$, and for $[L]$-connected pair $V^i_\mu \subset U^i_\lambda$, by lemma, we can fix $[L]$-extension $e^i_\mu : (\widetilde{V}^i_\mu, W^i_\mu) \to (V^i_\mu, U^i_\lambda)$;

(5) for every $i \geqslant 0$ and every $\lambda \in \Lambda_{i+1}$ there exists $\mu \in M_i$ such that $\operatorname{St}(u^{i+1}_\lambda, u^{i+1}) \subset v^i_\mu$ and every rectangle $u^{i+1}_\gamma \times U^{i+1}_\gamma$ is contained in the rectangle $v^i_\mu \times V^i_\mu$ provided $u^{i+1}_\gamma \cap u^{i+1}_\lambda \neq \emptyset$;

  Choice 2. For given $i \geqslant 0$ and $\lambda \in \Lambda_{i+1}$ we fix such a $\mu = \mu(\lambda)$.

First, we construct a family $\{u^0_\lambda \times U^0_\lambda\}_{\lambda \in \Lambda_0}$. Put $\Lambda_0 = X$ and for a point $x \in X$ consider a rectangle $u_x \times U^0_x \subset \mathcal{U}$ such that $F(x) \subset U^0_x$ (existence of such a rectangle follows from compactness of $F(x)$). Since F is u.s.c., we can choose a neighborhood $u^0_x \subset u_x$ of the point x such that $F(u^0_x) \subset U^0_x$.

The construction of families of rectangles is performed by induction on i. All steps of induction are similar to the first one. Here we only show how to perform the first step and to construct the families $\{v^0_\mu \times V^0_\mu\}_{\mu \in M_0}$ and $\{u^1_\lambda \times U^1_\lambda\}_{\lambda \in \Lambda_1}$.

Put $M_0 = X$ and for a point $x \in X$ consider a rectangle $u^0_\lambda \times U^0_\lambda$ containing $\{x\} \times F(x)$. By $UV^{[L]}$-property of $F(x)$ we find a closed neighborhood V^0_x of $F(x)$ such that the pair $V^0_x \subset U^0_\lambda$ is $[L]$-connected. Since F is u.s.c., we can choose a closed neighborhood $v^0_x \subset u^0_\lambda$ of the point x such that $F(v^0_x) \subset \operatorname{Int} V^0_x$.

Now we construct a family $\{u^1_\lambda \times U^1_\lambda\}_{\lambda\in\Lambda_1}$. Let α be a locally finite open cover of X refining v^0. For every element $A \in \alpha$ take an index $\mu \in M_0$ such that $A \subset v^0_\mu$ and denote $W_A = \operatorname{Int} V^0_\mu$. Then $A \times W_A$ lies in $v^0_\mu \times V^0_\mu$. Let $u^1 = \{u^1_\lambda\}_{\lambda\in\Lambda_1}$ be an open cover of X which is star-refined into α. Define

$$U^1_\lambda = \bigcap\bigl\{W_A \mid \operatorname{St}\bigl(u^1_\lambda, u^1\bigr) \subset A \in \alpha\bigr\}.$$

To verify (5), consider $u^1_{\lambda'} \in u^1$ such that $u^1_{\lambda'} \cap u^1_\lambda \neq \emptyset$. Then $u^1_{\lambda'} \subset \operatorname{St}(u^1_\lambda, u^1) \subset A$ for some $A \in \alpha$ and by definition $U^1_{\lambda'} \subset W_A$. Thus, $u^1_{\lambda'} \times U^1_{\lambda'} \subset A \times W_A \subset v^0_\mu \times V^0_\mu$.

3.2. Construction of the map f

Since X is a paracompact C-space, there exists a locally finite open cover Σ of X of the form $\Sigma = \bigcup_{i=1}^\infty \sigma_i$ such that for $i \geqslant 1$, σ_i is pairwise disjoint collection refining u^i. For every integer $k \geqslant 0$ denote by $\Sigma^{(k)}$ the set of points $x \in X$ such that the cover Σ has order $\leqslant k+1$ at x. Note that $X = \bigcup_{i=0}^\infty \Sigma^{(k)}$ and $\Sigma^{(k)}$ is closed in X. We will construct f inductively extending it over sets $\Sigma^{(k)}$.

For any element s of the cover Σ we denote by $\sigma(s)$ the integer number such that $s \in \sigma_{\sigma(s)}$.

Choice 3. For any element $s \in \Sigma$ we fix $\lambda(s) \in \Lambda_{\sigma(s)}$ such that $s \subset u^{\sigma(s)}_{\lambda(s)}$.

Let $s_0, s_1, \dots, s_n$ be elements of the cover Σ such that $s_0 \cap s_1 \cap \dots \cap s_n \neq \emptyset$. Then this set of elements could be ordered according to the order of numbers $\sigma(s_0), \sigma(s_1), \dots, \sigma(s_n)$, and the smallest element of the set $\{s_0, s_1, \dots, s_n\}$ is called the *basic element*. We always assume that s_0 is the basic element of the set $\{s_0, s_1, \dots, s_n\}$. We will use the following notations

$$[s_0, s_1, \dots, s_n] = X \setminus \bigcup\bigl\{\Sigma \setminus \{s_0, s_1, \dots, s_n\}\bigr\},$$

$$\langle s_0, s_1, \dots, s_n\rangle = (s_0 \cap s_1 \cap \dots \cap s_n) \cap \Sigma^{(n)}.$$

One should understand the set $[s_0, \dots, s_n]$ as closed n-dimensional "simplex" with interior $\langle s_0, s_1, \dots, s_n\rangle$ and boundary $\bigcup_{m=0}^n [s_0, s_1, \dots, \widehat{s_m}, \dots, s_k]$. It is easy to check that $\Sigma^{(n)} = \bigcup [s_{i_0}, s_{i_1}, \dots, s_{i_n}]$ and

$$[s_0, \dots, s_n] = \bigcup_{m=0}^n \bigl[s_0, \dots, \widehat{s_m}, \dots, s_k\bigr] \cup \langle s_0, \dots, s_n\rangle.$$

Let us construct the mapping f on the set $\Sigma^{(0)}$ which is a discrete collection of sets of the type $[s_0]$. We define f independently on every such a set. For a set $[s_0]$ we take a point $p \in F([s_0])$ and put $f([s_0]) = p$.

Let us extend f to arbitrary nonempty set $\langle s_0, s_1\rangle$. For $i = 0, 1$ we have $\langle s_i\rangle \subset u^{\sigma(s_i)}_{\lambda(s_i)}$ and then $f(\langle s_i\rangle) \subset U^{\sigma(s_i)}_{\lambda(s_i)}$ by property (3). According to the choice 2, we take $\mu \in M_{\sigma(s_0)-1}$ such that

$$[s_0, s_1] \subset \overline{\operatorname{St}\bigl(u^{\sigma(s_0)}_{\lambda(s_0)}, u^{\sigma(s_0)}\bigr)} \subset v^{\sigma(s_0)-1}_\mu \quad \text{and}$$

$$f\bigl([s_0]\bigr) \cup f\bigl([s_1]\bigr) \subset V^{\sigma(s_0)-1}_\mu.$$

Choice 1 gives us $\lambda = \lambda(\mu)$, a set $U_\lambda^{\sigma(s_0)-1}$ and $[L]$-extension

$$e_\mu^{\sigma(s_0)-1} : \bigl(\widetilde{V}_\mu^{\sigma(s_0)-1}, W_\mu^{\sigma(s_0)-1}\bigr) \to \bigl(V_\mu^{\sigma(s_0)-1}, U_\lambda^{\sigma(s_0)-1}\bigr).$$

Since the mapping $e_\mu^{\sigma(s_0)-1}|_{\widetilde{V}_\mu^{\sigma(s_0)-1}}$ is $[L]$-soft, we can lift the map $f|_{[s_0]\cup[s_1]} : [s_0] \cup [s_1] \to V_\mu^{\sigma(s_0)-1}$ to a map $g : [s_0] \cup [s_1] \to \widetilde{V}_\mu^{\sigma(s_0)-1}$. Now extend g to a mapping $\widetilde{g} : [s_0, s_1] \to W_\mu^{\sigma(s_0)-1}$ and define $f|_{[s_0,s_1]}$ as $e_\mu^{\sigma(s_0)-1} \circ \widetilde{g}$.

We can continue our construction so that the extension to a set $\langle s_0, s_1, \ldots, s_m\rangle$ uses $[L]$-extension $e_\mu^{\sigma(s_0)-1}$ and goes through $W_\mu^{\sigma(s_0)-1}$ resulting as $f|_{[s_0,\ldots,s_m]} = e_\mu^{\sigma(s_0)-1} \circ \widetilde{g}$. Therefore, the set $f([s_0, \ldots, s_m])$ is contained in $U_\lambda^{\sigma(s_0)-1}$ while the set $[s_0, \ldots, s_m]$ lies in $u_\lambda^{\sigma(s_0)-1}$. Note that both indexes λ and μ depend only on the basic element s_0 and do not depend on m. So, $[L]$-extension $e_\mu^{\sigma(s_0)-1}$ is a "rule" for constructing mapping on each set $\langle s_0, \ldots, s_m\rangle$ with basic element s_0.

Suppose that the map f is constructed on $\Sigma^{(k-1)}$. Let us extend f independently to every set of type $\langle s_0, \ldots, s_k\rangle$. Since the difference $\Sigma^{(k)} \setminus \Sigma^{(k-1)}$ is covered by a discrete family of such sets, it follows that the so obtained extension of f to $\Sigma^{(k)}$ would be continuous. Assume that s_1 is basic element of the set $\{s_1, s_2, \ldots, s_k\}$. Then the set $f(\langle s_1, \ldots, s_k\rangle)$ lies in some $U_{\lambda_1}^{\sigma(s_1)-1}$ and $u_{\lambda_1}^{\sigma(s_1)-1}$ contains $[s_1, \ldots, s_k]$. Since $\sigma(s_1) - 1 \geqslant \sigma(s_0)$, the set $f(\langle s_1, \ldots, s_k\rangle)$ lies in $V_\mu^{\sigma(s_0)-1}$ by property (5). Let

$$G = \bigcup_{1\leqslant m\leqslant k} \bigl[s_0, s_1, \ldots, \widehat{s}_m, \ldots, s_k\bigr].$$

Then, by our construction, $f|_G$ has a lift $g : G \to W_\mu^{\sigma(s_0)-1}$. Note that

$$f\bigl(\overline{\langle s_1, \ldots, s_k\rangle} \cap G\bigr) \subseteq V_\mu^{\sigma(s_0)-1} = \overline{V_\mu^{\sigma(s_0)-1}}.$$

Since the mapping $e_\mu^{\sigma(s_0)-1} : \widetilde{V}_\mu^{\sigma(s_0)-1} \to V_\mu^{\sigma(s_0)-1}$ is $[L]$-soft, we extend the lift g to the set $\overline{\langle s_1, \ldots, s_k\rangle}$. Now extend it to a mapping $g : [s_0, \ldots, s_k] \to W_\mu^{\sigma(s_0)-1}$ and define $f|_{[s_0,\ldots,s_k]}$ as the composition $e_\mu^{\sigma(s_0)-1} \circ g$.

It only remains to note that the local finiteness of Σ guarantees the continuity of the above constructed map f. Proof is completed.

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References

[1] F.D. Ancel, The role of countable dimensionality in the theory of cell-like embedding relations, Trans. Amer. Math. Soc. 287 (1985) 1–40.

[2] A. Chigogidze, Infinite dimensional topology and shape theory, in: R. Daverman, R. Sher (Eds.), Handbook of Geometric Topology, to appear.

[3] A. Chigogidze, V. Valov, Universal metric spaces and extension dimension, Topology Appl. 113 (2001) 23–27.

[4] A.N. Dranishnikov, The Eilenberg–Borsuk theorem for maps in an arbitrary complex, Math. Sb. 185 (4) (1994) 81–90 (in Russian); Translation in: Russian Acad. Sci. Sb. Math. 81 (2) (1995) 467–475.

[5] A.N. Dranishnikov, J. Dydak, Extension dimension and extension types, Trudy Mat. Inst. Steklov. 212 (1996) 61–94.

[6] L. Gorniewicz, A. Granas, W. Kryszewski, On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighborhood retracts, J. Math. Anal. Appl. 161 (1991) 457–473.

[7] W. Kryszewski, Graph-approximation of set-valued maps. A survey, in: Differential Inclusions and Optimal Control, in: Lecture Notes in Nonlinear Analysis, Vol. 2, 1998, pp. 223–235.

[8] W. Kryszewski, Graph-approximation of set-valued maps on noncompact domains, Topology Appl. 83 (1998) 1–21.

[9] R.C. Lacher, Cell-like mappings and their generalizations, Bull. Amer. Math. Soc. 83 (1977) 495–552.

[10] E.V. Ščepin, N.B. Brodsky, Selections of filtered multivalued mappings, Trudy Mat. Inst. Steklov. 212 (1996) 220–240.