Sliding mode synchronization of an uncertain fractional order chaotic system

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Abstract
Synchronization of chaotic and uncertain Duffing–Holmes system has been done using the sliding mode control strategy. Regarding the synchronization task as a control problem, fractional order mathematics is used to express the system and sliding mode for synchronization. It has been shown that, not only the performance of the proposed method is satisfying with an acceptable level of control signal, but also a rather simple stability analysis is performed. The latter is usually a complicated task for uncertain nonlinear chaotic systems.

1. Introduction
The problem of designing a system (response or slave system), which mimics the behaviour of another one (drive or master system) is called synchronization. Nowadays, synchronization of chaotic systems has been found to be more attractive due to its potential applications in secure communication and signal processing [1]. There is a close relationship between synchronization and control. From the viewpoint of control, synchronization of chaotic systems is a hard task because of their nonlinear behaviour and sensitivity to the initial values. The stability analysis of such a system could be mentioned as one of most important open areas of research in this field. This is due to the fact that for chaotic systems, with complex mathematical expression, stability analysis methods like Lyapunov are difficult to be implemented.

Fractional order calculus (FOC), an old mathematical topic from the 17th century, has recently attracted researchers in dynamic systems with chaotic behaviour, such as Chua circuit [2], Duffing system [3], jerk model [4], Chen system [5], Lü dynamics [6], Rössler model [7], Arneodo system [8] and Newton–Leipnik model [9]. Over the past two decades, due to the pioneering work of Ott et al. [10], synchronization of chaotic systems has become more and more interesting for researchers in different areas such as chaos synchronization of two Lü systems [11].

Providing a new mathematical structure to model the physical systems, FOC could attract researchers in control task in two different ways. The first attraction is providing a new era in controller design. This is due to the fact that, in most cases, controller design procedure is heuristic. FOC could provide a mathematical structure in which many characteristics of system behaviour is simply related to less number of parameters (like fractional commensurate order). In this way the paradigm of design could be changed. The second reason of attraction is related to the performance analysis of the designed system. FOC could provide a compact mathematical structure which ease stability analysis, in many cases.

Sliding mode control (SMC) is well known as a robust nonlinear control technique [12]. The main feature of SMC is that it can switch the control law very fast to drive the states of the system from any initial states onto a user-specified sliding
Let (3.2)

Fig. 1

Fractional order expression would provide better conditions both in instability analysis and design procedure. In other words, in comparison with integer order expression with the same resolution, fractional order expression would provide better conditions both in stability analysis and design procedure.

2. Basic definition and preliminaries

Three most commonly used definitions in Fractional calculus are Riemann–Liouville, Grunwald–Letnikov, and Caputo definitions.

Definition 1. Let \( m - 1 < \alpha < m, m \in N \), the Riemann–Liouville fractional derivative of order \( \alpha \) of any function \( f(t) \) is defined as follows [16]:

\[
D^\alpha_t f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(q) dq}{(t-q)^{\alpha+m-1}}
\]

(2.1.1)

where \( \Gamma \) is the Gamma function.

Definition 2. Let \( D^{(\alpha)}_t f(t) = \frac{d^m f(t)}{dt^m} \) denotes the Riemann–Liouville fractional derivative of \( f(t) \) of order \( \alpha \), the fractional derivative of Grunwald–Letnikov definition [16] is given by:

\[
D^n_\alpha f(t) \equiv \lim_{N \to \infty} \left\{ \left( \frac{N}{t} \right) \sum_{j=1}^{N} \left( \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha) \Gamma(j+1)} \right) \left( f \left( \frac{(N-j) t}{N} \right) \right) \right\}
\]

(2.1.2)

Definition 3. Let \( f \in C^m_{-1}, m \in N \). Then (left sided) Caputo fractional differential equation of \( f(x) \) is defined by [16]:

\[
D^n_\alpha f(t) = \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau \right\} \quad m-1 < \alpha < m \quad \alpha = m \in N
\]

(2.1.3)

where, \( m \), similar to \( n \) is the smallest integer number, larger than \( \alpha \).

3. Stability analysis for fractional order systems

Fractional order differential equations are at least as stable as their integer orders counterparts, because systems with memory are typically more stable than their memory–less alternatives [17]. It has been shown that the autonomous dynamic \( D^\alpha x = Ax \), \( x(0) = x_0 \) is asymptotically stable if the following condition is met [18]:

\[
|\arg(eig(A))| > q \pi /2,
\]

(3.1)

where, \( 0 < q < 1 \) and \( eig(A) \) represents the eigenvalues of matrix \( A \). In this case, each component of states decays towards 0, like \( t^{-q} \). Furthermore, the system is stable if \( |\arg(eig(A))| \geq q \pi /2 \) and those critical eigenvalues which satisfy \( |\arg(eig(A))| = q \pi /2 \) have geometric multiplicity of 1. The stability region for \( 0 < q < 1 \) is shown in Fig. 1. Now, consider the following autonomous commensurate order of fractional system:

\[
D^\alpha x = f(x),
\]

(3.2)

where \( 0 < \alpha < 1 \) and \( x_2 \in R^n \). The equilibrium points of system (3.2) are found by solving the equation:

\[
f(x) = 0.
\]

(3.3)

These points are locally and asymptotically stable if all eigenvalues of the Jacobian matrix \( A = \frac{\partial f}{\partial x} \), which are evaluated at the equilibrium points satisfy the following condition [17, 18]:

\[
|\arg(eig(A))| > q \pi /2.
\]

(3.4)

The main advantage of fractional expression of system in stability analysis is; all parameters of system (including the region of stability) could be affected by \( q \). This means more compactness in the system representation will be achieved rather than classic representation of systems. In other words, in comparison with integer order expression with the same resolution, fractional order expression would provide better conditions both in stability analysis and design procedure.
4. Sliding mode synchronization of fractional order systems

In synchronization task, there are a particular dynamic system as master and another different dynamic as slave. From the viewpoint of control, the task is to design a nonlinear controller which obtains signals from the master to tune the behaviour of the slave. Let us consider master and slave with fractional order derivative equations (1) and (2) respectively:

\[
\begin{align*}
D_q^{\alpha} x_1 &= x_2 \\
D_q^{\alpha} x_2 &= f(X,t)
\end{align*}
\]

\[
\begin{align*}
D_q^{\alpha} y_1 &= y_2 \\
D_q^{\alpha} y_2 &= f(Y,t) + \Delta f(Y,t) + d(t) + u(t)
\end{align*}
\]

where \( X = [x_1, x_2]^T \) and \( Y = [y_1, y_2]^T \) are the states of systems in (1) and (2) and \( 0 < \alpha \leq 1 \). \( f(Y,t) \) is an unknown nonlinear function which expresses system dynamics, \( \Delta f(Y,t) \) denotes uncertainty and \( d(t) \) is an acting disturbance against the performance of the system. A sliding mode control is proposed to provide the control input \( u(t) \). Indeed, the design procedure of sliding mode control has the following steps:

1- Constructing a sliding surface which represents a desired system dynamics.
2- Developing a switching control law to make the sliding mode possible on every point in the sliding surface. Any states outside the surface are driven to reach the surface in a finite time.

However, to achieve the control law, \( u(t) \), the synchronization error is defined as:

\[
e_i = x_i - y_i.
\]

The sliding surface should be designed accordingly by:

\[
S(t) = c_1 e_1 + c_2 e_2
\]

where \( c_1 \) and \( c_2 \) will be chosen in such a way that dynamic of the sliding surface will be vanished quickly. As soon as the state reaches to the surface, it should remain unchanged. This effect is usually called; the sliding mode is taken place. At this stage, the dynamic of the overall system will be controlled by the dynamic of sliding mode. Consequently, \( c_1 \) and \( c_2 \) must be chosen in such a way that the surface behaves a desired dynamic [19]. Correspondingly, the sliding mode control will be designed in two phases:

1. The reaching phase when \( S(t) \neq 0 \) and
2. The sliding phase by \( S(t) = 0 \).

A sufficient condition for the error to move from the first phase to the second one, is as follows:

\[
S(t) \dot{S}(t) \leq 0.
\]

This condition is called the sliding condition. In absence of uncertainty and external disturbance, the corresponding equivalent control force \( u_{eq}(t) \), can be obtained by \( \dot{S}(t) = 0 \). The following equation fractionalizes the classic derivative into a fractional type.

\[
\dot{S}(t) = D^{1-\alpha}(D^{\alpha}(S(t))) = 0 \rightarrow D^{\alpha}(S(t)) = 0.
\]

Control signals in the following equation drives the dynamic to reach to the sliding surface:
\[ D^q(S(t)) = c_1 D^q e_1 + c_2 D^q e_2 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} D^q(y_1 - x_1) \\ D^q(y_2 - x_2) \end{bmatrix} \]
\[ = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \left[ f(Y) - f(X) + u_{eq}(t) \right] = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \left[ f(Y) - f(X) \right] + c_2 u_{eq}(t). \tag{4.7} \]

Therefore, the equivalent control law is obtained as follows:
\[ u_{eq}(t) = \begin{bmatrix} c_1/c_2 & 1 \end{bmatrix} \begin{bmatrix} -e_2 \\ f(X) - f(Y) \end{bmatrix} = \frac{c_1}{c_2} e_2 + f(X) - f(Y). \tag{4.8} \]

To improve the robustness against model uncertainties and external disturbances such that states stay on the sliding surface, a switching control action can be integrated as:
\[ u(t) = u_{eq}(t) - KD^{q-1}(\text{sgn}(S)). \tag{4.9} \]

Candidate a Lyapunov function \( V_{SMC} = \frac{1}{2} \beta^2 \). Then, the reaching condition can be guaranteed for,
\[ K > |D^{1-q}(\Delta f(Y, t))| + |D^{1-q}(d(t))| \tag{4.10} \]
where \( |D^{1-q}(\Delta f(Y, t))| \) and \( |D^{1-q}(d(t))| \) are assumed to be bounded (i.e. \( |D^{1-q}(\Delta f(Y, t))| \leq \alpha \) and \( |D^{1-q}(d(t))| \leq \beta \)). Hence,
\[ \dot{V}_{SMC}(t) = S(t) \dot{S}(t) = S(t) D^{1-q}(c_1 e_2 + c_2) \left[ f(Y) - f(X) + \Delta f(Y, t) + d(t) + u_{eq} - KD^{q-1}(\text{sgn}(S)) \right] \]
\[ = S(t) D^{1-q}(\Delta f(Y, t) + d(t) - KD^{q-1}(\text{sgn}(S))).c_2 \]
\[ \leq c_2 \left[ -S.K(\text{sgn}(S) + |S| + \beta |S|) = -c_2 S(|K - \alpha - \beta| \leq 0. \tag{4.11} \right. \]

Eq. (4.11) confirms the existence of sliding mode dynamics. Therefore, the system is globally asymptoticaly stable. From (4.8) and (4.9), final control effort can be represented as:
\[ u(t) = -\frac{c_1}{c_2} e_2 + f(X) - f(Y) - KD^{q-1}(\text{sgn}(S)). \tag{4.12} \]

Therefore, the sliding mode controller design for fractional-order chaotic systems is accomplished.

5. Synchronization of uncertain fractional order Duffing–Holmes chaotic system

5.1. System description

In [20] an integer type of chaotic Duffing–Holmes is studied. Let us consider fractional order description of the system as:
\[
\begin{align*}
D^q x_1 &= x_2 \\
D^q x_2 &= x_1 - \alpha x_2 - x_1^3 + \beta \cos(t).
\end{align*} \tag{5.1.1}
\]

Initial conditions of master and slave are considered as: \( x_1(0) = 0.2 \), \( x_2(0) = 0.2 \) and \( y_1(0) = 0.1 \), \( y_2(0) = -0.2 \), respectively. Phase portrait of dynamic (5.1.1) for different values of \( q \) are shown in Fig. 2. Parameters \( \alpha \) and \( \beta \) are chosen 0.25 and 0.3, respectively. When \( q \) is reduced the chaos is seen reduced accordingly and the behaviour tends to become oscillatory. Practically, Duffing–Holmes slave system is perturbed by uncertainty and disturbance, which is as follows:
\[
\begin{align*}
D^q y_1 &= y_2 \\
D^q y_2 &= y_1 - \alpha y_2 - y_1^3 + \beta \cos(t) + \Delta f(Y, t) + d(t) + u(t)
\end{align*} \tag{5.1.2}
\]

where \( \Delta f(Y, t) \) and \( d(t) \) are chosen \( 0.1 \sin(t) \sqrt{y_1^2 + y_2^2} \) and \( 0.1 \sin(t) \) respectively.

5.2. Controller design

Regarding to Eq. (4.8) for the mentioned initial values, the equivalent control law will be of the following form:
\[ u_{eq}(t) = \frac{c_1}{c_2} e_2 + e_1 - \alpha x_2 - x_1^3 + \alpha y_2 + y_1^3 - 0.1 \sin(t) \left( \sqrt{y_1^2 + y_2^2} + 1 \right). \tag{5.2.1} \]

Eq. (5.2.1) results the control law, which is as follows:
\[ u(t) = \frac{c_1}{c_2} e_2 + e_1 - \alpha x_2 - x_1^3 + \alpha y_2 + y_1^3 - 0.1 \sin(t) \left( \sqrt{y_1^2 + y_2^2} + 1 \right) + K_s \text{sat}(S(t)). \tag{5.2.2} \]
Fig. 2. Phase portrait of Duffing–Holmes system for different values of the fraction commensurate.

The simulation result is shown in Fig. 3, when parameters are chosen as $c_1 = c_2 = 1$ and $K_s = 10$. The control signal, sliding surface and synchronization of states $X$ and $Y$ for $q = 0.98$ are also shown in Fig. 3(a) and (b). Similarly, the results for different values of $q = 0.96$ and $q = 0.9$ are shown in Fig. 3(c)–(f), respectively. It should be noted that the control is activated at $t = 20$ s. The significance of the sliding mode control in three simulations for different values of $q$ is obvious and a fast synchronization of master and slave is achieved. This also verifies the robustness of the designed controller.

5.3. Discussion

Some remarkable points could be immediately found from the results:

1. The performance of the designed procedure (synchronization) is satisfactory.
2. The phase portraits show how the proposed system managed to control the chaotic behaviour.
3. The low amplitude of the control signal, implies that, the implementation of the controller will not be faced any practical restrictions.
4. Stability analysis could be meaningfully done by finding appropriate Lyapunov function as a fractional function.
6. Conclusion

In this paper, a method for synchronization of chaotic and uncertain Duffing–Holmes system is addressed. The method is based on fractional order expression of the system and sliding mode controller designation. The main advantage of the proposed method is to provide the stability analysis. Furthermore, the performance of the system (synchronization) is satisfactory with acceptable amplitude of control signal. The robustness is also verified by means of different simulations.

References


