Effect of the medium viscosity on sound propagation and attenuation in ducts

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Abstract


The viscosity of the medium plays an important role in defining the characteristics of sound wave propagation in ducts. This effect, due to the difficulty of analysis, has been either completely neglected in the literature or considered only approximately. One common assumption has been that viscosity would affect the inviscid acoustic wave indirectly through its effect on the mean flow.

In this study, a mathematical model is constructed to describe the physical problem in its general form without imposing assumptions a priori. The set of equations describing the model are solved in a two-dimensional duct. The losses due to thermal conductivity of the medium are neglected in order to focus on that due to viscosity. The physical quantities have been conveniently expressed in only two nondimensional quantities, a frequency or wave number $bK$ and a Reynolds number $Re$. The effect of these two parameters on the propagation and attenuation constants was studied in detail for a wide range of $bK$ and $Re$. Results were obtained for the zeroth- and higher-order modes. The huge amount of numerical results obtained led to rather interesting conclusions.

Keywords: Duct acoustics; viscosity effect; propagation and attenuation of sound waves.

1. Introduction

Viscosity plays an important role in the field of duct acoustics. The field of duct acoustics concerns itself with the study of the propagation and attenuation characteristics of sound waves in ducts in all cases of the presence or absence of a mean flow.

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The viscosity of the medium inside the duct imposes a no-slip condition at the duct boundaries. Moreover, viscosity of the medium together with its thermal conductivity are responsible for the natural attenuation of the sound waves.

The effect of viscosity on the attenuation of acoustic modes in ducts was first discussed by Rayleigh in 1945 [13].

During the early 1950s, the problem of the effect of viscosity and heat conductivity on the acoustic wave propagation in circular, rectangular and two-dimensional ducts that carry no mean flow received considerable attention.

Extension of Rayleigh's work, alternative formulation of the problem and experimental results appeared in the literature of that period ([1-3,15] are good examples).

Later, during the 1960s and early 1970s, the interest focused on the solution of the duct acoustic problem in the presence of mean flow in the duct. The interest in this area was promoted by the need to reduce the aircrafts' engine noise that became a major public and governmental concern.

The author [4-8], among others, studied sound propagation in lined ducts (i.e., ducts with finite admittance at the walls) in the presence of viscous mean flow.

The excellent review [11] refers to over 150 references and covers all aspects of duct acoustics.

In the bulk of the literature so far, the effect of viscosity on the acoustics in ducts that carry mean flow was limited to its effect on the velocity profile of the mean flow. The acoustic disturbance was considered inviscid with the unrealistic slip along the duct boundaries. One exception to this is [10] in which the acoustic quantities are expressed as the sum of an inviscid part and a boundary layer part insignificantly outside a thin layer next to the wall. The problem is reduced to solving the inviscid acoustic case but with a modified specific wall admittance.

The approach taken in the current research is the formulation of the problem in its general form to include all variables without imposing assumptions a priori. The goal is to provide depth to the understanding of the transmission phenomena and to examine the role played by the different parameters and their effects on the characteristics of the acoustic wave.

In the present study, a two-dimensional duct is considered. The duct consists of two parallel, infinite, rigid planes at a distance $2b$ as shown in Fig. 1.

The choice of this particular geometry provides simplicity in setting the boundary conditions; meanwhile, all the trends of the effect of the different parameters would be representative of any other duct geometry as experience in this field indicates.

The physical quantities have been expressed conveniently in only two nondimensional quantities, a frequency number $bK$ and a Reynolds number $Re$. The effect of these two parameters on the propagation number and the attenuation number have been investigated for zeroth- and higher-order symmetric and anti-symmetric modes.
2. Analytical model

2.1. Basic equations

The flow field inside the duct is governed by five basic equations [14].

(1) *Continuity equation:*
\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} = 0,
\]
where \( \rho \) is the density, \( t \) is the time, \( u \) and \( v \) are the velocities in the \( x \)- and \( y \)-directions respectively.

(2) *Navier–Stokes equation: \( x \)-component of the momentum:*
\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( 2\mu \frac{\partial u}{\partial x} - \frac{2}{3} \mu \nabla \cdot \omega \right) + \frac{\partial}{\partial y} (\mu \nabla \cdot \omega);
\]
(3) *Energy equation: the energy equation for a perfect gas can be written as*
\[
\rho C_p \frac{dT}{dt} = \frac{\partial p}{\partial t} + \left\{ \frac{\partial}{\partial x} \left( k' \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k' \frac{\partial T}{\partial y} \right) \right\} + \mu \phi_{00},
\]
where \( k' \) is the thermal conductivity of the gas, \( C_p \) is the specific heat at constant pressure, \( T \) is the temperature and \( \phi_{00} \) is the dissipation function
\[
2 \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - \frac{2}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2.
\]

In order to focus on the effects of viscosity on the attenuation of the acoustic waves, we choose to neglect the effect of thermal conductivity of the gas, and in what follows we set
\[
k' = 0.
\]
Then, the energy equation for a perfect nonheat conducting gas would be
\[
\rho C_v \frac{dT}{dt} = \frac{dp}{dt} + \mu \phi_{00}.
\]
(4) *State equation:*
\[
P = \rho RT,
\]
where \( R \) is the gas constant.

The steady state (mean) quantities are \( \rho_0, U, V, P_0, T_0 \) and \( \phi_{00} \).
2.2. Perturbation of the flow

Assume that an acoustic perturbation is superimposed over the flow, e.g., from a fan, a microphone or any other acoustic source; hence the new variables will be

\[
\begin{align*}
\rho &= \rho_0 + \bar{\rho}, \\
u &= U + \bar{u}, \\
v &= V + \bar{v}, \\
P &= P_0 + \bar{P}, \\
T &= T_0 + \bar{T}, \\
\phi_{00} &= \Phi_{00} + \bar{\phi}_{00},
\end{align*}
\]

where the overbar denotes acoustic perturbation.

The equations (1)–(5) governing the flow including the acoustic wave components, after regrouping the terms for convenience, can be written as follows.

Equation (1) becomes

\[
\frac{\partial \rho_0}{\partial t} + \frac{\partial}{\partial x} (\rho_0 U) + \frac{\partial}{\partial y} (\rho_0 V) + \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\bar{\rho} \bar{u}) + \frac{\partial}{\partial y} (\bar{\rho} \bar{v}) \\
+ \frac{\partial}{\partial x} (\bar{\rho} \bar{u}) + \frac{\partial}{\partial y} (\bar{\rho} \bar{v}) = 0.
\]

Equation (2) becomes

\[
\rho_0 \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right) + \rho_0 \frac{\partial U}{\partial x} - \frac{\partial}{\partial x} \left( 2 \mu \frac{\partial U}{\partial x} - \frac{2}{3} \mu \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right) \\
- \frac{\partial}{\partial y} \left( \mu \frac{\partial U}{\partial y} + \mu \frac{\partial V}{\partial x} \right) \\
+ \frac{\partial \bar{P}}{\partial x} - \frac{\partial}{\partial x} \left( 2 \mu \frac{\partial \bar{u}}{\partial x} - \frac{2}{3} \mu \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right) - \frac{\partial}{\partial y} \left( \mu \frac{\partial \bar{u}}{\partial y} + \mu \frac{\partial \bar{v}}{\partial x} \right) \\
+ \rho_0 \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + \rho \left( \frac{\partial \bar{u}}{\partial t} + U \frac{\partial \bar{u}}{\partial x} + V \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \\
+ \rho \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0.
\]

Equation (3) becomes

\[
\rho_0 \left( \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right) + \frac{\partial P_0}{\partial y} + \frac{\partial}{\partial y} \left( 2 \mu \frac{\partial V}{\partial y} - \frac{2}{3} \mu \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \right) - \frac{\partial}{\partial x} \left( \mu \frac{\partial U}{\partial y} + \mu \frac{\partial V}{\partial x} \right)
\]
Equation (4) becomes

\[ dT + \frac{dP}{dP} + V \frac{dV}{dP} + U \frac{dU}{dP} + \frac{\partial}{\partial P} \left( \frac{\partial V}{\partial x} + V \frac{dV}{dP} \right) \]

\[ - \frac{\partial}{\partial y} \left( 2\mu \frac{\partial T}{\partial y} + \frac{\partial P}{\partial y} \right) - \frac{\partial}{\partial x} \left( \mu \frac{\partial V}{\partial y} + \mu \frac{\partial U}{\partial y} \right) \]

\[ + \left[ \rho_0 \left( \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) + \frac{\partial}{\partial P} \right] \left( \frac{\partial V}{\partial x} + V \frac{dV}{dP} + U \frac{dU}{dP} + \frac{\partial V}{\partial y} + V \frac{dV}{dP} + U \frac{dU}{dP} \right) = 0. \]

(13)

Equation (5) becomes

\[ \left[ \rho_0 C_p \frac{dT}{dt} - \frac{dP}{dt} - \mu \Phi_0 \right] + \left[ \rho_0 C_p \frac{d\bar{T}}{dt} + \bar{\rho} C_p \frac{dT}{dt} - \frac{d\bar{P}}{dt} - \mu \bar{\Phi}_0 \right] = 0. \]

(14)

Equation (5) becomes

\[ [P_0 - \rho_0 R T_0] + [\bar{P} - \rho_0 R \bar{T} - \bar{\rho} R T_0] - \left[ (\bar{\rho} R \bar{T}) \right] = 0. \]

(15)

Simplify equations (11)–(15) taking into account that

(i) the equilibrium quantities satisfy the basic equations, hence the expression in the first
square brackets in each equation vanishes;

(ii) we only consider a first-order perturbation, hence the product of perturbed quantities
will be neglected, i.e., the expression in third square brackets in each equation will, also, be
neglected;

(iii) \( \frac{\partial}{\partial t} (\rho_0, U, T_0) = 0, \quad \frac{\partial}{\partial y} \rho_0 = \frac{\partial}{\partial x} \rho_0 = 0; \)

(iv) the mean flow is parallel, i.e., \( V = 0; \)

(v) \( U = U(Y); \)

(vi) \( \mu = \text{constant}. \)

Further, let us consider harmonic variation in time so that the basic acoustic quantities can
be written as

\[ \bar{\rho} = \rho(y) \exp(i\omega t - \gamma x), \]

\[ \bar{u} = u(y) \exp(i\omega t - \gamma x), \]

\[ \bar{v} = v(y) \exp(i\omega t - \gamma x), \]

\[ \bar{P} = P(y) \exp(i\omega t - \gamma x), \]

\[ \bar{T} = T(y) \exp(i\omega t - \gamma x), \]

(16)

(17)

(18)

(19)

(20)

where \( \omega \) is the acoustic frequency, \( \gamma \) is the complex wave number \( \alpha' + i\beta' \), \( \alpha \) is the attenuation
coefficient, \( \beta' \) is the phase difference coefficient, \( i \) is the imaginary number \( \sqrt{-1} \). Substituting
(16)–(20) in (11)–(15) and dropping the argument \( y \), we get

\[ \Omega \rho = \rho_0 \left( \gamma u - \frac{dv}{dy} \right), \]

(21)
\[
\frac{d^2 u}{dy^2} + \frac{4}{3} \mu \gamma^2 u - \rho_0 \Omega u = \left( \rho_0 \frac{du}{dy} \right) v + \frac{1}{3} \mu \gamma \frac{dv}{dy} - \gamma P,
\]
(22)

\[
\frac{4}{3} \mu \frac{d^2 v}{dy^2} + \mu \gamma^2 v - \rho_0 \Omega v = \frac{1}{3} \gamma \mu \frac{du}{dy} + \frac{dP}{dy},
\]
(23)

\[
\Omega T = \frac{\Omega y}{\rho_0 C_p} P + 2 \frac{\mu}{\rho_0 C_p} \left( \frac{du}{dy} - \gamma v \right) \frac{dU}{dy}
\]
(24)

and

\[
P = R(\rho_0 T + T_0 \rho),
\]
(25)

where \( \Omega = i \omega - \gamma U \).

2.3. Normalization

It is much more practical to obtain solutions in nondimensional form. For this reason, our variables are expressed in terms of convenient basic geometric and ambient constant quantities.

2.3.1. Basic normalization quantities

- **Speed**: ambient speed of sound \( C_a = (\bar{\gamma}RT_a)^{1/2} \), where \( \bar{\gamma} \) is the ratio of specific heats \( C_p/C_v \), \( C_v \) is the specific heat at constant volume and \( T_a \) is the static ambient temperature.
- **Length**: \( b \) is the half width of the duct.
- **Time**: \( b/C_a \).
- **Density**: \( \rho_a \) is the ambient density.
- **Pressure**: \( p_a C_a^2 \).
- **Temperature**: \( T_a \) is the static ambient temperature.

2.3.2. Normalized parameters

- **Normalized velocities**: \( x \)-component \( u_n = u(y)/C_a \); \( y \)-component \( v_n = v(y)/C_a \); mean flow \( M = U(y)/C_a \) is the Mach number.
- **Normalized length**: \( y_n = y/b \).
- **Normalized density**: \( \rho_n = \rho(y)/\rho_a \), \( \rho_{on} = \rho_0/\rho_a \).
- **Reynolds number**: \( Re = \rho_a C_a b/\mu \).
- **Normalized pressure**: \( P_n = P(y)/\rho_a C_a^2 \).
- **Normalized temperature**: \( T_n = T(y)/T_a \), \( T_{0n} = T_0/T_a \) and \( \Omega(y) = i \omega - \gamma U = i \omega (1 - kM) \) where \( k = \gamma/iK \) and \( K = \omega/C_a \). Let \( \Omega_n = \Omega(y)b/C_a^2 \); then \( \Omega_n = i b K (1 - kM) \).

Note that the presence of the imaginary number \( i \) will cause the basic parameters \( u, v, P, \ldots \), etc. to be complex.

2.3.3. Normalized governing equations

Substituting the normalized parameters in the basic governing equations (21)–(25), we get the five governing equations — after dropping the subscript \( n \) — in the form

\[
\Omega \rho = \rho_0 \left( \delta u - \frac{dv}{dy} \right),
\]
(26)
\[
\frac{1}{\text{Re}} \frac{d^2 u}{dy^2} + \left( \frac{4}{3} \frac{\delta^2}{\text{Re}} - \rho_0 \Omega \right) u = \frac{\delta M}{\text{Re} \ dy} + \frac{\delta}{\text{Re} \ dy} - \delta P,
\]

(27)

\[
\frac{4}{3 \text{Re}} \frac{d^2 u}{dy^2} + \left( \frac{\delta^2}{\text{Re}} - \rho_0 \Omega \right) v = \frac{\delta}{\text{Re} \ dy} + \frac{d P}{dy},
\]

(28)

\[
\rho_0 \Omega T \frac{1}{(\gamma - 1)} = \Omega P + \frac{2}{\text{Re}} \left( \frac{du}{dy} - \delta v \right) \frac{dM}{dy},
\]

(29)

\[
P = \frac{1}{\gamma} \left( \rho_0 T + T_0 \rho \right),
\]

(30)

where

\[
\delta = \gamma b = i(bK)k = \delta_1 + i\delta_2.
\]

(31)

These are the set of five simultaneous differential equations in the five normalized acoustic parameters \( P, u, v, T \) and \( \rho \).

The problem can be further reduced by substituting (26) and (29) into (30) to eliminate \( \rho \) and \( T \), and we end up with only three equations in the three unknowns \( P, u \) and \( v \), as follows:

\[
\frac{1}{\text{Re}} \frac{d^2 u}{dy^2} + \left( \frac{4}{3} \frac{\delta^2}{\text{Re}} - \rho_0 \Omega \right) u = \frac{\delta M}{\text{Re} \ dy} + \frac{\delta}{\text{Re} \ dy} - \delta P,
\]

(32)

\[
\frac{4}{3 \text{Re}} \frac{d^2 u}{dy^2} + \left( \frac{\delta^2}{\text{Re}} - \rho_0 \Omega \right) v = \frac{\delta}{\text{Re} \ dy} + \frac{d P}{dy},
\]

(33)

\[
\Omega P - \left( \frac{2(\gamma - 1)}{\text{Re}} \frac{dM}{dy} \right) \frac{du}{dy} + \left( \delta u - \frac{dv}{dy} \right) \rho_0 T_0 - \left( \frac{2}{\text{Re}} \frac{dM}{dy} \right) \delta v.
\]

(34)

2.3.4. Important special cases

The case of inviscid acoustic waves can be obtained by assuming \( \mu = 0 \) and hence \( \text{Re} \rightarrow \infty \). The equation for pressure (34), using (32) and (33), can be written as

\[
\frac{d^2 P}{dy^2} + \frac{2k}{1 - kM} \frac{dM}{dy} \frac{dP}{dy} + (bK)^2 \left( (1 - kM)^2 - k^2 \right) P = 0,
\]

which is the same equation obtained in [12] and which was subsequently used in most studies that assumed inviscid acoustic disturbance, including [4–8].

Further, if we assume no mean flow, then

\[
M = \frac{dM}{dy} = 0,
\]

and the above pressure equation is reduced to

\[
\frac{d^2 P}{dy^2} + (bK)^2 \left( 1 - k^2 \right) P = 0,
\]
which has the general solution

\[ P = C \cos\left(\frac{bK}{\sqrt{1 - k^2}} y + \tau\right), \]

since \( \nu = 0 \) at \( y = \pm 1 \); hence from (33) we get

\[ \frac{dP}{dy} = 0 \quad \text{at} \quad y = \pm 1, \]

which results in

\[ \gamma = \sqrt{\left(-K^2 + N^2\left(\frac{\pi}{2b}\right)^2\right)} . \]

This is the well-known solution, where \( N \) is the modal number that takes the values \( 0, 1, 2, \ldots \) etc. for the zeroth, first and second, etc. modes respectively.

Note that no propagation occurs unless the frequency is higher than the cut-off frequency, i.e., \( \gamma \) is imaginary. The condition of propagation is then

\[ bK > \frac{1}{2} N\pi. \]

3. Solution of the case of zero mean flow

In this case we have

\[ M = M_0 = 0, \quad \rho_0 = 1, \quad T_0 = 1; \]

hence \( \Omega = i bK(1 - kM) = i bK \). Therefore, (32)–(34) become

\[
\begin{align*}
\frac{1}{\text{Re}} \frac{d^2u}{dy^2} + \left(\frac{4}{3} \frac{\delta^2}{\text{Re}} - i bK\right)u - \frac{\delta}{\text{Re}} \frac{dv}{dy} + \delta P &= 0, \quad (35) \\
\frac{4}{3} \frac{d^2v}{dy^2} + \left(\frac{\delta^2}{\text{Re}} - i bK\right)v - \frac{\delta}{\text{Re}} \frac{du}{dy} - \frac{dP}{dy} &= 0, \quad (36) \\
(i bK)P - \delta u + \frac{dv}{dy} &= 0. \quad (37)
\end{align*}
\]

Equation (37) gives \( P \) in terms of \( u, v \) as follows:

\[ P = \frac{\delta u}{i bK} - \frac{1}{i bK} \frac{dv}{dy}. \quad (38) \]

This, when substituted back in equations (35) and (36), results in two simultaneous equations in \( u, v \) as follows:

\[
\begin{align*}
\frac{1}{\text{Re}} \frac{d^2u}{dy^2} + \left(\frac{4}{3} \frac{\delta^2}{\text{Re}} - i bK + \frac{\delta^2}{i bK}\right)u - \left(\frac{\delta}{3 \text{Re}} + \frac{\delta}{i bK}\right)\frac{dv}{dy} &= 0, \quad (39) \\
\left(\frac{4}{3 \text{Re}} + \frac{1}{i bK}\right) \frac{d^2v}{dy^2} - \left(\frac{\delta}{3 \text{Re}} + \frac{\delta}{i bK}\right)\frac{du}{dy} + \left(\frac{\delta^2}{\text{Re}} - i bK\right)v &= 0. \quad (40)
\end{align*}
\]
After a lengthy mathematical manipulation, a solution for $u$, $v$ is found to be of the form

$$u(y) = A_1 e^{ay} + A_2 e^{-ay} + A_3 e^{by} + A_4 e^{-by}$$

and

$$v(y) = \lambda \{ \psi A_1 e^{ay} - \psi A_2 e^{-ay} + \phi A_3 e^{by} - \phi A_4 e^{-by} \},$$

where

$$\alpha^2 = \left( \frac{1}{4} C_2^2 - C_0 \right)^{1/2} - \frac{1}{2} C_2, \quad \beta^2 = -\left( \frac{1}{4} C_2^2 - C_0 \right)^{1/2} - \frac{1}{2} C_2,$$

$$C_2 = \frac{a_2}{a_4}, \quad C_0 = \frac{a_0}{a_4},$$

$$a_4 = \frac{4}{3 \Re} + \frac{1}{\Im (bK)}, \quad a_2 = \frac{8 \delta^2}{3 \Re^2} + \frac{2 \delta^2}{\Im (bK)} - \frac{7 (bK)}{3 \Re} - 1,$$

$$a_0 = \frac{4 \delta^2}{3 \Re^2} \cdot \frac{7 \delta^2 (bK)}{3 \Re} + \frac{\delta^4}{\Im (bK)} - (bK)^2 - \delta^2,$$

$$\lambda = \left( \frac{\delta}{3 \Re} + \frac{\delta}{i bK} \right)^{-1},$$

$$\psi = \frac{\alpha}{\Re} + \frac{1}{\alpha} \left( \frac{4 \delta^2}{3 \Re} - i bK + \frac{\delta^2}{i bK} \right), \quad \phi = \frac{\beta}{\Re} + \frac{1}{\beta} \left( \frac{\delta^2}{3 \Re} - i bK + \frac{\delta^2}{i bK} \right).$$

To get the values of the constants $A_1 - A_4$ and the value of $\delta = \delta_1 + i \delta_2$, we make use of the boundary conditions

$$u(\pm 1) = 0, \quad v(\pm 1) = 0.$$

Applying (41) and (42) at the boundaries, we get

$$A_1 e^{a} + A_2 e^{-a} + A_3 e^{b} + A_4 e^{-b} = 0,$$

$$A_1 e^{-a} + A_2 e^{a} + A_3 e^{-b} + A_4 e^{b} = 0,$$

$$A_1 e^{a} \psi - A_2 e^{-a} \psi + A_3 e^{b} \phi - A_4 e^{-b} \phi = 0,$$

$$A_1 e^{-a} \psi - A_2 e^{a} \psi + A_3 e^{-b} \phi - A_4 e^{b} \phi = 0.$$

Hence, we have four homogeneous equations in four unknowns, namely $A_1 - A_4$. Therefore, to have a nontrivial solution, the determinant $\Delta$ of the coefficients of these unknowns must vanish, where

$$\Delta = \begin{vmatrix}
  e^a & e^{-a} & e^b & e^{-b} \\
  e^{-a} & e^a & e^{-b} & e^b \\
  e^a \psi & e^{-a} \psi & e^b \phi & e^{-b} \phi \\
  e^{-a} \psi & e^a \psi & e^{-b} \phi & e^b \phi \\
\end{vmatrix} = 0.$$ (45)

Expanding and simplifying, the above condition can be written as

$$\Delta = 4 \{ \phi \sinh \alpha \cosh \beta - \psi \cosh \alpha \sinh \beta \} \{ \psi \sinh \alpha \cosh \beta - \phi \cosh \alpha \sinh \beta \} = 0.$$ (46)
For given parameters $bK$ and $Re$, there are discrete-eigenvalues-values for $\delta = \delta_1 + i\delta_2$ that satisfy (46). A Newton–Raphson iteration scheme was used to numerically find $\delta_1$ and $\delta_2$. Whenever the iteration converges to eigenvalues $\delta_1$ and $\delta_2$, they are used to find out the corresponding eigenfunctions $u$, $v$ and $p$ [9] to insure that the solution converges to the right mode.

The following section describes and discusses the results obtained for both $\delta_1$ and $\delta_2$ for the different propagation modes and for a wide range of the two basic parameters $Re$ and $bK$.

4. Analysis of results

As a reminder, we should recall that according to our model the acoustic wave is expressed in the nondimensional form

$$S = S(y) \exp[i(Kbt - \delta_2x - \delta_1x)],$$

where $S(y)$ is a complex quantity that varies across the duct width. $S$ represents any physical quantity associated with the sound wave, i.e., it could be sound pressure $P$, axial particle velocity $u$, transverse particle velocity $V$, temperature $T$ or density $\rho$.

During the course of this study a huge amount of numerical data was generated. A range of $bK$ from 10 to 25 and a range of $Re$ from 500 to $10^7$ were considered. For a wide range of combinations of these two physical parameters the eigenvalues $\delta_1$ (the attenuation number) and $\delta_2$ (the propagation number) were computed for the zeroth, first, second, third and fourth modes.

In the following, the effects of the medium viscosity, as represented by the Reynolds number $Re$, on the propagation and attenuation characteristics of the sound wave are discussed.

4.1. The effect of viscosity on the propagation number $\delta_2$

From the analysis presented above, it is shown that for inviscid medium, i.e., as $Re \to \infty$, the propagation constant $\delta_2$ can be computed from the formula

$$\delta_2^* = \sqrt{(bK)^2 - \left(\frac{1}{2}N\pi\right)^2}.$$

<table>
<thead>
<tr>
<th>$bK = 15$ ($\delta_2^* = 15.0000$)</th>
<th>$bK = 20$ ($\delta_2^* = 20.0000$)</th>
<th>$bK = 25$ ($\delta_2^* = 25.0000$)</th>
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<tr>
<td>$Re$</td>
<td>$\delta_2$</td>
<td>$\delta_2/\delta_2^*$</td>
</tr>
<tr>
<td>750</td>
<td>15.0053</td>
<td>1.0004</td>
</tr>
<tr>
<td>850</td>
<td>15.0118</td>
<td>1.0008</td>
</tr>
<tr>
<td>1000</td>
<td>15.0173</td>
<td>1.0012</td>
</tr>
<tr>
<td>1250</td>
<td>15.0213</td>
<td>1.0014</td>
</tr>
<tr>
<td>1500</td>
<td>15.0227</td>
<td>1.0015</td>
</tr>
<tr>
<td>1650</td>
<td>15.0230</td>
<td>1.0015</td>
</tr>
</tbody>
</table>
Table 2
The propagation number $\delta_2$ for the first mode

<table>
<thead>
<tr>
<th>$bK$ = 10 ($\delta_{2,1}^* = 9.8759$)</th>
<th>$bK$ = 15 ($\delta_{2,1}^* = 14.9175$)</th>
<th>$bK$ = 25 ($\delta_{2,1}^* = 24.9506$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,1}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,1}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,1}^*$</td>
</tr>
<tr>
<td>1650 9.9284 1.0053</td>
<td>1650 14.9777 1.0040</td>
<td>500 24.7015 0.9900</td>
</tr>
<tr>
<td>3000 9.9155 1.0040</td>
<td>3000 14.9647 1.0032</td>
<td>1000 24.9718 1.0008</td>
</tr>
<tr>
<td>10^4 9.8978 1.0022</td>
<td>10^4 14.9444 1.0018</td>
<td>4000 25.0102 1.0024</td>
</tr>
<tr>
<td>10^5 9.8828 1.0007</td>
<td>10^5 14.9261 1.0006</td>
<td>10^5 24.9617 1.0004</td>
</tr>
<tr>
<td>10^6 9.8781 1.0002</td>
<td>10^6 14.9202 1.0002</td>
<td>10^6 24.9541 1.0001</td>
</tr>
<tr>
<td>10^7 9.8766 1.0001</td>
<td>10^7 14.9184 1.0001</td>
<td>10^7 24.9517 1.0000</td>
</tr>
</tbody>
</table>

Table 3
The propagation number $\delta_2$ for the second mode

<table>
<thead>
<tr>
<th>$bK$ = 10 ($\delta_{2,2}^* = 9.4937$)</th>
<th>$bK$ = 15 ($\delta_{2,2}^* = 14.6673$)</th>
<th>$bK$ = 20 ($\delta_{2,2}^* = 19.7517$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,2}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,2}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,2}^*$</td>
</tr>
<tr>
<td>1150 9.5481 1.0057</td>
<td>1150 14.7400 1.0050</td>
<td>1150 19.8527 1.0051</td>
</tr>
<tr>
<td>10^4 9.5150 1.0022</td>
<td>10^4 14.6943 1.0018</td>
<td>10^4 19.7837 1.0016</td>
</tr>
<tr>
<td>10^5 9.5004 1.0007</td>
<td>10^5 14.6758 1.0006</td>
<td>10^5 19.7616 1.0005</td>
</tr>
<tr>
<td>10^6 9.4958 1.0002</td>
<td>10^6 14.6700 1.0002</td>
<td>10^6 19.7548 1.0002</td>
</tr>
<tr>
<td>10^7 9.4944 1.0001</td>
<td>10^7 14.6682 1.0001</td>
<td>10^7 19.7527 1.0000</td>
</tr>
</tbody>
</table>

Table 4
The propagation number $\delta_2$ for the third mode

<table>
<thead>
<tr>
<th>$bK$ = 15 ($\delta_{2,3}^* = 14.2406$)</th>
<th>$bK$ = 20 ($\delta_{2,3}^* = 19.4369$)</th>
<th>$bK$ = 25 ($\delta_{2,3}^* = 24.5519$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,3}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,3}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,3}^*$</td>
</tr>
<tr>
<td>750 14.3374 1.0068</td>
<td>750 19.5618 1.0068</td>
<td>750 24.7003 1.0060</td>
</tr>
<tr>
<td>1500 14.3085 1.0044</td>
<td>1500 19.5208 1.0043</td>
<td>1500 24.6543 1.0042</td>
</tr>
<tr>
<td>10^5 14.2432 1.0002</td>
<td>10^5 19.4400 1.0002</td>
<td>10^5 24.5553 1.0001</td>
</tr>
<tr>
<td>10^7 14.2413 1.0001</td>
<td>10^7 19.4379 1.0001</td>
<td>10^7 24.5529 1.0000</td>
</tr>
</tbody>
</table>

Table 5
The propagation number $\delta_2$ for the fourth mode

<table>
<thead>
<tr>
<th>$bK$ = 15 ($\delta_{2,4}^* = 13.6206$)</th>
<th>$bK$ = 20 ($\delta_{2,4}^* = 18.9874$)</th>
<th>$bK$ = 25 ($\delta_{2,4}^* = 24.1976$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,4}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,4}^*$</td>
<td>Re  $\delta_2$  $\delta_2/\delta_{2,4}^*$</td>
</tr>
<tr>
<td>750 13.7093 1.0065</td>
<td>750 19.0966 1.0058</td>
<td>750 24.3272 1.0054</td>
</tr>
<tr>
<td>1500 13.6844 1.0047</td>
<td>1500 19.0655 1.0041</td>
<td>1500 24.2891 1.0038</td>
</tr>
<tr>
<td>3000 13.6660 1.0033</td>
<td>3000 19.0426 1.0029</td>
<td>3000 24.2615 1.0026</td>
</tr>
<tr>
<td>10^4 13.6455 1.0018</td>
<td>10^4 19.0175 1.0016</td>
<td>10^4 24.2321 1.0014</td>
</tr>
<tr>
<td>10^6 13.6231 1.0002</td>
<td>10^6 18.9904 1.0002</td>
<td>10^6 24.2010 1.0001</td>
</tr>
<tr>
<td>10^7 13.6214 1.0001</td>
<td>10^7 18.9884 1.0001</td>
<td>10^7 24.1986 1.0000</td>
</tr>
</tbody>
</table>
For finite Re, $\delta_2$ is computed numerically and shown in Tables 1–5 for the first five modes. These results show that the propagation number is a weak function of Re. At high Re (higher than $10^4$) as Re increases, $\delta_2$ approaches $\delta_2^\star$ (its inviscid value) asymptotically from above.

For all practical purposes, the effect of viscosity on the wave propagation is negligible. This means that the speed of the wave propagation is, to a large extent, independent of the viscosity of the medium.

These findings extend and enhance the earlier conclusions of [6,8] that the wave propagation number is independent of not only viscosity of the medium but also the duct wall admittance and the profile of the viscous mean flow. The conclusion is that the propagation number for a given mode is dependent only of the nondimensional frequency $bK$ and the average Mach number of the mean flow.

4.2. The effect of viscosity on the attenuation number $\delta_1$

The effect of the medium viscosity on the attenuation of the zeroth mode is shown in Figs. 2 and 3.

Figure 2 shows that the attenuation of the sound wave due to the medium viscosity is substantial at relatively low Re. In order to clarify the significance of the numbers in these figures, we pay attention to the fact that a value of $\delta_1$ of 0.35 would mean that the sound level
would drop to less than half its value in a distance equivalent to the duct width $2b$. Hence, neglecting the medium viscosity $\rho$ in any situation of low $Re$ would result in significant errors. Figure 2 also shows that attenuation decreases as $Re$ increases. $\delta_1$, and hence attenuation, is
higher at high frequency $bK$. This fact is further illustrated in Fig. 3. At the same Re, attenuation increases rapidly as the sound frequency increases.

Figure 4 shows considerable attenuation of the first mode at $Re = 500$ and $bK = 25$, the level of the sound wave will drop to 15% of its value in a distance equivalent to one duct width. It, also, shows that attenuation decreases very rapidly as Re increases.

For a wider range of both Re and $bK$, Figs. 5 and 6 further illustrate the variation of $\delta_1$. Figure 5 shows that as $bK$ increases, the attenuation $\delta_1$ increases. The rate of increase is much higher at lower Re. Figure 6 shows that for $Re > 10^6$ the attenuation tends to zero as Re increases. At these high Re the attenuation is practically independent of the frequency $bK$. For higher modes, the results of this study show that not only the trends shown in Figs. 6 and 7 are repeated, but also the rather very interesting fact that the attenuation values do not vary much at the same Re and $bK$.

Tables 6–9 show that
(a) at low Re and high $bK$, higher modes would suffer slightly more attenuation;
(b) at high Re, higher modes have slightly less attenuation at the same $bK$.

Table 6
The attenuation number $\delta_1$, at $bK = 10$

<table>
<thead>
<tr>
<th>Re</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1650</td>
<td>0.1093</td>
<td>0.0979</td>
<td>0.0960</td>
<td>0.0960</td>
</tr>
<tr>
<td>3000</td>
<td>0.0703</td>
<td>0.0639</td>
<td>0.0621</td>
<td>0.0610</td>
</tr>
<tr>
<td>4000</td>
<td>0.0574</td>
<td>0.0524</td>
<td>0.0508</td>
<td>0.0494</td>
</tr>
<tr>
<td>5000</td>
<td>0.0492</td>
<td>0.0451</td>
<td>0.0436</td>
<td>0.0421</td>
</tr>
<tr>
<td>$10^4$</td>
<td>0.0310</td>
<td>0.0288</td>
<td>0.0276</td>
<td>0.0262</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0.0079</td>
<td>0.0075</td>
<td>0.0070</td>
<td>0.0064</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.0023</td>
<td>0.0022</td>
<td>0.0021</td>
<td>0.0018</td>
</tr>
<tr>
<td>$10^7$</td>
<td>0.0007</td>
<td>0.0007</td>
<td>0.0006</td>
<td>0.0006</td>
</tr>
</tbody>
</table>
Table 7
The attenuation number $\delta_1$, at $bK = 15$

<table>
<thead>
<tr>
<th>Re</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1650</td>
<td>0.1911</td>
<td>0.1638</td>
<td>0.1621</td>
<td>0.1630</td>
</tr>
<tr>
<td>3000</td>
<td>0.1173</td>
<td>0.1030</td>
<td>0.1016</td>
<td>0.1015</td>
</tr>
<tr>
<td>4000</td>
<td>0.0936</td>
<td>0.0831</td>
<td>0.0817</td>
<td>0.0814</td>
</tr>
<tr>
<td>5000</td>
<td>0.0788</td>
<td>0.0749</td>
<td>0.0693</td>
<td>0.0688</td>
</tr>
<tr>
<td>$10^4$</td>
<td>0.0473</td>
<td>0.0432</td>
<td>0.0423</td>
<td>0.0417</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0.0106</td>
<td>0.0101</td>
<td>0.0098</td>
<td>0.0095</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.0029</td>
<td>0.0028</td>
<td>0.0028</td>
<td>0.0027</td>
</tr>
<tr>
<td>$10^7$</td>
<td>0.0009</td>
<td>0.0009</td>
<td>0.0008</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

Table 8
The attenuation number $\delta_1$, at $bK = 20$

<table>
<thead>
<tr>
<th>Re</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>0.3281</td>
<td>0.2622</td>
<td>0.2637</td>
<td>0.2663</td>
</tr>
<tr>
<td>3000</td>
<td>0.1471</td>
<td>0.1508</td>
<td>0.1492</td>
<td>0.1496</td>
</tr>
<tr>
<td>4000</td>
<td>0.1493</td>
<td>0.1201</td>
<td>0.1186</td>
<td>0.1186</td>
</tr>
<tr>
<td>5000</td>
<td>0.1167</td>
<td>0.1009</td>
<td>0.0995</td>
<td>0.0994</td>
</tr>
<tr>
<td>$10^4$</td>
<td>0.0673</td>
<td>0.0598</td>
<td>0.0589</td>
<td>0.0586</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0.0135</td>
<td>0.0128</td>
<td>0.0126</td>
<td>0.0124</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.0035</td>
<td>0.0034</td>
<td>0.0033</td>
<td>0.0033</td>
</tr>
<tr>
<td>$10^7$</td>
<td>0.0013</td>
<td>0.0010</td>
<td>0.0010</td>
<td>0.0010</td>
</tr>
</tbody>
</table>

Table 9
The attenuation number $\delta_1$, at $bK = 25$

<table>
<thead>
<tr>
<th>Re</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>–</td>
<td>–</td>
<td>0.3673</td>
<td>0.3744</td>
</tr>
<tr>
<td>3000</td>
<td>–</td>
<td>0.2046</td>
<td>0.2054</td>
<td>0.2067</td>
</tr>
<tr>
<td>4000</td>
<td>–</td>
<td>0.1631</td>
<td>0.1620</td>
<td>0.1625</td>
</tr>
<tr>
<td>5000</td>
<td>–</td>
<td>0.1364</td>
<td>0.1350</td>
<td>0.1352</td>
</tr>
<tr>
<td>$10^4$</td>
<td>–</td>
<td>0.0791</td>
<td>0.0780</td>
<td>0.0778</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0.0167</td>
<td>0.0156</td>
<td>0.0154</td>
<td>0.0152</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.0041</td>
<td>0.0039</td>
<td>0.0039</td>
<td>0.0039</td>
</tr>
<tr>
<td>$10^7$</td>
<td>0.0017</td>
<td>0.0012</td>
<td>0.0012</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

5. Conclusions

(1) In order to study the effect of medium viscosity on the characteristics of sound wave propagation and attenuation in ducts a mathematical model was constructed without introducing any assumptions a priori.

The set of equations describing our model is solved analytically as far as possible. The eigenvalues that represent the modal propagation and attenuation numbers were obtained numerically using FORTRAN 77 with real numbers represented in double-precision mode to
insure maximal computational accuracy. Numerous numerical computational problems were encountered and were overcome.

(2) The physical quantities that govern the problem were conveniently grouped only in two nondimensional quantities, a Reynolds number Re and a frequency (or wave) number \( bK \). The effect of these two parameters on the model propagation and attenuation characteristics is presented in both graphical and tabular form.

(3) The medium viscosity affects the propagation number only slightly. This is in agreement with earlier findings, see [6].

(4) The medium viscosity affects the wave attenuation significantly in particular at high frequency and low Reynolds number. Neglecting viscosity in such a situation would lead to significant errors.

(5) At the same Re and \( bK \), the attenuation of different modes varies slightly. This leads us to expect that the profile of the sound wave changes slowly and only moderately along the duct length.

References