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# From coherent structures to universal properties

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**Abstract**

Given a 2-category $\mathscr{K}$ admitting a calculus of bimodules, and a 2-monad $\mathsf{T}$ on it compatible with such calculus, we construct a 2-category $\mathscr{L}$ with a 2-monad $\mathsf{S}$ on it such that

- $\mathsf{S}$ has the adjoint-pseudo-algebra property.
- The 2-categories of pseudo-algebras of $\mathsf{S}$ and $\mathsf{T}$ are equivalent.

Thus, coherent structures (pseudo-$\mathsf{T}$-algebras) are transformed into universally characterised ones (adjoint-pseudo-$\mathsf{S}$-algebras). The 2-category $\mathscr{L}$ consists of lax algebras for the pseudo-monad induced by $\mathsf{T}$ on the bicategory of bimodules of $\mathscr{K}$. We give an intrinsic characterisation of pseudo-$\mathsf{S}$-algebras in terms of *representability*. Two major consequences of the above transformation are the classifications of lax and strong morphisms, with the attendant coherence result for pseudo-algebras. We apply the theory in the context of internal categories and examine monoidal and monoidal globular categories (including their *monoid classifiers*) as well as pseudo-functors into $\mathscr{C}at$. 

*MSC:* 18D05; 18D10; 18D30; 18D35; 18D50

## 1. Introduction

In the categorical approach to algebraic structures we deal with them in terms of algebras for a monad. In the study of structure borne by a category we are led to consider 2-monads on $\mathscr{C}at$, the 2-category of categories, functors and natural transformations, cf. [6]. However, the strict associativity axiom for algebras is too restrictive to

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deal with the structures of interest, e.g. completeness or cocompleteness, in which the operations are associative only up to isomorphism. This prompts the consideration of *pseudo-algebras* for a 2-monad T, where the usual T-algebra axioms for a monad are weakened by the introduction of *structural isomorphisms*, subject to *coherence conditions*. Likewise, the morphisms of interest are the *strong* ones (or pseudo-morphisms) which preserve the operations only up to coherent isomorphism. When the structure of interest is actually determined by *universal properties*, e.g. a category $\mathbb{C}$ possesing certain limits or colimits, the structural isomorphisms in the pseudo-algebra are uniquely determined by such properties (they are *canonical*) and their coherence conditions are automatically satisfied. In other situations, such as $\mathbb{C}$ having a monoidal structure $(I, \otimes, \alpha, \rho, \lambda)$, there is nothing universal (in general) about the operations $(I, \otimes)$ *with respect to* $\mathbb{C}$ to imply the existence of the structural isomorphisms (associativity $\alpha$, left unit $\lambda$ and right unit $\rho$), and therefore they must be provided as part of the data, and their coherence conditions verified. However, if we regard such monoidal structure as algebraic structure on a multicategory $\mathbb{M}$ (with underlying category $\mathbb{C}$), it becomes a universal property, namely that $\mathbb{M}$ be *representable* in the sense of [11].

The developments in [11] lead us to seek a systematic way of achieving such a transformation. That is, given the 2-monad T on $\mathscr{C}at$ whose pseudo-algebras are monoidal categories (thus $T\mathbb{C}$ = free monoid on $\mathbb{C}$), we would like to derive the 2-category $\mathscr{M}ulticat$ of multicategories and the 2-monad T′ on it whose pseudo-algebras are again monoidal categories, where furthermore T′ has the *adjoint-pseudo-algebra property*. This latter means that $x : T'X \to X$ bears a pseudo-algebra structure if and only if $x \dashv \eta_X$, i.e. the structure is left adjoint to the unit. This property has been analysed in [14], and amounts to requiring $\mu \dashv \eta T'$, i.e. the adjoint condition holds for the free algebra. Thus, for a given $X$ there is up to isomorphism only one possible pseudo-algebra structure on it, if any such exists, which is completely characterised by the adjunction condition.

To formulate the question of 'transforming coherent structures into universal properties' formally, we make the following identifications:

| coherent structure ≡ pseudo-algebra for a 2-monad |
|---|

| structure with universal property ≡ (adjoint-) pseudo-algebra for a 2-monad with the adjoint-pseudo-algebra property. |
|---|

The problem can be now precisely stated as follows: given a 2-monad T on a 2-category $\mathscr{K}$, can we find a 2-category $\mathscr{K}'$ and a 2-monad T′ on it such that

1. T′ has the adjoint-pseudo-algebra property.

2. The 2-categories of pseudo-algebras, strong morphisms and transformations of $\mathsf{T}$ and $\mathsf{T}'$ are equivalent?

The main goal of this paper is to give a positive answer to this question. We do so under the following hypotheses:

- $\mathscr{K}$ admits a calculus of bimodules (i.e. pullbacks of fibrations and cofibrations, pullback stable coidentifiers and Kleisli objects for monads on bimodules, cf. Section 2).
- $\mathsf{T}$ is cartesian, preserves bimodules, their composites and identities, and their Kleisli objects, cf. Section 3.

Under these assumptions the 2-monad $\mathsf{T}$ induces a pseudo-monad $\mathsf{Bimod}(\mathsf{T})$ on $\mathsf{Bimod}(\mathscr{K})$. We show that taking $\mathscr{K}' = \mathsf{lax}\text{-}\mathsf{Bimod}(\mathsf{T})\text{-}\mathsf{alg}$, the 2-category of normal lax algebras for this pseudo-monad (with representable bimodules as morphisms) provides another basis for the axiomatisation of $\mathsf{T}$-$\mathsf{alg}$, i.e. the 2-category of $\mathsf{T}$-algebras is monadic over $\mathscr{K}'$. The 2-monad $\mathsf{T}'$ induced on $\mathscr{K}'$ by this adjunction satisfies (1) and (2) above.

The question we posed above is of a foundational nature, as vehemently argued in [3]. For a structure characterised by a universal property, it is its mere *existence* which matters, regardless of any actual *choice of representative* for the operations. Thus our result provides an alternative approach to 'avoiding the axiom of choice in category theory' to that given in [17], which considers all possible choices of representatives simultaneously.

We will further establish two important consequences from our construction of $\mathscr{K}'$ and $\mathsf{T}'$:

- *Classification of strong morphisms*: under a mild additional hypothesis on $\mathscr{K}$ and $\mathsf{T}$ (see Section 7) we can reflect pseudo-algebras and strong morphisms into the strict ones. The associated 'strong morphism classifier' is a strict $\mathsf{T}$-algebra *equivalent* to the given pseudo-algebra. This is therefore a strong coherence result.
- *Classification of lax morphisms*: the monadic adjunction $\mathsf{T}\text{-}\mathsf{alg} \to \mathscr{K}'$ induces a 2-comonad on $\mathsf{T}$-$\mathsf{alg}$ whose Kleisli 2-category corresponds to that of $\mathsf{T}$-algebras and *lax morphisms* between them, cf. Section 6. We thus get an effective and simple description of the classifier of lax morphisms out of a (pseudo-)$\mathsf{T}$-algebra, and we recover some important monoid classifiers, cf. Sections 9.1 and 10.3.

Our consideration of bimodules was motivated by the pursuit of the theory of representable multicategories in [11]. In trying to understand how to deal with the 'hom' of a multicategory $\mathbb{M}$, we observed that the hom-sets of (multi)arrows $\mathbb{M}(\vec{x}, y)$ could be organised into a bimodule $M : T\bar{\mathbb{M}} \nrightarrow \bar{\mathbb{M}}$, where $\bar{\mathbb{M}}$ is the category of linear morphisms (those with a singleton domain) of $\mathbb{M}$ and $T\bar{\mathbb{M}}$ is the free strict monoidal category on it. Furthermore, the identities and composition of $\mathbb{M}$ endow the bimodule $M$ with the structure of a normal $\mathsf{Lax}\text{-}\mathsf{Bimod}(\mathsf{T})$-algebra (or equivalently, with a monad structure as we will see in Section 4), and $\mathbb{M}$ can be recovered from such data. Thus we arrive at three alternative views of the structure 'monoidal category':

1. As a pseudo-algebra for the free-monoid monad on $\mathscr{C}at$.
2. As an adjoint pseudo-algebra on a bimodule $M : T\bar{\mathbb{M}} \nrightarrow \bar{\mathbb{M}}$ with a monad structure.
3. As a representable multicategory, where a multicategory is a monoid structure on a multigraph.

All three views are valuable. The first is the traditional one, which by virtue of the classical 'all diagrams commute' result of MacLane [16] admits a *finite presentation*. The other two approaches have the advantage of enabling us to reason by universal arguments and dispense with the coherence axioms. The third approach is technically the simplest and we will exploit it in Section 9.1 to give an explicit description of the monoid classifier. The second approach is the one that leads to the general theory we develop in the present paper and exhibits the correspondence with pseudo-algebras in a more penetrating way than the account based on multigraphs. The fact that the bimodule so constructed from a multicategory $\mathbb{M}$ corresponds to its 'hom' is made precise in [10], which exhibits it as part of a Yoneda structure on multicategories, and develops a theory for fibrations for them. Without indulging in details here, let us point out that this provides a natural example of a 2-category where the appropriate notion of fibration is *not* the representable one advocated in [21].

In fact, the second and third approaches above can be formally related (and shown equivalent) in the context of internal category theory (see Section 8). The second part of the paper is devoted to three basic examples in this context: the one already mentioned of monoidal categories, monoidal globular categories and pseudo-functors into $\mathscr{C}at$. We show their coherence results (via a technique introduced in [11] specifically for adjoint-pseudo-algebras and developed here in Section 7) and in the case of monoidal and monoidal globular categories give explicit descriptions of their monoid classifiers.

*Overview of the paper*: Part I deals with the general construction producing a 2-monad with the adjoint pseudo-algebra property from a given one, while Part II instantiates this general framework in the context of internal category theory and studies three basic examples.

*Part I*: In Section 2 we recall the basic results about bimodules (2-sided discrete fibrations) in a 2-category which we need. In Section 2.1 we review the explicit description of Kleisli objects for monads on bimodules in $\mathscr{C}at$, while in Section 2.2 we give a construction of the kind of Kleisli objects we need (which we christened *representable* since we require universality only in the context of representable bimodules) in terms of coinserters and coequifiers.

In Section 3 we introduce the kind of 2-monad we deal with, namely *cartesian* ones in which the 2-functor preserves comma-objects and coidentifiers. We derive some intrinsic properties concerning the behaviour of such monad in the context of bimodules (Lemma 3.1) which allows us to deduce an induced pseudo-monad on the bicategory of bimodules (Corollary 3.2).

In Section 4 we introduce a 2-category of lax algebras relative to the pseudo-monad previously constructed (Definition 4.3), the particular point of interest being the definition of 2-cells. We provide an alternative characterisation of this 2-category (Definition 4.4 and Proposition 4.6) as the 2-category of monads in a Kleisli bicategory of bimodules and give an algebraic description of its 2-cells in terms of equivariant morphisms.

Section 5 is the technical core of the paper. In Section 5.1 we set up the fundamental 2-adjunction between $\mathsf{T}$-alg and Lax-Bimod($T$)-alg (Proposition 5.2), the main point of note being the use of representable Kleisli objects to obtain a free $\mathsf{T}$-algebra on a lax Bimod($T$)-algebra. In Section 5.2 we show that the 2-monad induced by this adjunction has the adjoint-pseudo-algebra property (Proposition 5.3), the first important intrinsic result of the present work, whose proof involves some technical delicacy. We then establish one of our main results (Theorem 5.4), viz. the characterisation of lax Bimod($T$)-algebras which bear an adjoint-pseudo-algebra structure in terms of *representability* of their underlying bimodules and the invertibility of their structural associator 2-cell. In Section 5.3 we complete our basic general theory establishing the monadicity of the fundamental 2-adjunction (Theorem 5.6) and the correspondence of its psuedo-algebras with pseudo-$\mathsf{T}$-algebras. The proof of the theorem relies in the representable characterisation of adjoint-pseudo-algebras mentionedabove.

In Section 6 we exhibit an interesting byproduct of the setup in Section 5, namely an explicit construction of the classifier of lax morphisms between (pseudo-)$\mathsf{T}$-algebras (Theorem 6.1).

Section 7 deals with the classification of strong morphisms in terms of strict ones. In order to do so, we establish a couple of technical results regarding Kleisli objects (Propositions 7.2 and 7.3) as well as recalling a key technical lemma (Lemma 7.1) which shows that we can construct the relevant coinverter (for the classification of strong morphisms) using Kleisli objects. Theorem 7.4 is our final main result of the general theory, showing that the classification sought yields the coherence result, namely every pseudo-algebra is equivalent to a strict one (Corollary 7.5). Notice that we provide a full-fledged coherence statement, encompassing morphisms and 2-cells.

*Part II*: In Section 8 we show that given a category with pullbacks $\mathbb{B}$ and a cartesian monad $\mathsf{T}$ on it, the 2-category $\mathscr{C}at(\mathbb{B})$ of internal categories and the induced 2-monad $\mathscr{C}at(\mathsf{T})$ satisfy the hypothesis of our general theory. This requires a review of internal bimodules, in particular the construction of Kleisli objects (in Section 8.1). The main original result here is Theorem 8.2 which shows that in this context we can dispense with bimodules in favour of spans and their simple pullback composition. This in turn enables an easy explicit description of monoid classifiers for monoidal and monoidal globular categories in the subsequent sections.

Section 9 reviews the basic results of [11] as consequences of the theory in Part I. The main novelty is the recovery of the well-known monoid classifier for monoidal categories (i.e. the category of finite ordinals and monotone maps) from our general classification of lax morphisms (Theorem 9.2). Section 10 shows that Batanin's monoidal globular categories fit into our framework, since they are pseudo-algebras for a 2-monad induced by a cartesian monad on the category of $\omega$-graphs (Section 10.1). Hence we can give a universal characterisation of monoidal-globular categories in terms of *representable multicategories* on $\omega$-graphs (Corollary 10.1), recover their basic coherence result (Corollary 10.2), and their globular monoid classifier (Corollary 10.4, the basic result of [7]).

Section 11 shows how the other basic example of a coherent structure, namely pseudo-functors into $\mathscr{C}at$, are dealt with in our present theory. An interesting aspect of this example is that it exhibits clearly the representable characterisation of adjoint-pseudo-algebras (Remark 11.3). It also shows how lax functors into $\mathsf{Bimod}(\mathscr{C}at)$ arise naturally in this context, which explains their relevance for fibred category theory (Remark 11.2).

# Part I: General theory

## 2. Preliminaries on bimodules

Since there is no comprehensive account of the elementary properties of bicategories of bimodules internal to a 2-category, we collect some basic facts in this section. Background material on bimodules can be found in [21,22,26,27,8] and [2, Section 1]. In order to build a bicategory of bimodules internal to a given 2-category $\mathscr{K}$, we assume the following:

1. $\mathscr{K}$ admits pullbacks of fibrations and cofibrations.
2. $\mathscr{K}$ admits comma-objects.
3. $\mathscr{K}$ admits coidentifiers, stable under pullback along fibrations.

**2.1. Remark.** If $\mathscr{K}$ admits pullbacks, it also admits comma-objects if and only if it admits cotensors with the arrow category $^{\rightarrow}$.

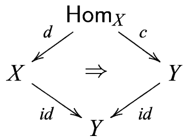
It is convenient to view a bicategory of bimodules as a 2-dimensional version of a bicategory of relations. From this perspective, our assumptions on $\mathscr{K}$ amount to *regularity* for an ordinary category. With $\mathscr{K}$ as above, recall that a *bimodule* or *discrete fibration* from $X$ to $Y$ (objects of $\mathscr{K}$) is a span

$$\begin{array}{ccc} & R & \\ {\scriptstyle d_R}\swarrow & & \searrow{\scriptstyle c_R} \\ X & & Y \end{array}$$

which is discrete as an object in $\mathbf{Spn}(\mathscr{K})(\mathbf{X},\mathbf{Y})$, i.e. has discrete fibres, and bears an algebra structure for the monad

$$\mathsf{Hom}_X \circ (\_) \circ \mathsf{Hom}_Y : \mathbf{Spn}(\mathscr{K})(\mathbf{X},\mathbf{Y}) \to \mathbf{Spn}(\mathscr{K})(\mathbf{X},\mathbf{Y})$$

where $\_ \circ \_$ is composition of spans (via pullbacks) and the following diagram is a comma-object:



Therefore $\mathsf{Hom}_X$ amounts to the cotensor $X^{\to}$. Such an algebra structure on the span $R$ means that $d_R : R \to X$ is a split fibration, $c_R : R \to Y$ is a split cofibration and their actions are compatible. Hence, such an algebra structure is unique up to isomorphism if it exists.

**2.2. Remark.** (*Fibrations in* $\mathscr{C}at$)**.** The internal definition of fibrations in a 2-category, as introduced in [21], reflects the situation in $\mathscr{C}at$: given a functor $p : \mathbb{E} \to \mathbb{B}$, the free fibration over it (in $\mathscr{C}at/\mathbb{B}$) is given by the projection $\bar{p} : id_{\mathbb{B}} \downarrow p \to \mathbb{B}$ out of the comma-object. This defines a 2-monad $id_{\mathbb{B}} \downarrow \_ : \mathscr{C}at/\mathbb{B} \to \mathscr{C}at/\mathbb{B}$ with the adjoint-pseudo-algebra property (cf. Section 1). Thus, $p$ is a fibration iff the unit $\eta_p : p \to id_{\mathbb{B}} \downarrow p$ has a right-adjoint. This latter amounts to a choice of *cleavage* for $p$. This characterisation of fibrations (and its simple extension to deal with 2-sided discrete fibrations or bimodules) can be internalised in any 2-category with comma-objects, and it is this internal version we work with throughout.

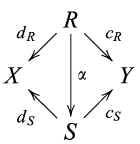
Now we can define the bicategory of bimodules in such a 2-category $\mathscr{K}$:

**2.3. Definition.** The *bicategory of bimodules* $\mathsf{Bimod}(\mathscr{K})$ consists of

*Objects*: those of $\mathscr{K}$.

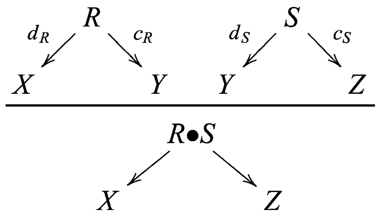
*Morphisms*: a morphism from $X$ to $Y$ is a bimodule from $X$ to $Y$, which we write $R : X \nrightarrow Y$.

*2-cells*: a 2-cell between morphisms is a morphism between the top objects of the spans, commuting with the domain and codomain morphisms and the actions of the fibrations and cofibrations:



In short, $\alpha$ is a $\mathsf{Hom}_X \circ (\_) \circ \mathsf{Hom}_Y$-algebra morphism.

The identity span on $X$ is $id \downarrow id = \mathsf{Hom}_X : X \nrightarrow X$. Composition is given by



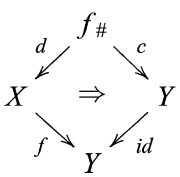

where $R \bullet S$ is given by the following coequalizer:

$$R \circ \mathsf{Hom}_Y \circ S \underset{R\circ r}{\overset{l\circ S}{\rightrightarrows}} R \circ S \twoheadrightarrow R \bullet S$$

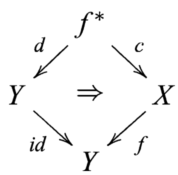
with $l$ being the action of the cofibration $c_R : R \to Y$ and $r$ being the action of the fibration $d_S : S \to Y$. Horizontal composition of 2-cells is canonically induced by that of morphisms, while their vertical composition is inherited from that of 1-cells in $\mathscr{K}$. When drawing diagrams, we display bimodules by bent arrows, reserving straight arrows for 1-cells in $\mathscr{K}$.

We recall the following properties of the bicategory $\mathsf{Bimod}(\mathscr{K})$:

1. A morphism $f : X \to Y$ in $\mathscr{K}$ gives rise to the *representable bimodule* $f_\# : X \nrightarrow Y$ defined by the comma-object




   and to $f^* : Y \nrightarrow X$ given by the comma-object




   Furthermore, $f_\# \dashv f^*$.
2. *Embeddings*: There is a homomorphism $(\_)_\# : \mathscr{K}^{\mathrm{co}} \to \mathsf{Bimod}(\mathscr{K})$: which sends a morphism $f : X \to Y$ to the representable bimodule $f_\# : X \nrightarrow Y$, and a homomorphism $(\_)^* : \mathscr{K}^{\mathrm{op}} \to \mathsf{Bimod}(\mathscr{K})$ with action $(f : X \to Y) \mapsto (f^* : Y \nrightarrow X)$.

Both these homomorphisms are locally fully faithful (that is, full and faithful on 2-cells).

3. There is a particular instance of composition which deserves special interest: given morphisms $f : X \to Z$ and $g : Y \to Z$, the composite bimodule $f_\# \bullet g^* : X \nrightarrow Y$ is given by the top span of the comma-object

$$\begin{array}{ccccc} & & f\downarrow g & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & \Rightarrow & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & Z & & \end{array}$$

meaning that the canonical comparison $f \downarrow g \to f_\# \bullet g^*$ is an isomorphism. We have therefore the following exactness property:

$$p^* \bullet q_\# \cong f_\# \bullet g^* \quad \text{canonically}$$

4. It follows from the previous item that, given a morphism $f : X \to Y$, the unit $\tilde{f} : \mathsf{Hom}_X \Rightarrow f_\# \bullet f^*$ of the adjunction $f_\# \dashv f^*$ is an isomorphism iff $f$ is (representably) fully faithful.
5. Every bimodule

$$\begin{array}{ccccc} & & R & & \\ & \swarrow^{d_R} & & \searrow^{c_R} & \\ X & & & & Y \end{array}$$

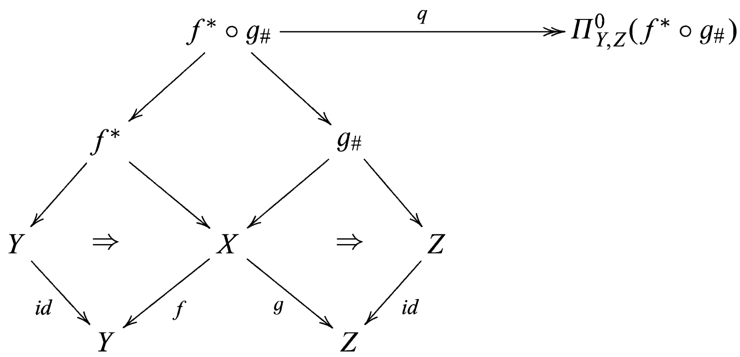
has a canonical factorisation $R \cong d_R^* \bullet (c_R)_\#$. This factorisation is compatible with 2-cells, in the sense that given

$$\begin{array}{ccccc} & & R & & \\ & \swarrow^{d_R} & \Big\downarrow & \searrow^{c_R} & \\ X & & \alpha & & Y \\ & \nwarrow_{d_S} & \Big\downarrow & \nearrow_{c_S} & \\ & & S & & \end{array}$$

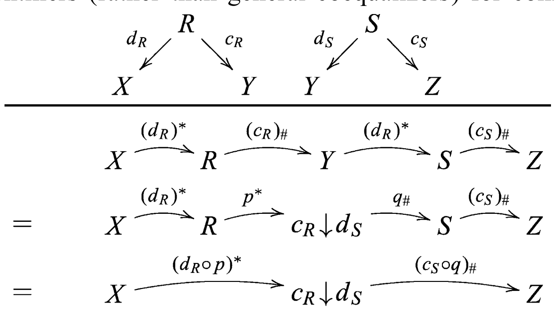
it induces $\alpha^* : d_R^* \to d_S^*$ and $\alpha_\# : (c_R)_\# \to (c_S)_\#$ by the universal property of comma-objects, and then

$$\begin{array}{ccc} R & \xrightarrow{\sim} & d_R^* \bullet (c_R)_\# \\ {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle\alpha^*\bullet\alpha_\#} \\ S & \xrightarrow{\sim} & d_S^* \bullet (c_S)_\# \end{array}$$

6. Let us analyse what the composition of a representable bimodule and a dual of such amount to



thus $f^* \bullet g_\# = \Pi^0_{Y,Z}(f^* \circ g_\#)$ where the latter is the ‘connected components’ (i.e. the reflection into a discrete object) of $f^* \circ g_\#$ in $\mathbf{Spn}(\mathscr{K})(\mathbf{Y},\mathbf{Z})$. Thus, we only require such coidentifiers (rather than general coequalizers) for composition of bimodules

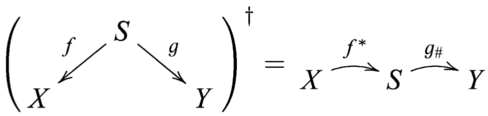


where $p$ and $q$ are the projections out of the comma-object $c_R \downarrow d_S$ as in (3).

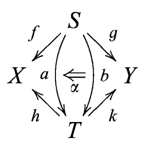
7. There is a reflection

$$\mathbf{Bimod}\,(\mathscr{K})(X,Y) \underset{\perp}{\overset{\dagger}{\rightleftarrows}} \mathbf{Spn}(\mathscr{K})(\mathbf{X},\mathbf{Y})$$

given as follows:



Therefore this reflection sends the 2-cells of $\mathbf{Spn}(\mathscr{K})(\mathbf{X},\mathbf{Y})$,



to identities, $\alpha^\dagger = id$: $a^\dagger = b^\dagger$.

8. *Duality*: Every bimodule $R : X \nrightarrow Y$ has associated its *dual* $R^{\mathrm{o}} : Y \nrightarrow X$ given by $R^{\mathrm{o}} = c_R^* \bullet (d_R)_\#$.
   The assignment $R \mapsto R^{\mathrm{o}}$ extends to an identity-on-objects biequivalence $(\_)^{\mathrm{o}} : \mathsf{Bimod}(\mathscr{K}) \to \mathsf{Bimod}(\mathscr{K})^{\mathrm{op}}$.
   To describe its action on 2-cells, given $\alpha : R \Rightarrow S$,
   $$\alpha^{\mathrm{o}} = \alpha^* \bullet \alpha_\# : c_R^* \bullet (d_R)_\# \;\Rightarrow\; c_S^* \bullet (d_S)_\#$$
   where $\alpha^* : c_R^* \Rightarrow c_S^*$ and $\alpha_\# : (d_R)_\# \Rightarrow (d_S)_\#$ are induced by universality of comma-objects, as above.
9. *Change of base*: Given a bimodule $R : X \nrightarrow Y$ and morphisms $f : X' \to X$ and $g : Y' \to Y$, we obtain a bimodule $f_\# \bullet R \bullet g^* : X' \nrightarrow Y'$. We thus have a *change of base* functor $(f,g)^* : \mathsf{Bimod}(\mathscr{K})(X,Y) \to \mathsf{Bimod}(\mathscr{K})(X',Y')$, whose action can be described more simply via pullbacks; the bimodule $(f,g)^*(R)$ is obtained as indicated in the following limit diagram:
   $$\begin{array}{ccccc} X' & \xleftarrow{d'} & (f,g)^*(R) & \xrightarrow{c'} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow[d]{} & R & \xrightarrow[c]{} & Y \end{array}$$
   In $\mathscr{K} = \mathscr{C}at$, we have $(f,g)^*(R)(x',y') = R(fx', gy')$.

### 2.1. Monads and Kleisli objects

In our intended application of bimodules, monads and their associated Kleisli objects in $\mathsf{Bimod}(\mathscr{K})$ play a central role. As a helpful analogy, recall that for a given commutative ring $R$, an $R$-algebra $A$ can be presented either as a monoid in $R$-mod or as a ring together with a ring homomorphism $R \to A$ (whose image lies in the center of $A$). This latter view amounts to taking the Kleisli object of the monad (in the bicategory of rings and bimodules) corresponding to the former presentation.

We recall the explicit description of Kleisli objects for monads on bimodules in $\mathscr{C}at$. Given a monad $(M : X \nrightarrow X, \eta, \mu)$ in $\mathsf{Bimod}(\mathscr{C}at)$, its Kleisli object is given by the category $\underline{M}$ with

*Objects*: those of $X$.

*Morphisms*: $\underline{M}(x,y) = M(x,y)$, the fibre of the bimodule $M$ over the objects $x, y$.

*Identities*: $\underline{id}_x = \eta_{x,x}(id_x) \in M(x,x)$

*Composition*:
$$\frac{f \in M(x,y) \qquad g \in M(y,z)}{g \circ f = \mu_{x,z}([f,g]) \in M(x,z)}$$
where $[f,g]$ represents the equivalence class of the pair $(f,g)$ in $M \bullet M$.

The unit $\eta : \mathsf{Hom}_X \Rightarrow M$ induces a functor $J : X \to \underline{M}$, which is the identity on objects. There is a 2-cell $\rho : M \bullet J_\# \Rightarrow J_\#$ which sets up a universal lax cocone. Notice that we recover the monad $M$ as the composite $J_\# \bullet J^* \cong J \downarrow J$, which is the resolution of the monad obtained via its Kleisli object. Just as in the case of algebras mentioned above, to give a monad on $X$ is equivalent then to give a category $M$ and an identity-on-objects functor $J:X \to M$.

**2.4. Remark.** It follows from the above explicit description that given a lax cocone $\gamma : M \bullet L \Rightarrow L : X \nrightarrow Y$ in $\mathsf{Bimod}(\mathscr{C}at)$, if $L : X \nrightarrow Y$ is representable ($L = l_\#$) so is the induced mediating bimodule $\hat{L} : \underline{M} \nrightarrow Y$ ($\hat{L} = \hat{l}_\#$), and this preservation of representability is 2-functorial in the sense that given a morphism $k : Y \to Z$ in $\mathscr{K}$, we have $\widehat{k \circ l} = \hat{k} \circ \hat{l}$ and similarly for 2-cells between such morphisms, cf. [27, Axiom 5]. We say that Kleisli objects in $\mathsf{Bimod}(\mathscr{C}at)$ *preserve representability*.

## *2.2. Kleisli objects for bimodules via coinserters and coequifiers*

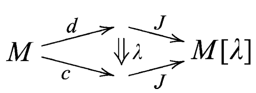
In order to set up the monadic adjunction in Section 5.1 below we must assume that Kleisli objects for monads in $\mathsf{Bimod}(\mathscr{K})$ behave like those in $\mathsf{Bimod}(\mathscr{C}at)$. In fact, we only need the universal property of Kleisli objects (i.e. initial lax cocone) with respect to lax cocones $\gamma : M \bullet L \Rightarrow L : X \nrightarrow Y$ with $L$ *representable*, with the induced mediating bimodules representable as well (as in Remark 2.4). We then say that $\mathsf{Bimod}(\mathscr{K})$ *admits representable Kleisli objects*.

**2.5. Assumption.** $\mathsf{Bimod}(\mathscr{K})$ admits representable Kleisli objects and $\mathsf{Bimod}(T)$: $\mathsf{Bimod}(\mathscr{K}) \to \mathsf{Bimod}(\mathscr{K})$ preserves them.

We give an explicit construction of such representable Kleisli objects in $\mathscr{K}$ using colimits, so as to have sufficient conditions on $\mathscr{K}$ and $T$ for the above assumption to hold. Before embarking on the abstract construction, it might be helpful for the reader to think of it as a 2-dimensional analogue of 'taking a quotient by an equivalence relation', which in a regular category is achieved by taking the coequalizer of the pair of morphisms (*domain*, *codomain*) of the relation. Here, we first add a 2-cell between such pair and then impose the 'lax cocone' equations (thus coequalizing at the level of 2-cells, rather than 1-cells). Thus our present work may be seen as some foundational 2-dimensional algebra intrinsic to a 'regular 2-category'. We rely on the notions of coinserter and coequifier in a 2-category, which can be found in the survey article [13].

**2.6. Proposition.** *If $\mathscr{K}$ admits coinserters and coequifiers, then* $\mathsf{Bimod}(\mathscr{K})$ *admits representable Kleisli objects.*

**Proof.** Consider a monad $(M : X \nrightarrow X, \eta, \mu)$ in $\mathsf{Bimod}(\mathscr{K})$. Let $M[\lambda]$ be the (object part of) the coinserter of the pair $d, c : M \to X$:



Notice that the 2-cell $\lambda$ sets up a lax cocone over $M \simeq d^* \bullet c_\#$ as follows:

$$\frac{\dfrac{\lambda : Jd \Rightarrow Jc}{\lambda_\# : c_\# \bullet J_\# \Rightarrow d_\# \bullet J_\#}}{\rho = (\check{\lambda_\#}) : J_\# \Rightarrow d^* \bullet c_\# \bullet J_\#}$$

But this lax cocone need not satisfy the equations involving $\eta$ and $\mu$, so we must enforce them. For the unit, pasting $\rho$ and $\eta$, we obtain the 2-cell $\rho \cdot J_\# \eta : J_\# \Rightarrow J_\#$, which by (contravariant) local full and faithfullness of $\_{}_\#$, corresponds to a (unique) 2-cell in $\mathscr{K}$, written $\kappa : J \Rightarrow J$. Consider the coequifier of this 2-cell and the identity:

$$X \underset{J}{\overset{J}{\rightrightarrows}} \Downarrow\kappa \; \Downarrow id \; M[\lambda] \xrightarrow{\pi} M[\lambda]_{/(\kappa \equiv id)}$$

We obtain thus a new cocone $\pi J : X \to M[\lambda]_{/(\kappa \equiv id)}$ with $\pi\rho : M \bullet \pi J_\# \Rightarrow \pi J_\#$. Again,[1] we must impose on this data the condition for $\mu$. We have two 2-cells

$$\pi\rho \cdot \pi\rho M : M \bullet M \Rightarrow M \qquad \pi\rho \cdot \mu : M \bullet M \Rightarrow M$$

Using the following comma square

$$\begin{array}{ccccc} & & c{\downarrow}d & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ M & & \Rightarrow & & M \\ & \underset{c}{\searrow} & & \underset{d}{\swarrow} & \\ & & X & & \end{array}$$

the above 2-cells determine, by adjoint transposition, 2-cells

$$(\pi\rho \cdot \check{\pi}\rho M) : q_\# \bullet c_\# \bullet J_\# \Rightarrow p_\# \bullet d_\# \bullet J_\# \qquad (\pi\check{\rho} \cdot \mu) : q_\# \bullet c_\# \bullet J_\# \Rightarrow p_\# \bullet d_\# \bullet J_\#$$

which by (contravariant) local full and faithfullness of $\_{}_\#$, correspond to two parallel 2-cells $\mu_l, \mu_r : Jdp \Rightarrow Jcq$, which we coequify:

$$c{\downarrow}d \underset{Jcq}{\overset{Jdp}{\rightrightarrows}} \Downarrow\mu_l \; \Downarrow\mu_r \; M[\lambda]_{/(\kappa \equiv id)} \xrightarrow{\varpi} M[\lambda]_{/(\kappa \equiv id, \mu_l \equiv \mu_r)}$$

so that $\underline{M} = M[\lambda]_{/(\kappa \equiv id, \mu_l \equiv \mu_r)}$ is the Kleisli object, with universal lax cocone $\varpi\pi J : X \to \underline{M}$ and $\varpi\pi\rho : M \bullet \varpi\pi J_\# \Rightarrow \varpi\pi J$. Universality and preservation of representability follow at once. □

**2.7. Corollary.** *If* $T : \mathscr{K} \to \mathscr{K}$ *preserves coinserters and coequifiers, then* $\mathsf{Bimod}(T) : \mathsf{Bimod}(\mathscr{K}) \to \mathsf{Bimod}(\mathscr{K})$ *preserves representable Kleisli objects.*

[1] If $\mathscr{K}$ admits sums, we could perform both ‘coequifications’ in one step.

**2.8. Remark.**

- Notice that when $\mathscr{K}$ admits a Yoneda structure (in the sense of [25]), so that a bimodule $M : X \nrightarrow Y$ is classified by a 1-cell $\hat{M} : Y \to PX$ (e.g. for $\mathscr{K} = \mathscr{C}at$, the Yoneda object is $P\mathbb{X} = [\mathbb{X}^{\mathrm{op}}, \mathscr{S}et]$), this restricted universal property with respect to representables entails the general one, so that $\mathsf{Bimod}(\mathscr{K})$ admits Kleisli objects in full generality (and moreover, representability is preserved).
- In Section 8.1 we show that for $\mathscr{K} = \mathscr{C}at(\mathbb{B})$, the 2-category of internal categories in $\mathbb{B}$, $\mathsf{Bimod}(\mathscr{K})$ admits Kleisli objects preserving representability, although it does not admit a Yoneda structure in general, i.e. internal bimodules are not classifiable by internal functors.

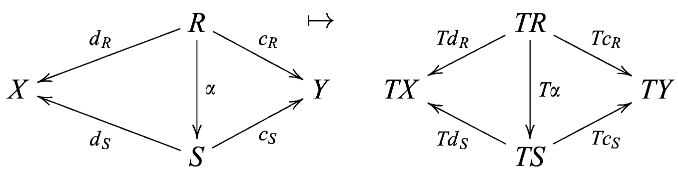
## 3. Bicategory of bimodules associated to a cartesian 2-monad

Given a 2-category $\mathscr{K}$ admitting a calculus of bimodules as in Section 2, we consider a 2-monad $(T, \eta, \mu)$ on it which induces a pseudo-monad on $\mathsf{Bimod}(\mathscr{K})$. More precisely, we assume

1. $T : \mathscr{K} \to \mathscr{K}$ preserves pullbacks, comma-objects and the coidentifiers $f^* \circ g_\# \to \Pi^0(f^* \circ g_\#)$ of Section 2(6).
2. $\eta : 1 \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are cartesian 2-natural transformations, i.e. the naturality squares are pullbacks.

Notice that by (1) $T$ preserves bimodules, their identities and composites. It does therefore induce a homomorphism

$$\mathsf{Bimod}(T) : \mathsf{Bimod}(\mathscr{K}) \longrightarrow \mathsf{Bimod}(\mathscr{K})$$




We want such a 2-monad to induce both a pseudo-monad $(\mathsf{Bimod}(T), \eta_\#, \mu_\#)$ and a pseudo-comonad $(\mathsf{Bimod}(T), \eta^*, \mu^*)$ on $\mathsf{Bimod}(\mathscr{K})$ (see e.g. [9,15,18] for details on pseudo-monads). Hence we must show that the cartesian 2-natural transformations $\eta$ and $\mu$ induce pseudo-natural transformations. This follows from the technical lemma below.

**3.1. Lemma.** *Let $F, G : \mathscr{L} \to \mathscr{K}$ be 2-functors, with $G$ preserving cotensors with $^{\to}$, and $\alpha : F \Rightarrow G$ be a cartesian 2-natural transformation. The following properties hold*:

1. *Every component $\alpha_x : Fx \to Gx$ is both a split fibration and cofibration.*

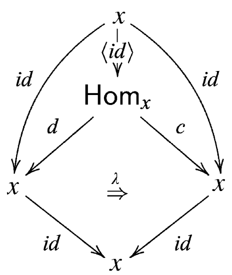
2. *For $f : x \to y$, the naturality square*

$$\begin{array}{ccc} Fx & \xrightarrow{F_y} & Fy \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ Gx & \xrightarrow[G_f]{} & Gy \end{array}$$

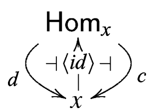
*induces an isomorphism $\theta_f : (\alpha_x)_\# \bullet Gf_\# \xrightarrow{\sim} Ff_\# \bullet (\alpha_y)_\#$.*
*Its adjoint transpose $\check{\theta}_f : (Ff)^* \bullet (\alpha_x)_\# \to (\alpha_y)_\# \bullet (Gf)^*$ is also an isomorphism.*

**Proof.**

1. Recall that for a given object $x$




we have the string of adjunctions

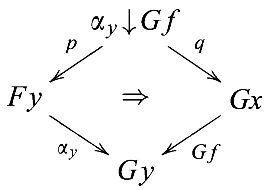



For $\alpha_x : Fx \to Gx$, cartesianness implies that

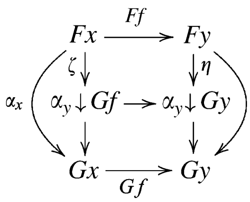
$$\begin{array}{ccc} F\mathsf{Hom}_x & \xrightarrow{Fc} & F_x \\ {\scriptstyle {}^{\alpha}\mathsf{Hom}_X}\downarrow & & \downarrow{\scriptstyle \alpha_x} \\ G\mathsf{Hom}_x & \xrightarrow{Gc} & Gx \\ {\scriptstyle Gd}\downarrow & {\scriptstyle G\lambda}\Nearrow & \downarrow{\scriptstyle id} \\ Gx & \xrightarrow[id]{} & Gx \end{array}$$

is the comma object $Gx \downarrow \alpha_x$. Thus since $F\langle id\rangle : Fx \to F\,\mathsf{Hom}\,x$ is left adjoint to $Fc : F\,\mathsf{Hom}\,x \to Fx$, also $\eta : Fx \to Gx \downarrow \alpha_x$ is left adjoint to the projection $Gx \downarrow \alpha_x \to Fx$; and so $\alpha_x$ is a split fibration. The same argument in $\mathscr{K}^{co}$ yields the split cofibration structure.

2. Recall that $(\alpha_y)_\# \bullet (Gf)^* : Fy \nrightarrow Gx$ is the comma-object




and therefore the naturality square induces a canonical morphism $\zeta : Fx \to \alpha_y \downarrow Gf$ in $\mathbf{Spn}(\mathscr{K})(Fy, Gx)$. Since $\alpha_y$ is a split cofibration, the canonical morphism $\eta : Fy \to \alpha_y \downarrow Gy$ has a left adjoint $l \dashv \eta$ over $Gy$. Considering the following diagram




where both squares and the outer rectangle are pullbacks, we see that $\zeta$ is the pullback of $\eta$ along $Gf$, and therefore[2] has a left adjoint $Gf^*(l) \dashv \zeta$. This adjunction is taken by the reflection Section 2(7) to an isomorphism $\zeta^\dagger : (Ff)^* \bullet (\alpha_x)_\# \xrightarrow{\sim} (\alpha_y)_\# \bullet (Gf)^*$ which corresponds to $\check{\theta}_f$. □

**3.2. Corollary.** *A cartesian* 2*-natural transformation* $\alpha : F \Rightarrow G$ *as in Lemma* 3.1, *with both* $F, G : \mathscr{L} \to \mathscr{K}$ *preserving pullbacks, comma-objects and the relevant coidentifiers, induces pseudo-natural transformations*

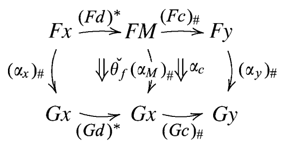
$$\mathsf{Bimod}(\alpha_\#) : \mathsf{Bimod}(F) \Rightarrow \mathsf{Bimod}(G)$$

*and*

$$\mathsf{Bimod}(\alpha^*) : \mathsf{Bimod}(G) \Rightarrow \mathsf{Bimod}(F).$$

**Proof.**

- $\mathsf{Bimod}(\alpha_\#)$: Given a bimodule $M \cong d^* \bullet c_\# : X \nrightarrow Y$ define the invertible 2-cell $\alpha_M : FM \bullet (\alpha_y)_\# \Rightarrow (\alpha_x)_\# \bullet GM$ as the pasting composite




  where the left isomorphism is that of Lemma 3.1(2) and the right one is given by the homomorphism $(\_)_\#$ applied to the naturality square for $\alpha$.

[2] Pullback along $Gf$ yields a 2-functor $Gf^* : \mathscr{K}/Gy \to \mathscr{K}/Gx$ which takes the left adjoint $l$ to a left adjoint $Gf^*l$ of $Gf^*(\eta) = \zeta$.

- $\mathsf{Bimod}(\alpha^*)$: Use the same argument in $\mathscr{K}^{\mathrm{co}}$. □

**3.3. Corollary.** *Given $(T, \eta, \mu)$ a cartesian 2-monad on $\mathscr{K}$, with $T$ preserving comma-objects, it induces both a pseudo-monad* $(\mathsf{Bimod}(T), \eta_\#, \mu_\#)$ *and a pseudo-comonad* $(\mathsf{Bimod}(T), \eta^*, \mu^*)$ *on* $\mathsf{Bimod}(\mathscr{K})$.

**3.4. Remark.** (*Coherence assumptions*). We make the following simplifying assumptions allowed by coherence for bicategories:
- We regard $\mathsf{Bimod}(\mathscr{K})$ as a 2-category, thereby ignoring the associativity constraints.
- We regard $\mathsf{Bimod}(T) : \mathsf{Bimod}(\mathscr{K}) \to \mathsf{Bimod}(\mathscr{K})$, $(\_)_\# : \mathscr{K}^{\mathrm{co}} \to \mathsf{Bimod}(\mathscr{K})$ and $(\_)^* : \mathscr{K}^{\mathrm{op}} \to \mathsf{Bimod}(\mathscr{K})$ as 2-functors, thereby ignoring the coherent structural isomorphisms for composites and identities.

Now we can consider the Kleisli bicategory $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$ associated to this pseudo-comonad, which we describe explicitly as follows:

*Objects*: those of $\mathscr{K}$.

*Morphisms*: a morphism from $X$ to $Y$ is a span

$$TX \xleftarrow{\;d_R\;} R \xrightarrow{\;c_R\;} Y$$

which is a discrete fibration from $TX$ to $Y$, or more simply a 1-cell in $\mathsf{Bimod}(\mathscr{K})$ between these objects.

*2-cells*: a 2-cell is a morphism of bimodules so that

$$\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})(X, Y) = \mathsf{Bimod}(\mathscr{K})(TX, Y).$$

The identity span on $X$ is $\eta_X^* : TX \nrightarrow X$ and composition is given by

$$\frac{TX \xrightarrow{\;R\;} Y \qquad\qquad TY \xrightarrow{\;S\;} Z}{TX \xrightarrow{\;\mu_X^*\;} T^2X \xrightarrow{\;TR\;} TY \xrightarrow{\;S\;} Z}$$

Horizontal composition of 2-cells is clearly induced by that of morphisms, while the vertical composition is inherited from $\mathsf{Bimod}(\mathscr{K})$.

**3.5. Remark.** The bicategory $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$ is analogous to our bicategory $\mathbf{Spn}_{\mathbf{T}}(\mathbb{B})$ associated to a cartesian monad on a category with pullbacks $\mathbb{B}$ in [11, Appendix A].

We record the following properties of the unit of a cartesian 2-monad:

**3.6. Proposition.** *Consider a 2-monad $(T, \eta, \mu)$ on a 2-category $\mathscr{K}$ admitting a calculus of bimodules*

1. *If $\eta$ is cartesian, then it is monic.*
2. *If $T$ preserves cotensors with $^{\to}$ and $\eta$ is cartesian, then $\eta$ is fully faithful.*

**Proof.**
1. The square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle T\eta_X} \\ TX & \xrightarrow[\eta_{TX}]{} & T^2X \end{array}$$

is a pullback and the lower arrow is a split monic ($\mu_X \eta_{TX} = id$), hence the top arrow is monic.

2. Given 'global generic' elements $a, b : Z \to X$, we must stablish a 1–1 correspondence

$$(\alpha : a \Rightarrow b) \quad \leftrightarrow \quad (\eta_X \alpha : \eta_X a \Rightarrow \eta_X b).$$

The 2-cells into $X$ are classified by 1-cells into $\mathsf{Hom}_X$ and those into $TX$ are classified by 1-cells into $\mathsf{Hom}_{TX} \cong T(\mathsf{Hom}_X)$. In the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{d} & \mathsf{Hom}_X & \xrightarrow{c} & X \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{\mathsf{Hom}_X}} & & \downarrow{\scriptstyle T\eta_X} \\ TX & \xleftarrow[Td]{} & T\mathsf{Hom}_X & \xrightarrow[Tc]{} & TX \end{array}$$

both squares are pullbacks and $\eta_{\mathsf{Hom}_X}$ is monic, which yields the desired correspondence. □

## 4. Lax algebras vs. monads

Given the pseudo-monad $(\mathsf{Bimod}(T), \eta_\#, \mu_\#)$ on $\mathsf{Bimod}(\mathscr{K})$, we consider lax algebras for it, as in e.g. [21].

**4.1. Definition.** A *lax* $\mathsf{Bimod}(T)$*-algebra* consists of an object $X$ of $\mathscr{K}$, a bimodule $M : TX \nrightarrow X$ and structural 2-cells

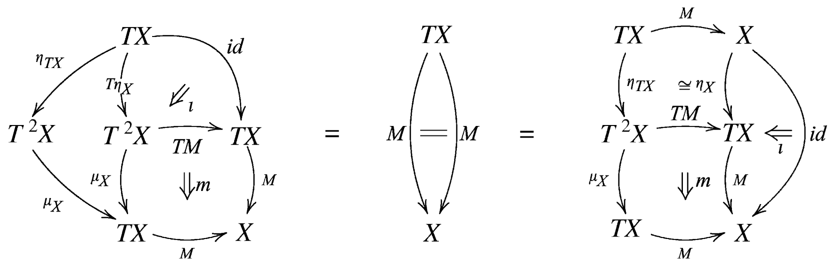
$$\iota : \mathsf{Hom}_X \Rightarrow (\eta_X)_\# \bullet M$$

and

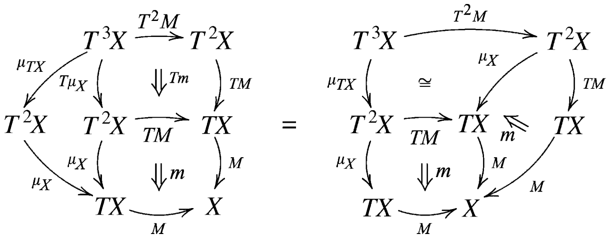
$$m : TM \bullet M \Rightarrow (\mu_X)_\# \bullet M$$

satisfying the following axioms:

- unit:



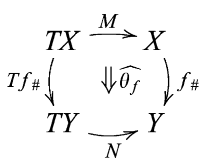

- associativity




When no confusion is likely we simply say that $M : TX \nrightarrow X$ is a lax algebra, understanding the rest of the data. The lax algebra is *normal* if $\iota = id_{\mathsf{Hom}_X}$ (which of course requires $\mathsf{Hom}_X = (\eta_X)_\# \bullet M : X \nrightarrow X$).

Given lax algebras $M : TX \nrightarrow X$ and $N : TY \nrightarrow Y$, a morphism between them is given by a morphism $f : X \to Y$ in $\mathscr{K}$ and a 2-cell $\theta_f : (Tf)^* \bullet M \Rightarrow N \bullet f^*$ in $\mathsf{Bimod}(\mathscr{K})(TY, X)$, compatible with the structural 2-cells.

**4.2. Remark.** The notion of morphism between lax algebras above is a special instance of the notion of lax morphism of lax algebras in [21], where we require $f$ to be a representable bimodule (one coming from a morphism in $\mathscr{K}$). Indeed, taking the adjoint mate of $\theta_f$ across $Tf_\# \dashv (Tf)^*$ and $f_\# \dashv f^*$ we get



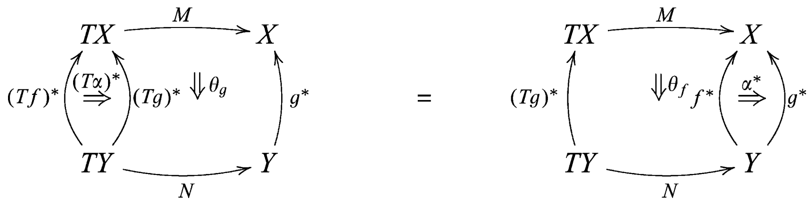

but the above presentation makes it easier to fit 2-cells. In our constructions however, we will use whichever direction is more convenient, without making a fuss over it.

**4.3. Definition.** The 2-category $\mathsf{Lax\text{-}Bimod}(T)$-alg of lax algebras for the pseudo-monad $(\mathsf{Bimod}(T), \eta_\#, \mu_\#)$ on $\mathsf{Bimod}(\mathscr{K})$ consists of:

*Objects*: normal lax algebras $M : TX \nrightarrow X$.

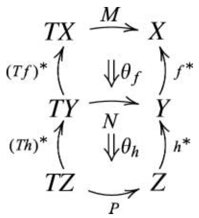
*Morphisms*: morphisms of lax algebras $(f, \theta_f) : M \to N$.

*2-cells*: given morphisms $(f, \theta_f), (g, \theta_g) : M \to N$, a 2-cell $\alpha : (f, \theta_f) \Rightarrow (g, \theta_g)$ consists of a 2-cell $\alpha : f \Rightarrow g$ in $\mathscr{K}$ such that



*Identities*: $(id_X, id_M) : M \to M$.

*Composition*: composition of 1-cells is given by composition of 1-cells in $\mathscr{K}$ and the pasting



while compositions of 2-cells are inherited from $\mathscr{K}$.

The 2-category Lax-Bimod($T$)-alg would be the basis on which pseudo-$T$-algebras become properties. We will give an alternative description of this 2-category in terms of $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$, which, besides being of interest in itself, will be helpful to recapture our guiding example of multicategories, cf. Theorem 8.2 in Part II.

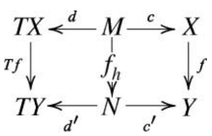
**4.4. Definition.** The 2-category $\mathsf{Mnd}(\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$ of (normal) *monads* in $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$ consists of:

*Objects*: monads in $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})$, that is

- An object $X$ in $\mathscr{K}$ and a bimodule $M : TX \nrightarrow X$
- Unit $\iota : \eta_X^* \Rightarrow M$ and multiplication $m : \mu_X^* \bullet TM \bullet M \Rightarrow M$ satisfying the unit and associativity axioms.

We furthermore require the following *normality* condition: the adjoint transpose of $\iota$ across $(\eta_X)_\# \dashv \eta_X^*$ is $id : \mathsf{Hom}_X \to \mathsf{Hom}_X$. Henceforth we call monads satisfying this condition *normal*.

*Morphisms*: a morphism from $M : TX \nrightarrow X$ to $N : TY \nrightarrow Y$ is given by a pair of morphisms $f : X \to Y$ and $f_h : M \to N$ in $\mathscr{K}$ such that



such $f_h$ preserves the bimodule structure and is compatible with the units and multiplications of $M$ and $N$.

*2-cells*: Given morphisms $(f, f_h), (g, g_h) : M \to N$, a 2-cell $\phi : (f, f_h) \Rightarrow (g, g_h)$ consists of a 2-cell $\phi : M \Rightarrow (Tf, g)^*N$ in $\mathsf{Bimod}(\mathscr{K})(TX, X)$ which is a morphism of $(M, M)$-bimod (left $M$-right $M$ modules). This last requirement makes sense: $M$, being a monoid in $\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})(X, X)$ acts on itself by composition, while $N$ has a similar $(N, N)$-bimod structure, which is transferred by change-of-base along the morphisms $(f, f_h), (g, g_h) : M \to N$.

**4.5. Remark.**

- Our definition of morphism of monads above is a special case of that of [20], where we have restricted the 1-cells to be representable bimodules. This is in accordance with the similar restriction we imposed on morphisms of lax algebras.
- Our definition of 2-cells mimics that of transformation between morphisms of multicategories [11, Definition 6.6].

Consistently with our treatment of morphisms of multicategories in [11], we say that a 1-cell in $\mathsf{Mnd}(\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K}))$ as above is *full and faithful* if the corresponding morphism of bimodules $f_h : M \to Tf_\# \bullet N \bullet g^*$ is an isomorphism, so that we have a change of base situation.

We now set about to show that the 2-categories $\mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}$ and $\mathsf{Mnd}(\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K}))$ are essentially the same. The only subtlety in this correspondence is the identification of 2-cells. Given a (normal) lax algebra $M : TX \nrightarrow X$ with structural 2-cells $\imath : \mathsf{Hom}_X \Rightarrow (\eta_X)_\# \bullet M$ and $m : TM \bullet M \Rightarrow (\mu_X)_\# \bullet M$, we obtain a (normal) monad $M : TX \nrightarrow X$ with unit and multiplication obtained from $\imath$ and $m$ by transposing across the adjunctions $(\eta_X)_\# \dashv \eta_X^*$ and $(\mu_X)_\# \dashv \mu_X^*$. Let us denote the resulting data $(\check{M}, \check{\imath}, \check{\mu})$.

**4.6. Proposition.** *There is an isomorphism of* 2*-categories*

$$(\check{\_}) : \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg} \xrightarrow{\sim} \mathsf{Mnd}(\mathsf{Bimod}_{\mathbf{T}}(\mathscr{K})).$$

**Proof.** We must first verify that $(\check{M}, \check{\imath}, \check{\mu})$ is indeed a monad. This follows by a routine calculation using the unit and associativity axioms for a lax algebra (Definition 4.1). The normality condition is straightforward. It is furthermore clear that starting with a normal monad $(M, \imath, m)$ we obtain the data for a normal lax algebra by taking adjoint transposes of $\imath$ and $m$, and this correspondence is inverse to $(\check{\_})$.

As for morphisms, to give

$$\begin{array}{ccccc} TX & \xleftarrow{d} & M & \xrightarrow{c} & X \\ {\scriptstyle Tf}\downarrow & & {\scriptstyle f_h}\downarrow & & \downarrow{\scriptstyle f} \\ TY & \xleftarrow[d']{} & N & \xrightarrow[c']{} & Y \end{array}$$

between monads is the same as to give a morphism of bimodules $\hat{f}_h : M \Rightarrow (Tf, f)^*(N)$ (by change-of-base). But $(Tf, f)^*(N) = (Tf)_\# \bullet N \bullet f^*$, and so we have the following correspondence

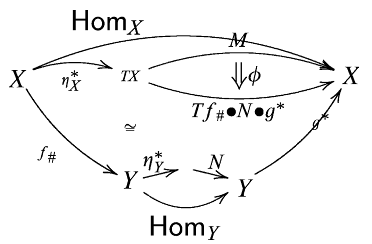
$$\frac{\hat{f}_h : M \Rightarrow (Tf)_\# \bullet N \bullet f^*}{\theta_f : (Tf)^* \bullet M \Rightarrow N \bullet f^*} \; Tf_\# \dashv Tf^*$$

which sets up the bijective correspondence between morphisms of lax algebras $(f, \theta_f): M \to N$ and morphisms of monads $(f, f_h): \check{M} \to \check{N}$.

As for 2-cells, given one in Lax-Bimod($T$)-alg, $\alpha : f \Rightarrow g$ for parallel morphisms $(f, \theta_f), (g, \theta_g): M \to N$, we get one in Mnd(Bimod$_{\mathbf{T}}(\mathscr{K})$) by precomposing with the adjoint transpose of $\theta_f$:

$$\underline{\alpha} \equiv M \overset{\check{\theta}_f}{\Rightarrow} Tf_\# \bullet N \bullet f^* \overset{Tf_\# \bullet N \bullet \alpha^*}{\Rightarrow} Tf_\# \bullet N \bullet g^*$$

and in the opposite direction, given $\phi : M \Rightarrow Tf_\# \bullet N \bullet g^*$ the composite




where the isomorphism is that of pseudo-naturality of $\eta^*$, yields a 2-cell $\bar{\phi}$: Hom$_X \Rightarrow f_\# \bullet g^* = f \downarrow g$ and therefore a 2-cell $\bar{\bar{\phi}} : f \Rightarrow g$ in $\mathscr{K}$ which is furthermore well-defined as a 2-cell in Lax-Bimod($T$)-alg. These correspondences of 2-cells are readily verified to be mutually inverse. □

**4.7. Remark.**

- The best way to understand the correspondence at the level of 2-cells in the above proof is to look at the analysis of ordinary natural transformations in $\mathscr{C}at$ in [11, Section 6.1].
- The correspondence at the object level, namely lax algebras versus monads, relies purely on the fact that the unit and multiplication of Bimod($T$) have right adjoints. However, the rest of the setup relies heavily on the relationship between $\mathscr{K}$ and Bimod($\mathscr{K}$), construing morphisms of the former as maps in the latter. Notice in particular that we define 2-cells for monads by means of ‘tents’ (commuting diagrams of 1-morphisms) using the fact that bimodules are spans.

## 5. Pseudo-algebras and properties

Having introduced our basic new gadget, namely the 2-category Mnd(Bimod$_T(\mathscr{K})$) (and its equivalent Lax-Bimod($T$)-alg), we now proceed to our main point: the 2-category of T-algebras is *monadic* over Mnd(Bimod$_T(\mathscr{K})$) and furthermore this monad has the *adjoint-pseudo-algebra* property. We then show that a pseudo-T-algebra amounts to a *universal property* of a lax-Bimod($T$)-algebra, namely that the unit of the monadic adjunction have a left adjoint. We write T-alg for the 2-category of T-algebras, strict morphisms and 2-cells compatible with such (i.e. modifications).

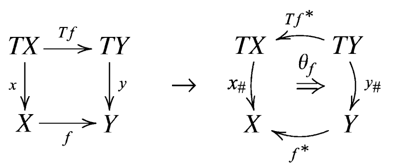
### 5.1. *The adjunction between* Lax-Bimod($T$)-alg *and* T-alg

*From* T-alg *to* Lax-Bimod($T$)-alg: Given a T-algebra $x: TX \to X$, the representable bimodule $x_\# : TX \nrightarrow X$ has a (normal) lax Bimod($T$)-algebra structure:

- The (identity) unit is $\mathsf{Hom}_X = (x \circ \eta_X)_\# = (\eta_X)_\# \bullet x_\#$, by the unit equation for $x$.
- The multiplication is similarly obtained from the associativity for $x$:

$$Tx_\# \bullet x_\# = (x \circ Tx)_\# = (x \circ \mu_X)_\# = (\mu_X)_\# \bullet x_\#$$

A morphism of T-algebras $f : x \to y$ (with $y : TY \to Y$) induces a morphism of lax algebras



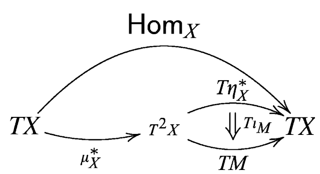

by adjoint transposition and similarly a 2-cell $\alpha : f \Rightarrow g$ between two such morphisms induces one $\alpha : f \Rightarrow g$ in Lax-Bimod($T$)-alg. We have thus defined a 2-functor

$$R : \mathsf{T\text{-}alg} \to \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}$$

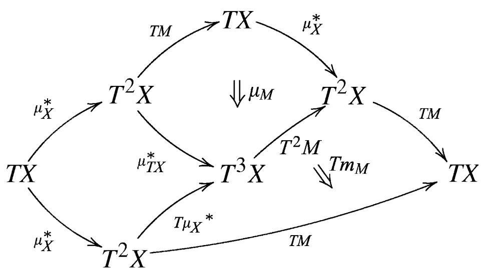
*The free* T-*algebra on a lax* Bimod($T$)-*algebra*: Given a (normal) lax algebra $M : TX \nrightarrow X$, we consider it as a (normal) monad, according to Proposition 4.6

**5.1. Lemma.** *Given a (normal) monad* $(M : TX \nrightarrow X, \iota_M, m_M)$ *(in* $\mathsf{Bimod}_T(\mathscr{K})$*), the composite bimodule* $\mu_X^* \bullet TM : TX \nrightarrow TX$ *has a monad structure in* $\mathsf{Bimod}(\mathscr{K})$

**Proof.** *Unit*:

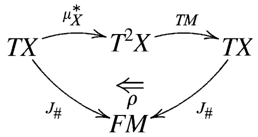


*Multiplication*:

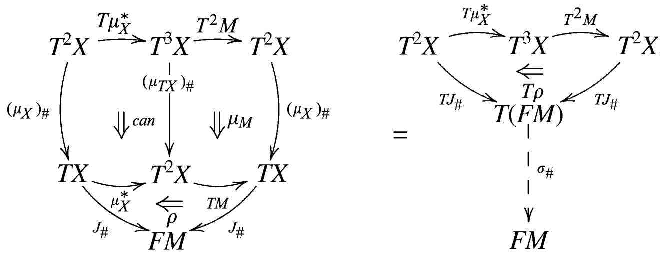


The monad axioms follow from those of $(TM, T\imath_M, Tm_M)$ and pseudo-naturality of $\mu^*$. $\square$

Given a (normal) monad $(M : TX \nrightarrow X, \imath_M, m_M)$, let



be the (representable) Kleisli object of the monad of Lemma 5.1. By Assumption 2.5, Bimod($T$) preserves such Kleisli object. Hence, by universality, we have a unique mediating morphism $\sigma : T(FM) \to FM$ in

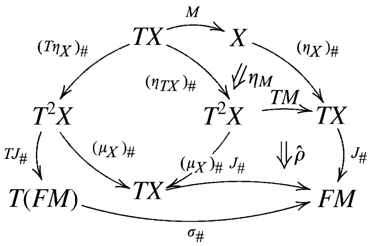


where *can* is the adjoint transpose of the associativity square for $\mu_\#$ and $\mu_M$ is the invertible 2-cell corresponding to the pseudo-naturality of $\mu_\#$. It follows by uniqueness of mediating morphisms between universal lax cocones (Kleisli objects) that $\sigma : T(FM) \to FM$ is a T-algebra, using the 2-functorial preservation of representability of Assumption 2.5. Using the same assumption, we can extend the action of $F$ to morphisms and 2-cells, thereby defining a 2-functor

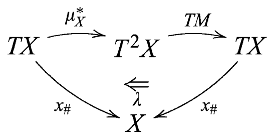
$$F : \text{Lax-Bimod}(T)\text{-alg} \to \text{T-alg}$$

**5.2. Proposition.** *The above* 2*-functors set up an adjunction of* 2*-categories* $F \dashv R : \text{T-alg} \to \text{Lax-Bimod}(T)\text{-alg}$ *whose unit is full and faithful.*

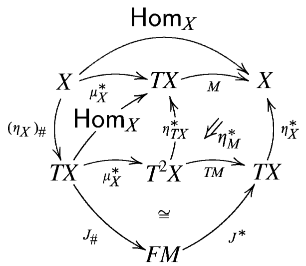
**Proof.** *Unit*: Given $(M : TX \nrightarrow X, \imath_M, m_M)$, define $\zeta_M : M \Rightarrow RF(M)$ as follows:



*Counit*: Given a T-algebra $x : TX \to X$, we have a lax cocone



where $\lambda$ is the adjoint transpose of $Tx_\# \bullet x_\# = (\mu_X)_\# \bullet x_\#$ (associativity for $x$). Thus we have an induced morphism $\hat{x} : FM \to X$ from the Kleisli object $FM$. Once again, associativity for $x$ and universality imply that this induced morphism is a T-algebra morphism from $\sigma : T(FM) \to FM$ to $x : TX \to X$, which is the desired counit $\varepsilon_x = \hat{x}$. To show $\zeta_M$ fully faithful, first let us notice that the underlying morphism $J\eta_X$ in $\mathscr{K}$ is fully faithful as shown by the following diagram:



Finally, full and faithfulness of $\zeta_M$ follows similarly, using that of $\eta_X$ (cf. Proposition 3.6(2)). □

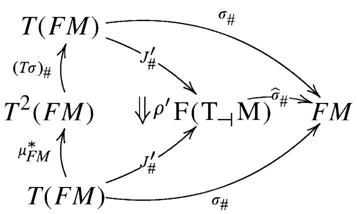
### 5.2. *Adjoint psuedo-algebras and representable bimodules*

Let $\mathsf{T}_\dashv = (T_\dashv : \mathsf{Lax\text{-}Bimod}(T)\text{-alg} \to \mathsf{Lax\text{-}Bimod}(T)\text{-alg}, \zeta, v)$ be the 2-monad induced by the adjunction of Proposition 5.2. In this subsection we will show that its pseudo-algebras are characterised as left adjoints to units and give an intrinsic characterisation of them in terms of representability.

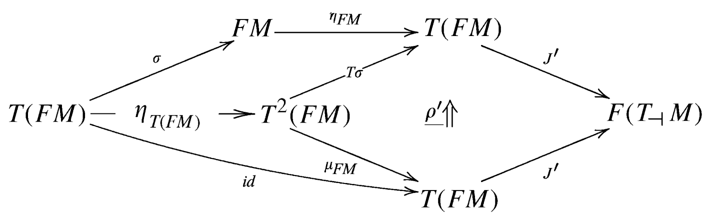
**5.3. Proposition.** *The* 2-*monad* $(T_{\dashv}, \zeta, v)$ *has the adjoint-pseudo-algebra property*, *i.e.* $v_M \dashv \zeta_{T_{\dashv}M}$ *for all normal lax algebras* $M$.

**Proof.** Since the counit of the required adjunction is the identity, we must simply define the unit.

Notice that $T_{\dashv}M = (\sigma)_{\#} : T(FM) \nrightarrow FM$ and $v_M : T_{\dashv}^2 M \to T_{\dashv}M$ is the morphism of algebras $\hat{\sigma} : F(T_{\dashv}M) \to FM$ uniquely determined in



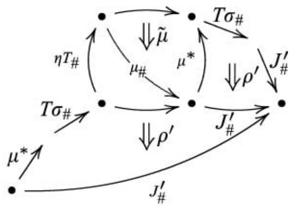

by the lax cocone $\alpha : \mu_{FM}^* \bullet (T\sigma)_{\#} \bullet (\sigma)_{\#} \Rightarrow (\sigma)_{\#}$ given by the adjoint transpose of $id : (T\sigma)_{\#} \bullet (\sigma)_{\#} \Rightarrow (\mu_{FM})_{\#} \bullet (\sigma)_{\#}$, while the underlying morphism of $\zeta_{T_{\dashv}M} : T_{\dashv}M \Rightarrow T_{\dashv}^2 M$ is $J'\eta_{FM} : FM \to F(T_{\dashv}M)$. To give a 2-cell $1 \Rightarrow J'\eta_{FM}\hat{\sigma}$ amounts to give, by the 2-dimensional universal property of the Kleisli object, a modification $J' \Rightarrow J'\eta_{FM}\sigma$ between the respective lax cocones. To define such modification, notice that the adjoint transpose $\underline{\rho'}_{\#} : (T\sigma)_{\#} \bullet J'_{\#} \Rightarrow (\mu_{FM})_{\#} \bullet J'_{\#}$ in $\mathsf{Bimod}(\mathscr{K})$ corresponds to a 2-cell $\underline{\rho'} : J'\mu_{FM} \Rightarrow J'T\sigma$. Define $\kappa : J' \Rightarrow J'\eta_{FM}\sigma$ as the composite




The verification that the 2-cell so defined is indeed a modification is quite delicate, so we outline the details. The equation

$$\rho' \circ (\mu_{FM}^* \bullet (T\sigma)_{\#} \bullet \kappa_{\#}) = \kappa_{\#} \circ (\alpha \bullet (\eta_{FM})_{\#} \bullet J'_{\#})$$

amounts to the equality (omitting objects to simplify the diagram)

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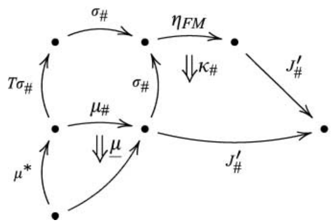


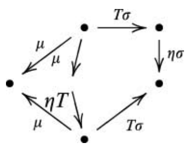

the upper pasting can be simplified using the fact that $\rho'$ is a lax cocone for the monad $\mu^*_X \bullet (T\sigma)_\#$:

$$\rho' \cdot \rho' T\sigma_\# \mu^*_X = \rho' \cdot J'_\# \bar{m},$$

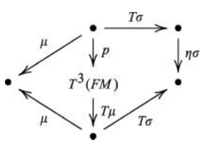
where $\bar{m} = Tm_\sigma \mu^*_X \cdot T\sigma_\# \mu_\sigma \mu^*_X$ is the multiplication for the monad $T\sigma_\# \mu^*_X$, cf. Lemma 5.1. So we will be done if we can show the following equality:

$$T\sigma_\#(T\underline{\mu}\mu^* \circ \mu_\sigma \mu^* \circ \tilde{\mu}(\mu^* \bullet T\sigma_\# \bullet \eta T_\#)) = T\sigma_\#(\tilde{\mu}\eta T_\# \circ \eta T_\# \underline{\mu})$$

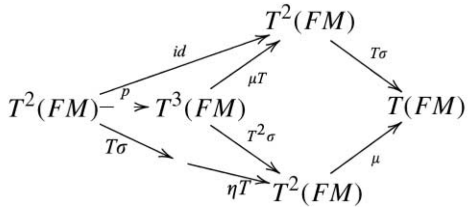
as pasting both sides with $\rho'$ gives the desired equality. The trick to prove this latter equality is to realise the morphisms of spans which induce these 2-cells. The left one is induced by




while the right one is induced by



where in turn



using cartesianness of $\mu$. Finally, naturality shows $\eta_{T(FM)} \circ \mu_{FM} = T\mu_{FM} \circ p$ and thus the morphisms of bimodules induced from those of spans are equal, as desired.

To conclude the proof, we must verify the adjunction equations for the 2-cell $\underline{\kappa} : 1 \Rightarrow J'\eta_{FM}\hat{\sigma}$ induced by the modification $\kappa$, namely

- $\hat{\sigma}\kappa = id$, which is immediate by the definition of the lax cocone with 2-cell $\alpha$ which induces $\hat{\sigma}$.
- $\kappa J'\eta_{FM} = id$, which is established using $\rho' \circ (\iota \bullet J'_{\#}) = id$ (lax cocone condition for the unit of the monad $\iota : \mathsf{Hom}_{T(FM)} \Rightarrow \mu^*_{FM} \bullet (T\sigma)_{\#}$).

Finally, universality of (representable) Kleisli objects guarantees that $\underline{\kappa}$ is a well-defined 2-cell in Lax-Bimod($T$)-alg. □

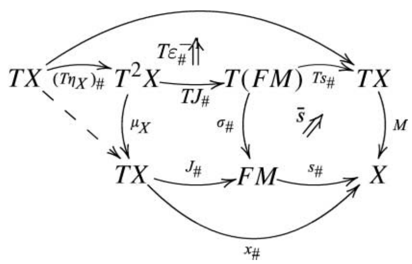
Now we establish an important intrinsic characterisation of the adjoint-pseudo-algebras for $(T_{\dashv}, \zeta, v)$. In fact, it was such a characterisation which motivated the present theory.

**5.4. Theorem.** *A normal lax algebra $M : TX \nrightarrow X$ with structure 2-cell $m : TM \bullet M \Rightarrow (\mu_X)_{\#} \bullet M$ is an adjoint-pseudo-algebra for the 2-monad $(T_{\dashv}, \zeta, v)$ (i.e. $\zeta_M : M \to T_{\dashv}M$ admits a left adjoint) iff the following two conditions hold:*

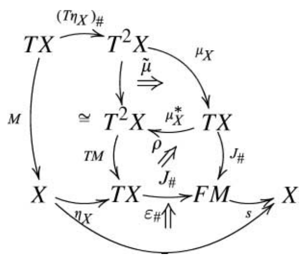
1. *$M : TX \nrightarrow X$ is a representable bimodule.*
2. *$m : TM \bullet M \Rightarrow (\mu_X)_{\#} \bullet M$ is an isomorphism.*

**Proof.** $\Rightarrow$ Let $M : TX \nrightarrow X$ have an adjoint-pseudo-algebra structure $(s, \bar{s}) : T_{\dashv}M \to M$, with $\tau : id \Rightarrow \zeta_M \circ s$ and $\varepsilon : s \circ \zeta_M \Rightarrow id$ the unit and (invertible) counit of the adjunction $s \dashv \zeta_M$.

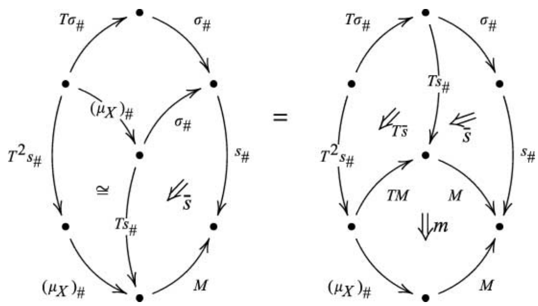
Recall that the underlying morphism of $\zeta_M$ is $X - \eta_X \to TX - J \to FM$. Define $x = s \circ J : TX \to X$. We now intend to show that $x$ represents $M$. Define $\theta : x_{\#} \Rightarrow M$ as the pasting



We claim $\theta$ is an isomorphism: its inverse $\theta^{-1}$ is given by the pasting



$\theta \circ \theta^{-1} = id$ because the counit is a 2-cell in Lax-Bimod($T$)-alg and $\theta^{-1} \circ \theta = id$ by the adjunction equations for $(J\eta_X)XS_\# \dashv s$ (since $(J\eta_X)_\# \varepsilon^{-1} = \tau(J\eta_X)XS_\#$). We have thus shown that $M$ is representable. To see that its structure 2-cell $m$ is an isomorphism, we use the fact that $(s, \bar{s}) : \sigma_\# \to M$ commutes which the structural 2-cells of these lax algebras. We have therefore



Since $\theta$ is an isomorphism, so is $\theta(Ts)_\#$, as well as $(Ts)_\# T\tau_\#$ by adjointness, because $\varepsilon$ is an isomorphism and therefore $\varepsilon^{-1}s = s\tau$. From this adjointness we also conclude that the pasting of $\theta(Ts)_\#$ and $(Ts)_\# T\tau_\#$ is $\bar{s}$. Hence,

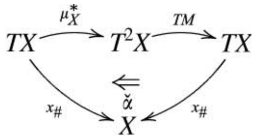
$$\bar{s} : \sigma_\# \bullet s_\# \Rightarrow Ts_\# \bullet M \text{ is an isomorphism.} \tag{1}$$

Thus, the equality of the pasting diagrams above imply that $mT^2 s_\#$ is an isomorphism and so is $mT^2 s_\# (T^2 J T^2 \eta_X)_\#$. Therefore,

$$(mT^2 s_\# (T^2 J T^2 \eta_X)_\#)(\mu_X)_\# T^2 \varepsilon_\# = m(MTMT^2 \varepsilon_\#)$$

is an isomorphism. Finally, $MTMT^2 \varepsilon_\#$ being an isomorphism allows us to conclude that $m$ is one as well.

$\Leftarrow$: Given $x : TX \to X$, the isomorphisms $\theta : x_\# \Rightarrow M$ and $m : TM \bullet M \Rightarrow (\mu_X)_\# \bullet M$ endow $x$ with a pseudo-T-algebra structure. We should therefore show that any such pseudo-algebra does endow $x_\# : TX \nrightarrow X$ with a pseudo-$\mathsf{T}_\dashv$-algebra structure (which a fortriori would be an adjoint one, by Proposition 5.3). Let $\iota : id \Rightarrow x \circ \eta_X$ and $\alpha : x \circ Tx \Rightarrow x \circ \mu_X$ be the structural isomorphisms. The adjoint transpose of $\alpha_\#$ produces a lax cocone




and we get an induced morphism $\hat{x}: F(x_\#) \to X$, which is equipped with structural isomorphisms

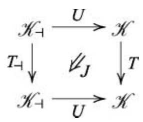
*Unit*: $\iota_\#^{-1}$ : $\mathrm{Hom}_X \Rightarrow (x \circ \eta_X)_\# = (J\eta_X)_\# \bullet \hat{x}_\#$.

*Associativity*: $\hat{\alpha}: T_\dashv \hat{x}_\# \bullet \hat{x}_\# \Rightarrow v_{x_\#} \bullet \hat{x}_\#$ uniquely determined by the given $\alpha: x \circ Tx \Rightarrow x \circ \mu_X$ and the 2-dimensional universal property of the Kleisli objects involved. Furthermore, universality of Kleisli objects ensure that the induced canonical isomorphisms satisfy the pseudo-T-algebra axioms. □

A clear example of this theorem is given in the case of pseudo-functors into $\mathscr{C}at$ in Part II, cf. Remark 11.3.

### 5.3. Monadicity

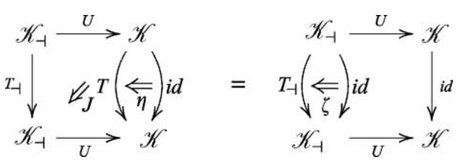
Let $\mathscr{K}_\dashv = \mathsf{Lax} - \mathsf{Bimod}(\mathsf{T})\text{-}\mathsf{alg}$. There is an evident 'underlying object' 2-functor $U: \mathscr{K}_\dashv \to \mathscr{K}$ with action $(M: TX \nrightarrow X) \mapsto X$. Furthermore, there is a 2-natural transformation



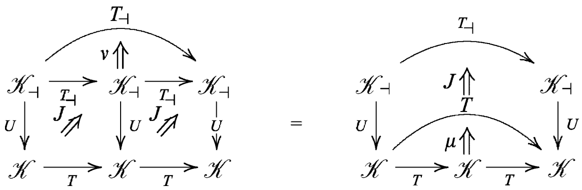
given by $J_M = TX \text{–} J \to FM$, the Kleisli morphism into the Kleisli object $FM = U(T_\dashv(M))$. The 2-naturality of $J$ follows from the universality of Kleisli objects (with the preservation of representability of our Assumption 2.5).

**5.5. Proposition.** *The pair* $(U, J)$ *is a morphism of* 2*-monads from* $(T_\dashv, \zeta, v)$ *to* $(T, \eta, \mu)$.

**Proof.** We must verify the compatibility with units and multiplications:



follows immediately from the definition of $\zeta$ that $U(\zeta_M) = J \circ \eta_X$ (for $M : TX \nrightarrow X$).



we have by definition of $v$ that the left pasting instantiated at $M : TX \nrightarrow X$ is the upper morphism in the following commuting diagram:

while the bottom map is the corresponding instance of the right pasting. □

**5.6. Theorem.** (Monadicity)**.** *The morphism of* 2*-monads* $(U,J) : \mathsf{T}_{\dashv} \to \mathsf{T}$ *induces* 2*-equivalences*

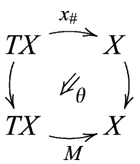
$$\boxed{(U,J)\text{-}\mathscr{A}lg : \mathsf{T}_{\dashv}\text{-}\mathscr{A}\mathsf{lg} \simeq \mathsf{T}\text{-}\mathscr{A}\mathsf{lg}}$$

$$\boxed{\mathsf{Ps}\text{-}(U,J)\text{-}\mathscr{A}lg : \mathsf{Ps}\text{-}\mathsf{T}_{\dashv}\text{-}\mathscr{A}\mathsf{lg} \simeq \mathsf{Ps}\text{-}\mathsf{T}\text{-}\mathscr{A}\mathsf{lg}}$$

**Proof.** The details are essentially contained in the proof of Theorem 5.4. Given a (pseudo-)T-algebra $x : TX \to X$ (with structural isomorphisms $\iota$ and $\alpha$) we have

$$TX \xrightarrow{J} FM \xrightarrow{\hat{x}} X = TX \xrightarrow{x} X$$

by definition of $\hat{x}$. In the other direction,



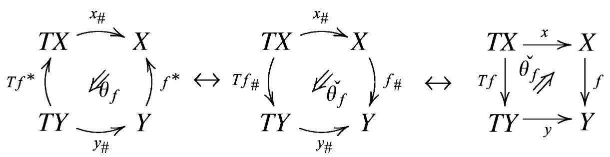
is an isomorphism of (pseudo-)$T_{\dashv}$-algebras. □

**5.7. Remark.** The underlying object 2-functor $U : \mathscr{K}_{\dashv} \to \mathscr{K}$ is locally conservative, a fibration at the 1-cell level (by change-of-base for bimodules) and has a left 2-adjoint with action $X \mapsto (\eta_X^* : TX \nrightarrow X)$.

## 6. Classification of lax morphisms

We use the adjunction $F \dashv R : \mathsf{T}\text{-alg} \to \mathsf{Lax\text{-}Bimod}(T)\text{-alg}$ to obtain an explicit classification of lax morphisms between (pseudo-)$T$-algebras.

Given (pseudo-)$T$-algebras $x : TX \to X$ and $y : TY \to Y$, consider a morphism $(f, \theta_f) : x_\# \to y_\#$ in Lax-Bimod($T$)-alg. We have the following correspondences:




the first one by taking adjoint mates and the second one using that $(\_)_\# : \mathscr{K}^{\mathrm{co}} \to$ Bimod($\mathscr{K}$) is locally fully faithful. The resulting data amounts to a *lax morphism* between the (pseudo-)$T$-algebras. There is a similar (and quite clear) identification at the level of 2-cells, so that we have the following result:

**6.1. Theorem.** (Classification of lax morphisms).

1. *Given $T$-algebras $x : TX \to X$ and $y : TY \to Y$, the adjunction*

   $$F \dashv R : \mathsf{T\text{-}alg} \to \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}$$

   *induces the following isomorphisms*:

   $$\mathsf{T\text{-}alg}_l(x, y) \cong \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}(x_\#, y_\#) \cong \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}(FRx, y)$$

   *and therefore the following isomorphism of 2-categories*:

   $$\mathsf{T\text{-}alg}_l \cong \mathsf{T\text{-}alg}_G,$$

   *where $G = FR$ is the 2-comonad induced on* T-alg *and the second 2-category above its associated Kleisli construction.*
2. *Given pseudo-$T$-algebras $x : TX \to X$ and $y : TY \to Y$, the biadjunction $F \dashv R$ :* Ps-T-alg $\to$ Lax-Bimod($T$)-alg *induces the following equivalences*:

   $$\mathsf{Ps\text{-}T\text{-}alg}_l(x, y) \simeq \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}(x_\#, y_\#) \simeq \mathsf{Lax\text{-}Bimod}(T)\text{-}\mathsf{alg}(FRx, y)$$

   *and therefore the following biequivalence of 2-categories*:

   $$\mathsf{Ps\text{-}T\text{-}alg} \simeq \mathsf{Ps\text{-}T\text{-}alg}_G$$

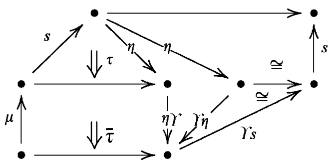
   *where $G = FR$ is the pseudo-comonad induced on* Ps-T-alg *and the second 2-category above its associated Kleisli construction.*

**6.2. Remark.** A classification of lax morphisms for $T$-algebras for 2-monads *with a rank* appears in [6]. Besides relying on a quite different hypothesis from ours, the work in *ibid.* does not provide any explicit description of the corresponding free objects (admitedly, that paper has an altogether different flavour and purpose from the present one). As we will see in Part II, our construction allows us to recover (or discover) *monoid classifiers*.

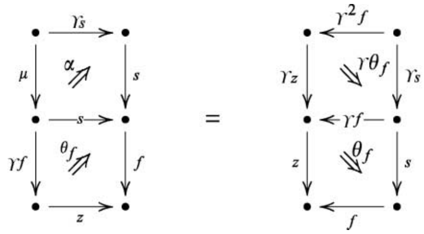
## 7. Classification of strong morphisms and coherence

Given a pseudo-$T_\dashv$-algebra, we want to freely associate a strict algebra to it. In other words, we want to construct a reflection for the inclusion $T_\dashv\text{-alg} \hookrightarrow \text{Ps-}T_\dashv\text{-alg}$. As we will see, we only get a 'bireflection' in the sense that we will get equivalences rather than isomorphisms at the level of the 'Hom' categories. Since $T_\dashv$ has the adjoint-pseudo-algebra property, we intend to apply the technique for strictification we introduced in [11, Section 10.2]. We reproduce the analysis of *ibid.* in this abstract setting.

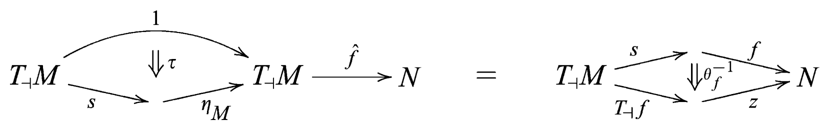
Let $M$ be an adjoint-pseudo-algebra, $s \dashv \zeta_M$ with unit $\tau : 1 \Rightarrow \eta_M s$. Suppose $N$ is a strict $T_\dashv$-algebra with structure $z : T_\dashv N \to N$, and let $(f, \theta_f) : M \to N$ a strong morphism. Recall from [21] that the associativity structure 2-cell $\alpha : s \circ \mu_M \Rightarrow s \circ Ts$ for $M$ is given by the pasting (we write $\Upsilon = T_\dashv$ to simplify notation)




where $\bar{\tau}$ is the unit of $\mu_M \dashv \eta_{T_\dashv M}$ and the isomorphisms are the counits of the adjunctions. The fact that $(f, \theta_f)$ is a strong morphism implies the equality

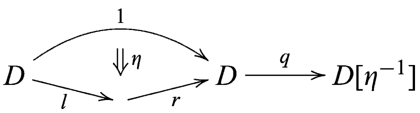



The morphism $f : M \to N$ induces $\hat{f} = z \circ Tf : TM \to N$. A little fiddling with the above diagrams shows that

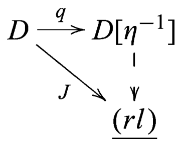



Hence $\hat{f} : T_\dashv M \to N$ inverts $\tau$. Notice that $T_\dashv M$ has a strict $T_\dashv$-algebra structure (the free such over $M$). So a good object candidate for the free strict $T_\dashv$-algebra for $M$ is the *coinverter* of $\tau$. It would give us the desired reflection if we can endow it with a $T_\dashv$-algebra structure. We reproduce, without proof, the key technical lemma [11, Lemma 10.4]:

**7.1. Lemma.** *Consider an adjunction* $\eta, \varepsilon : l \dashv r : C \to D$ *in a* 2*-category, with* $r$ *full and faithful* (*which is equivalent to* $\varepsilon$ *being an isomorphism*). *Consider the coinverter of the unit*



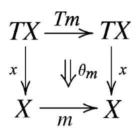

*and the unique morphism* $l' : D[\eta^{-1}] \to C$ *induced by* $l$.
1. *The morphisms* $l' : D[\eta^{-1}] \to C$ *and* $qr : C \to D[\eta^{-1}]$ *form an adjoint equivalence.*
2. *The coinverter* $D[\eta^{-1}]$ *is the Kleisli object* $\underline{(rl)}$ *of the* (*idempotent*) *monad* $rl : D \to D$ *induced on* $D$ *by the given adjunction, so that there is a canonical isomorphism*



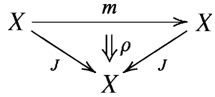

In order to apply this lemma, we must show that we have the relevant Kleisli objects. In principle we are working in the ambient 2-category Lax-Bimod($T$)-alg, but the object part of the monad whose Kleisli object we should provide is a (free) strict $T$-algebra. Hence the following result will suffice for our purposes:

**7.2. Proposition.** *Given a* 2*-monad* T *on* $\mathscr{K}$, *if* $\mathscr{K}$ *has Kleisli objects* (*for monads*) *and* $T : \mathscr{K} \to \mathscr{K}$ *preserves them, then the forgetful* 2*-functor* $U :$ T-alg$_l \to \mathscr{K}$ *creates Kleisli objects in the* 2*-category of algebras and lax morphisms.*

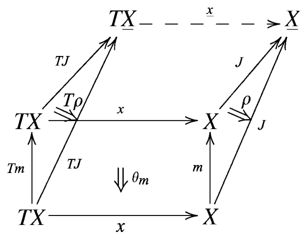
**Proof.** Let



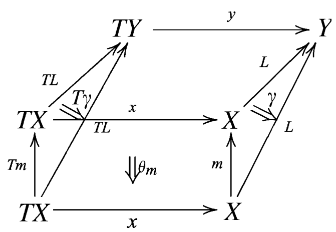

be an endomorphism in T-alg$_l$ with $((m, \theta_m), \eta, \mu)$ a monad. Let



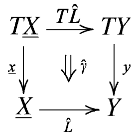

be the Kleisli object of $(m, \eta, \mu)$ in $\mathscr{K}$. Since $T$ preserves it, there is a uniquely determined morphism $\underline{x} : T\underline{X} \to \underline{X}$ mediating between lax cocones in the following diagram:



Uniqueness of mediating morphisms between lax cocones means that the algebra axioms for $\underline{x}: T\underline{X} \to \underline{X}$ follow from those of $x: TX \to X$. The above lax cocone in $\mathsf{T\text{-}alg}_l$ is the Kleisli object of $((m, \theta_m), \eta, \mu)$. To verify its universality, consider another lax cocone



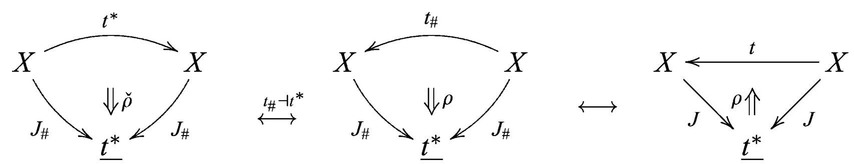
we have a mediating morphism $\hat{L}: \underline{X} \to Y$ induced by universality of $\underline{X}$ and a 2-cell



induced by 2-dimensional universal property of $T\underline{X}$, which furthermore guarantees the validity of the axioms making the above diagram a morphism in $\mathsf{T\text{-}alg}_l$. The 2-dimensional property of the Kleisli object follows similarly. □

**7.3. Proposition.** *If* $\mathsf{Bimod}(\mathscr{K})$ *admits representable Kleisli objects, then* $\mathscr{K}$ *admits Kleisli objects. Furthermore, if* $\mathsf{Bimod}(T)$ *preserves (representable) Kleisli objects, so does* $T: \mathscr{K} \to \mathscr{K}$.

**Proof.** Given a monad $(t: X \to X, \eta, \mu)$ in $\mathscr{K}$, we transform it into a monad $(t^*: X \nrightarrow X, \eta^*, \mu^*)$ in $\mathsf{Bimod}(\mathscr{K})$. We claim that the resulting representable Kleisli object $\underline{t^*}$ in $\mathsf{Bimod}(\mathscr{K})$ yields one for $t$ in $\mathscr{K}$. Indeed, we obtain a lax cocone for $t$ as follows:



the latter correspondence by $\_{\#} : \mathscr{K}^{co} \to \mathsf{Bimod}(\mathscr{K})$ being locally fully faithful. The universal property of such a representable Kleisli object in $\mathsf{Bimod}(\mathscr{K})$ restricts appropriately to $\mathscr{K}$ thanks to the preservation of representability, cf. Remark 2.4. Preservation by $T$ is now evident, since the action of $\mathsf{Bimod}(T)$ on maps (representable bimodules) is simply that of $T$ on the corresponding morphisms of $\mathscr{K}$. $\square$

We are finally in position to state the important classification theorem for strong morphisms in terms of strict ones:

**7.4. Theorem.** (Classification of strong morphisms)**.** *The inclusion* $J : T_{\dashv}\text{-}\mathsf{alg} \to \mathsf{Ps}\text{-}T_{\dashv}\text{-}\mathsf{alg}$ *has a left biadjoint* $(\_)^{\sigma} : \mathsf{Ps}\text{-}T_{\dashv}\text{-}\mathsf{alg} \to T_{\dashv}\text{-}\mathsf{alg}$ *whose unit is a (pseudo-natural) equivalence.*

**Proof.** Notice first that for a given pseudo-$T_{\dashv}$-algebra $M$, the (morphism part of the) monad $\zeta_M \circ s : (v_M)_{\#} \to (v_M)_{\#}$ corresponds to a morphism in $\mathsf{T}\text{-}\mathsf{alg}_l$. Now, our Assumption 2.5 and Proposition 7.3 allows us to apply Proposition 7.2. The corresponding Kleisli object yields the required left biadjoint by virtue of the analysis preceding Lemma 7.1, while this latter guarantees that the unit of the biadjunction is an equivalence, as required. $\square$

**7.5. Corollary.** (Coherence for pseudo-algebras)**.**
1. *Every pseudo-*$T_{\dashv}$*-algebra is equivalent to a strict one.*
2. *Every pseudo-*$T$*-algebra is equivalent to a strict one.*

**7.6. Remark.** Just about the only other result in the literature about coherence at this level of generality is that of [19]. The main difference in terms of prerequisites is that we assume the good behaviour of $T$ with respect to bimodules, which allows its transformation into a doctrine with the adjoint-pseudo-algebra property. Thus, we require $T$ to be cartesian and preserve $\mathsf{Hom}_{\_}$. Besides that, we require $\mathsf{Bimod}(T)$ to preserve representable Kleisli objects (which implies $T$ does as well), while Power requires $T$ to preserve *bijective-on-objects* functors. Since

'Kleisli morphism = bijective-on-objects + right adjoint'

whenever these terms make sense (e.g. in any 2-category endowed with a Yoneda structure [25]) and 2-functors preserve adjoints, our requirement is in principle formally weaker than that of Power (modulo our general assumption about bimodules). One advantage of our construction is that it usually provides an *explicit* description of the associated strict algebra, at least to the extent the relevant Kleisli object can be so described, which is certainly the case for our applications in Part II.

## Part II: Applications in internal category theory

In Part I we developed a theory allowing us to transform coherent structures into adjoint-pseudo-algebras, in the context of a 2-category $\mathscr{K}$ admitting a calculus of bimodules and a 2-monad $\mathsf{T}$ on it preserving bimodules, their composites and their Kleisli objects. We will now provide a general construction of an important kind of example of such a $(\mathscr{K}, \mathsf{T})$ pair: given a category with pullbacks $\mathbb{B}$ and a cartesian monad $\mathbf{T}$ on it, the 2-category $\mathscr{C}at(\mathbb{B})$ of internal categories equipped with the induced 2-monad $\mathscr{C}at(\mathbf{T})$ satisfies our hypothesis. We proceed to illustrate the resulting theory at work in this internal setting by examining three important examples: monoidal categories and pseudo-functors (into $\mathscr{C}at$) (the quintessential 'coherent structures') and the more novel *monoidal globular categories* introduced by Batanin for the development of weak higher-dimensional categorical structures.

## 8. Bimodules for internal categories

In this section we consider $\mathbb{B}$ a category with pullbacks, and $\mathbf{T} = (T, \eta, \mu)$ a *cartesian monad* on it, i.e.

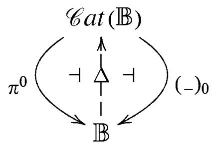
- The functor $T : \mathbb{B} \to \mathbb{B}$ preserves pullbacks.
- The transformations $\eta : id \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are cartesian, i.e. the naturality squares are pullbacks.

We get a cartesian 2-monad $\mathscr{C}at(\mathbf{T}) : \mathscr{C}at(\mathbb{B}) \to \mathscr{C}at(\mathbb{B})$ on the 2-category of internal categories, functors and natural transformations in $\mathbb{B}$, since we have a 2-functor

$$\mathscr{C}at(\_) : \mathbf{Pbk} \to \mathbf{2} - \mathscr{C}at$$

where $\mathbf{Pbk}$ is the 2-category of categories with pullbacks, pullback-preserving functors and cartesian transformations.

Provisionally, we also assume $\mathbb{B}$ has pullback-stable coequalizers and that $T$ preserves them, so we can apply our preceeding theory with $\mathscr{K} = \mathscr{C}at(\mathbb{B})$. In particular we have an adjoint string



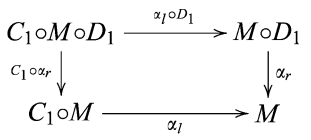

where $\Delta$ takes an object to the discrete category on it, and the 'connected components' of an internal category $\mathbf{C} = C_0 \leftarrow d - C_1 - c \rightarrow C_0$ is given by the coequalizer

$$C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0 \twoheadrightarrow \pi^0(C)$$

For a primer on internal bimodules, i.e. bimodules in $\mathscr{C}at(\mathbb{B})$, see [12, Chapter 2]. Given categories $\mathbf{C}$ and $\mathbf{D}$ in $\mathbb{B}$, a bimodule $M : \mathbf{C} \nrightarrow \mathbf{D}$ amounts to a span

$$C_0 \xleftarrow{\;s\;} M \xrightarrow{\;t\;} D_0$$

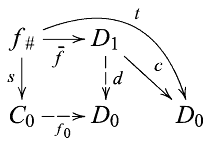
equipped with an action $\alpha : C_1 \circ M \circ D_1 \to M$ in $\mathbf{Spn}(\mathbb{B})(\mathbf{C}_0, \mathbf{D}_0)$, commuting with the monoid structure of $\mathbf{C}$ and $\mathbf{D}$. Alternatively, $M$ is equipped with a pair of actions $\alpha_l : C_1 \circ M \to M$ and $\alpha_r : M \circ D_1 \to M$ compatible in the sense that the following diagram commutes:



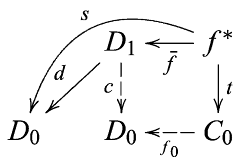

**8.1. Proposition.** *If $T$ preserves pullbacks, $\mathscr{C}at(T)$ preserves comma-objects. Furthermore, if $T$ preserves coequalizers, $\mathscr{C}at(T)$ preserves bimodule composition.*

**Proof.** For an internal category $\mathbf{C}$, the corresponding 'hom' bimodule is $\mathsf{Hom}_{\mathbf{C}} = C_0 \leftarrow d - C_1 - c \rightarrow C_0$ and therefore $T\mathsf{Hom}_{\mathbf{C}} = \mathsf{Hom}_{T\mathbf{C}}$. □

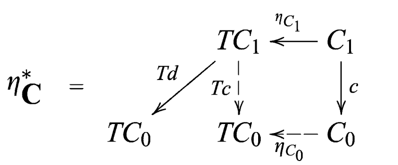
*Embeddings*: Given an internal functor $f : \mathbf{C} \to \mathbf{D}$, its associated representable bimodule $f_\# : \mathbf{C} \nrightarrow \mathbf{D}$ is given by the span

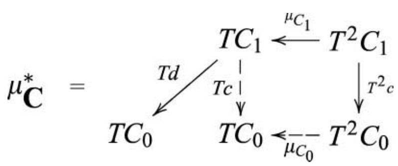



where the square is a pullback, and its dual $f^* : \mathbf{D} \nrightarrow \mathbf{C}$ has underlying span

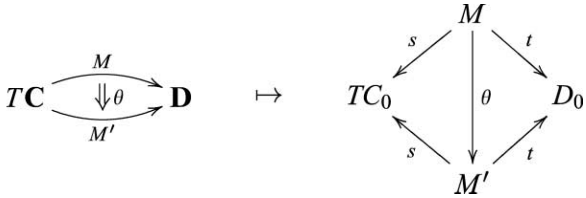



where the square is a pullback. In particular, for the cartesian transformations $\eta$ and $\mu$ we have

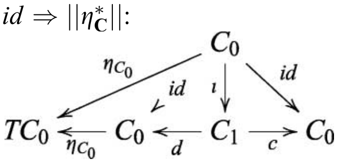


We want to relate the 2-category $\mathsf{Bimod}_{\mathscr{C}at(T)}(\mathscr{C}at(\mathbb{B}))$ with the simpler $\mathbf{Spn}_{\mathsf{T}}(\mathbb{B})$ of [11], which we have used as a framework for representable multicategories. There is in fact a lax functor $||\_||: \mathsf{Bimod}_{\mathscr{C}at(\mathbf{T})}(\mathscr{C}at(\mathbb{B})) \to \mathbf{Spn}_{\mathsf{T}}(\mathbb{B})$ with action




with structural 2-cells (for $N: T\mathbf{D} \nrightarrow \mathbf{E}$)
$\gamma_{\mathbf{C}}: id \Rightarrow ||\eta^*_{\mathbf{C}}||$:




$\delta_{M,N}: ||M|| \bullet ||N|| \Rightarrow ||M \bullet N||$: given by the coequalizer which realizes the composition $TM \bullet N$ in $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$

$$TM \circ TD_1 \circ N \underset{T\alpha^M_r \circ N}{\overset{TM \circ \alpha^N_l}{\rightrightarrows}} TM \circ N \xrightarrow{\delta_{M,N}} TM \bullet N$$

The lax functor $||\_||$ therefore takes monads to monads. In fact, it induces a 2-functor

$$\mathsf{Mnd}(||\_||): \mathsf{Mnd}(\mathsf{Bimod}_{\mathscr{C}at(T)}(\mathscr{C}at(\mathbb{B})) \to \mathsf{Mnd}(\mathbf{Spn}_{\mathbf{T}}(\mathbb{B}))$$

where in the target 2-category we drop the requirement of normality on the monads (which is not preserved by lax functors), so that

$$\mathsf{Mnd}(\mathbf{Spn}_{\mathbf{T}}(\mathbb{B})) \equiv \mathscr{M}ulticat_T(\mathbb{B})$$

the 2-category of multicategories, morphisms of such and transformations as defined in [11, Section 6] (where we had left implicit the monad $T$ under the assumption of being the free-monoid monad $M$, but the definition is purely formal relative to the cartesianness of $\mathsf{T}$). Here is where we exploit the description of 2-cells in $\mathsf{Mnd}(\mathsf{Bimod}_{\mathsf{T}}(\mathscr{K}))$ given in Definition 4.4.

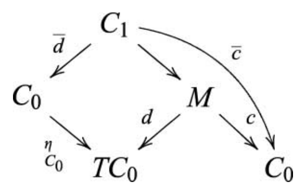
**8.2. Theorem.** *The* 2-*functor*

$$\mathsf{Mnd}(||\_||): \mathsf{Mnd}(\mathsf{Bimod}_{\mathscr{C}at(T)}(\mathscr{C}at(\mathbb{B}))) \to \mathscr{M}ulticat_T(\mathbb{B})$$

*is an isomorphism.*

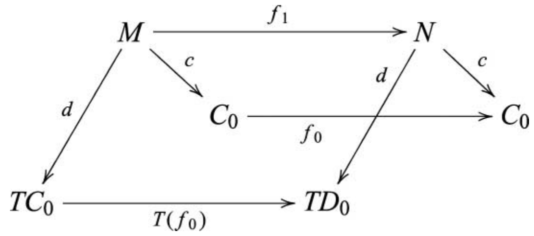
**Proof.** We define an explicit inverse $H: \mathscr{M}ulticat_T(\mathbb{B}) \to \mathsf{Mnd}(\mathsf{Bimod}_{\mathscr{C}at(T)}(\mathscr{C}at(\mathbb{B})))$ to the given 2-functor.

Given a multicategory $\mathbf{M} = TC_0 \leftarrow d - M - c \rightarrow C_0$ we obtain a category $\bar{\mathbf{M}}$ by pulling back along $\eta_{C_0}$:

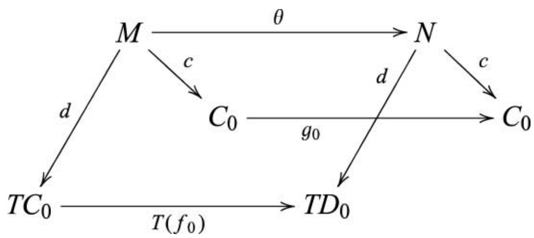



cf. [11, Definition 6.7]. We then construe $M$ itself as a bimodule $H(M) : T\bar{\mathbf{M}} \nrightarrow \bar{\mathbf{M}}$, whose action is obtained by 'restriction' of the composition operation of $M$. Furthermore, the monad structure of $M$ as a multicategory endows $H(M)$ with a monad structure in $\mathsf{Bimod}_{\mathscr{C}at(T)}(\mathscr{C}at(\mathbb{B}))$, which is normal by the very definition of $\bar{\mathbf{M}}$.

Given a morphism of multicategories




we obtain an internal functor $\bar{f} : \bar{\mathbf{M}} \rightarrow \bar{\mathbf{N}}$ (since $\overline{(\_)}$ is a 2-functorial construction) and a morphism of bimodules $\hat{f}_1 : H(M) \rightarrow (Tf_0, f_0)^*(H(N))$ commuting with the monads structures. Finally, a 2-cell



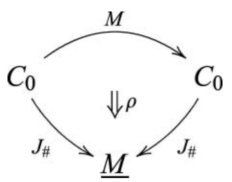

gives a 2-cell $H\theta : \bar{f} \Rightarrow \bar{g}$, namely $\hat{\theta} : M \rightarrow (Tf_0, g_0)^*(N)$, which being a morphism of $(M,M)$-bimod is also a morphism of $(H(M), H(M))$-bimod. □

In view of the above 2-isomorphism we can do without coequalizers in $\mathbb{B}$, and work simply with spans and their (pullback) composition, rather than the more complex composition of bimodules.

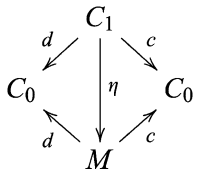
*8.1. Kleisli objects in* $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$

**8.3. Proposition.** $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$ *admits Kleisli objects which preserve representability* (*in the sense of Remark* 2.4).

**Proof.** Given a monad $(M : \mathbf{C} \nrightarrow \mathbf{C}, \eta, \mu)$, we obtain via the lax functor $||\_||$ a monad $(||M||, ||\eta||, ||\mu||)$ in $\mathbf{Spn}(\mathbb{B})$ which is therefore an internal category. This category $\underline{M}$ is the (vertex of the lax cocone of the) Kleisli object of $M$. To define a lax cocone

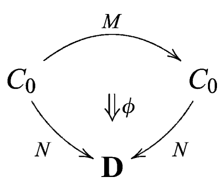



the 2-cell $\eta : \mathsf{Hom}_{\mathbf{C}} \Rightarrow M$ yields an identity-on-objects functor $J : \mathbf{C} \to \underline{M}$:

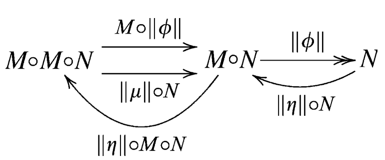



cf. the description of $\mathsf{Hom}_{\mathbf{C}}$ in the proof of Proposition 8.1. The 2-cell $\rho : M \bullet J_\# \Rightarrow J_\#$ is given by $\mu$ since $J_\#$ corresponds simply to the span $C_0 \leftarrow d - M - c \rightarrow C_0$, because $J$ is identity-on-objects (see the description of $f_\#$ above).

For universality, given another lax cocone




we can take the mediating bimodule $\hat{N} : \underline{M} \nrightarrow \mathbf{D}$ to be $N$ itself, as we have a split coequalizer:




Clearly, if $N$ is representable, so is the induced $\hat{N}$ and we can force this property to be 2-functorial. □

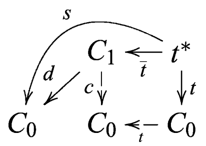
Given the explicit description of Kleisli objects in $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$ above, we obtain the following easy consequence.

**8.4. Corollary.** *If* $T:\mathbb{B}\to\mathbb{B}$ *preserves pullbacks and coequalizers, then the homomorphism* $\mathsf{Bimod}(T):\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))\to\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$ *preserves Kleisli objects.*

The above properties about Kleisli objects in $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$ show that $\mathscr{C}at(\mathbb{B})$ admits a suitable calculus of bimodules and a cartesian monad on it induces a well-behaved pseudo-monad on $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$. As we already mentioned, by virtue of Theorem 8.2 we can use spans instead of bimodules. Consequently we can dispense with Kleisli objects in the adjunction of Proposition 5.2 and use the simpler description of this adjunction in [11, Section 7]. However, we do require well-behaved Kleisli objects in $\mathscr{C}at(\mathbb{B})$ in our classification of strong morphisms and coherence (see Proposition 7.2). We obtain these quite simply (and elegantly) from those in $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$ as shown in Proposition 7.3.

**8.5. Corollary.**
- $\mathscr{C}at(\mathbb{B})$ *admits Kleisli objects.*
- *Given a pullback-preserving functor* $T:\mathbb{B}\to\mathbb{B}$, *the induced* 2*-functor* $\mathscr{C}at(T)$: $\mathscr{C}at(\mathbb{B})\to\mathscr{C}at(\mathbb{B})$ *preserves them.*

**8.6. Remark.** Notice that by the construction in the proof of Proposition 7.3, the internal category $\underline{t^*}$ corresponding to a monad $(t:\mathbf{C}\to\mathbf{C},\eta,\mu)$ in $\mathscr{C}at(\mathbb{B})$ has underlying graph




so that $\underline{t^*}(x,y)=C_1(x,ty)$, just as we expect from the usual construction in $\mathscr{C}at(\mathscr{S}et)$.

It is now clear that the construction of Kleisli objects in $\mathscr{C}at(\mathbb{B})$, which we obtain from those in $\mathsf{Bimod}(\mathscr{C}at(\mathbb{B}))$, does only involve pullbacks and no colimits. Hence, such Kleisli objects are preserved by $\mathscr{C}at(T):\mathscr{C}at(\mathbb{B})\to\mathscr{C}at(\mathbb{B})$, for a pullback-preserving $T:\mathbb{B}\to\mathbb{B}$.

## 9. Multicategories and monoidal categories

In this section we review our motivating example. Let $\mathbb{B}=\mathscr{S}et$ (we could work in far greater generality, e.g. a topos with a natural numbers object). Let $T=(\_^*):\mathscr{S}et\to\mathscr{S}et$ be the free-monoid monad, so that $X^*$ can be explicitly described as the set of finite sequences of elements of $X$. This monad is cartesian, cf. [4]. An algebra for the corresponding 2-monad $M = \mathscr{C}at(\_^*) : \mathscr{C}at \to \mathscr{C}at$ amounts to giving a strict monoidal category, an algebra morphism is a strict monoidal functor and a 2-cell a monoidal natural transformation. A pseudo-algebra is a monoidal category (with an infinite presentation) and a strong morphism is a strong monoidal functor. We do not elaborate here in the distinction between the 'usual' finite presentation of a monoidal category and its corresponding infinite presentation (with $n$-fold tensor products for every $n$ and coherent isomorphisms between their multicategory composites) and refer to [11, Section 9.1]. As indicated there, the distinction is inessential for our purposes.

A monad in $\mathbf{Spn}_{\mathsf{M}}(\mathscr{S}et)$ is a multicategory. We furthermore identify the corresponding morphisms and 2-cells so that $\mathsf{Mnd}(\mathbf{Spn}_{\mathsf{M}}(\mathscr{S}et)) = \mathscr{M}ulticat$ as in [11, Section 6]. Given a multicategory $\mathbb{C}$,

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle d_1}\swarrow & & \searrow{\scriptstyle c_1} \\ \mathbf{MC_0} & & C_0 \end{array}$$

with $\bar{\mathbb{C}}$ the corresponding category of linear morphisms ($\bar{\mathbb{C}}(x, y) = C_1(\langle x\rangle, y)$), the corresponding normal monad in $\mathsf{Bimod}_{\mathsf{M}}(\mathscr{C}at)$ is given by the bimodule $\mathsf{Hom}_{\mathbb{C}} : M\bar{\mathbb{C}} \nrightarrow \bar{\mathbb{C}}$ with fibres $\mathsf{Hom}_{\mathbb{C}}(\vec{x}, y) = C_1(\vec{x}, y)$ and action given by (multicategory) composition in $\mathbb{C}$.

The 2-category $\mathscr{M}onCat$ of strict monoidal categories and strict monoidal functors is monadic over $\mathscr{M}ulticat$ according to Theorem 5.6. The corresponding adjoint pseudo-algebras in $\mathscr{M}ulticat$ are the *representable multicategories*. These and their correspondence with monoidal categories, as well as the relevant coherence result were analysed in detail in [11]. We skip these topics here and turn our attention to the reconstruction of $\Delta$ as the monoid classifier for monoidal categories.

### *9.1. Monoid classifier*

Given strict monoidal categories $\mathbb{C}$ and $\mathbb{D}$, to give a lax monoidal functor $f : \mathbb{C} \to \mathbb{D}$ amounts to give a strict monoidal functor $f : FR(\mathbb{C}) \to \mathbb{D}$. A *monoid* in $\mathbb{C}$ amounts to a lax monoidal functor $(X, \cdot, e) : \mathbf{1} \to \mathbb{C}$ while a monoid morphism is a 2-transformation between such. In particular, the identity $id : FR(\mathbf{1}) \to FR(\mathbf{1})$ yields a monoid $G : \mathbf{1} \to FR(\mathbf{1})$ Thus we obtain the following classification:

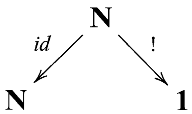
**9.1. Corollary.** (Monoid classifier for monoidal categories)**.** *Given a* (*strict*) *monoidal category* $\mathbb{C}$, *there is an* (*isomorphism*) *equivalence of categories*

$$\mathsf{Monoid}(\mathbb{C}) \equiv (\mathsf{Ps}-)\mathscr{M}on\mathscr{C}at(FR(\mathbf{1}), \mathbb{C})$$

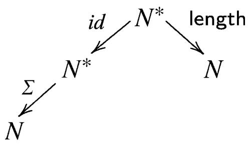
*realised* (*from right to left*) *by precomposition with the* generic monoid

$$G : \mathbf{1} \to FR(\mathbf{1}).$$

Let us give an explicit description of $FR(\mathbf{1})$. First $R(\mathbf{1})$ is the terminal multicategory, whose underlying multigraph can be identified with




where $\mathbf{N}$ is the set of natural numbers. Thus we have a unique arrow $n \succ \bullet$ for every $n$. Now $FR(\mathbf{1})$ has the following underlying graph:




Hence the objects of $FR(\mathbf{1})$ are natural numbers. To give an arrow between $n$ and $m$, we must give a 'partition' of $n = n_1 + \cdots + n_m$, which corresponds to the arrow

$$\langle (n_1 \succ \bullet), \ldots, (n_m \succ \bullet) \rangle : n \to m.$$

Considering $n$ and $m$ as finite ordinals, such an arrow is then a *monotone function* from $[n]$ to $[m]$. Conversely, a monotone function $h : [n] \to [m]$ yields a partition of $[n] = h^{-1}(1) + \cdots + h^{-1}(m)$ via its (possibly empty) fibres; the sum is now interpreted as ordinal sum.

**9.2. Theorem.** $\boxed{FR(\mathbf{1}) \equiv \Delta}$

*where $\Delta$ is the category of finite ordinals and monotone functions, with strict monoidal structure given by ordinal sum. The generic monoid is $1 \in \Delta$ with the unique morphisms $! : 0 \to 1$ and $! : 2 \to 1$.*

We have thus recovered the classical description of the monoid classifier for monoidal categories, usually credited to Lawvere.

## 10. Monoidal globular categories

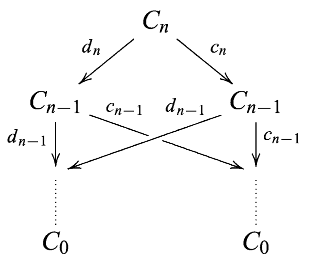
Monoidal globular categories were introduced in [1] as a framework for weak higher-dimensional categories. Briefly put, they allow the definition of higher-dimensional *operads* (see Remark 10.3 below) so that weak $n$-categories are defined as the algebras for a *contractible* operad in the monoidal globular category of spans.

In our brief incursion into monoidal globular categories we will exhibit them as pseudo-algebras on *globular categories* (Section 10.1), give an equivalent adjoint-pseudo-algebra version, namely *representable multicategories in the category* $\omega$-gph *of $\omega$-graphs* and formulate the attendant coherence result (Section 10.2). In Section

10.3 we analyse lax morphisms from the terminal monoidal globular category into a given monoidal globular category $\mathbb{C}$ (the *globular monoids* in $\mathbb{C}$ introduced in [7]) and give a description of their classifier.

### 10.1. Pseudo-algebras on globular categories

An $n$-graph $C$ is a commutative diagram of sets




which means that the following *globularity* condition is satisfied:

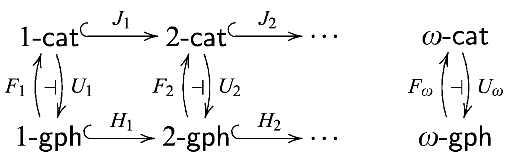
$$c_{i-1}c_i = c_{i-1}d_i$$

$$d_{i-1}c_i = d_{i-1}d_i \quad 2 \leq i \leq n.$$

This implies that we have well defined domain and codomain functions from any dimension $i$ to a lower one $j$, $d_i^j : C_i \to C_j$ and $c_i^j : C_i \to C_j$ by iterated composition of $d$'s and $c$'s respectively. We refer to the elements of $C_i$ as *i-cells*.

A morphism between $n$-graphs $C$ and $D$ is a collection of functions $f_i : C_i \to D_i$ commuting with $d_i$ and $c_i$. We thus have the category $n$-gph. Every $n$-category has an underlying $n$-graph, which yields the forgetful functor $U_n : n\text{-cat} \to n\text{-gph}$ from the category of $n$-categories and $n$-functors. This functor is monadic.

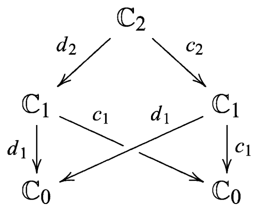
We want to consider the monad $T_\omega : \omega\text{-gph} \to \omega\text{-gph}$ on $\omega$-graphs induced by the monadic adjunction $F_\omega \dashv U_\omega : \omega\text{-cat} \to \omega\text{-gph}$. To analyse its behaviour, we look at its ‘finite approximations’ in the filtration (colimit sequence):



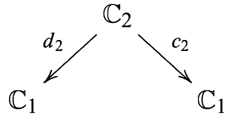

where $H_n : n\text{-gph} \to n{+}1\text{-gph}$ considers an $n$-graph as an $n{+}1$-graph with no $n{+}1$-cells ($C_{n+1} = \emptyset$) and $J_n : n\text{-cat} \to n+1\text{-cat}$ adds only identity $n+1$-cells to an $n$-category ($C_{n+1} = C_n$). All the categories $n$-gph and $\omega$-gph are presheaf categories, and hence have dimension-wise limits and coequalizers. The monad $T_n : n\text{-gph} \to n\text{-gph}$ is cartesian, very much like the free-category monad on a graph which is a free-monoid construction (see e.g. [11, Section 11]) and so is $T_\omega$ as indicated in [24,7]. Consequently we have the 2-monads $\mathscr{C}at(T_1) : \mathscr{C}at(1\text{-gph}) \to \mathscr{C}at(1\text{-gph})$ and $\mathscr{C}at(T_\omega) : \mathscr{C}at(\omega\text{-gph}) \to$

$\mathscr{C}at(\omega\text{-gph})$. Notice that $\omega$-gph = Glob, the category of globular sets in [24], and thus $\mathscr{C}at(\omega\text{-gph}) = \mathsf{GlobCat}$, the category of globular categories and globular functors. We want to analyse the pseudo-algebras of $\mathscr{C}at(T_\omega)$. We do so by analysing the algebras and pseudo-algebras of $\mathscr{C}at(T_2)$.

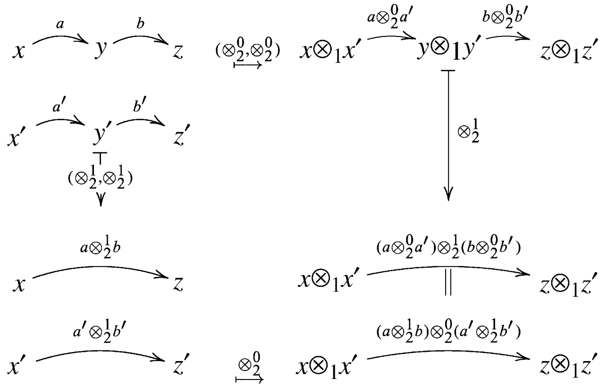
An object in $\mathscr{C}at(2\text{-gph})$ is a 2-graph of categories



To simplify the analysis, let us take $\mathbb{C}_0 = \mathbf{1}$, the terminal category, so that we are left with a graph of categories. A $\mathscr{C}at(T_2)$-algebra structure on this graph consists of strict monoidal structures $(\mathbb{C}_1, \otimes_1, I_1)$ and $(\mathbb{C}_2, \otimes_2^0, I_2^0)$, with $d_2$ and $c_2$ strict monoidal functors, and furthermore the span



has a monoid structure $(\mathbb{C}_2, \otimes_2^1, I_2^1)$ in $\mathbf{Spn}(\mathscr{C}at)(\mathbb{C}_1, \mathbb{C}_1)$ which commutes with the other monoidal structure on $\mathbb{C}_2$. Let us spell out this last condition: given an object $a$ in $\mathbb{C}_2$, write $x \overset{a}{\curvearrowright} y$ to indicate $d_2(a) = x$ and $c_2(a) = y$.



Now to give a pseudo-$\mathscr{C}at(T_2)$-algebra structure on the same graph we weaken all the monoids to pseudo-monoids (but $d_2$ and $c_2$ remain *strict* monoidal morphisms with respect to $\otimes_2^0$ and $\otimes_1$) and $\otimes_2^0$ and $\otimes_2^1$ pseudo-commute, in the sense that the equality in the above chasing-tensors diagram becomes an isomorphism $\gamma_{a,b}^{a',b'} : (a \otimes_2^1 b) \otimes_2^0 (a' \otimes_2^1 b') \xrightarrow{\sim} (a \otimes_2^0 a') \otimes_2^1 (b \otimes_2^0 b')$ coherent with respect to the pseudo-monoidal structures involved. This amounts to the fact that $\otimes_2^1$ is a strong monoidal functor $\mathbb{C}_2 \times_{\mathbb{C}_1} \mathbb{C}_2$ (which inherits a pairwise monoidal structure from $(\mathbb{C}_2, \otimes_2^0, I_2^0, \alpha, \lambda, \rho)$) to

$\mathbb{C}_2$. Similar considerations apply to the units. This is the point of view adopted in [5], which gives a sound conceptual account of interchange constraints and their attendant axioms. Extrapolating to the colimit, we conclude

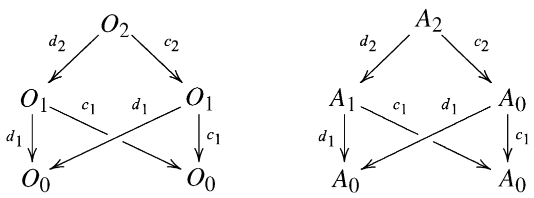
$$\boxed{\mathscr{C}at(T_\omega)\text{-alg} \equiv \mathit{Strict\text{-}Monoidal\text{-}Globular\text{-}Cat}}$$

$$\boxed{\text{Ps-}\mathscr{C}at(\mathbf{M})\text{-alg} \equiv \mathit{Monoidal\text{-}Globular\text{-}Cat}}$$

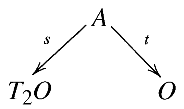
where the 2-categories on the right give the ‘infinitary’ presentations of the objects of the corresponding ones in [1].

## *10.2. Representable multicategories in* $\omega$-gph

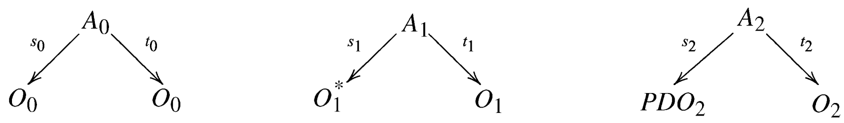
Consider an object in $\mathbf{Spn}_{\mathsf{T}_2}$(2-gph): it consists of two 2-graphs $O$ and $A$:




and a span

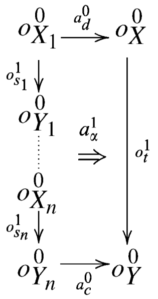



in 2-gph which in turn consists of 3 ordinary spans




compatible with $d_i, c_i$ $(i = 0, 1, 2)$, where $O_1^*$ is the set of composable 1-cells in $O$ and $PDO_2$ is the set of pastings of 2-cells in $O$. See [23] for a detailed description of the pasting diagrams of the related (and more complex) situation of computads and their role in the construction of free 2-categories.

A *multicategory* in 2-gph is therefore such a span endowed with a monoid structure. This means that the first span above sets up a category with object of objects $O_0$ and object of morphisms $A_0$. The second span is then a multicategory as follows: let us denote the elements of $A_i$ by symbols $a_x^i$ and adopt a similar convention for the elements of $O_i$. The combinatorial information of source, target, domain and codomain for an element of $A_1$ can be represented diagramatically as a 2-cell:



to indicate

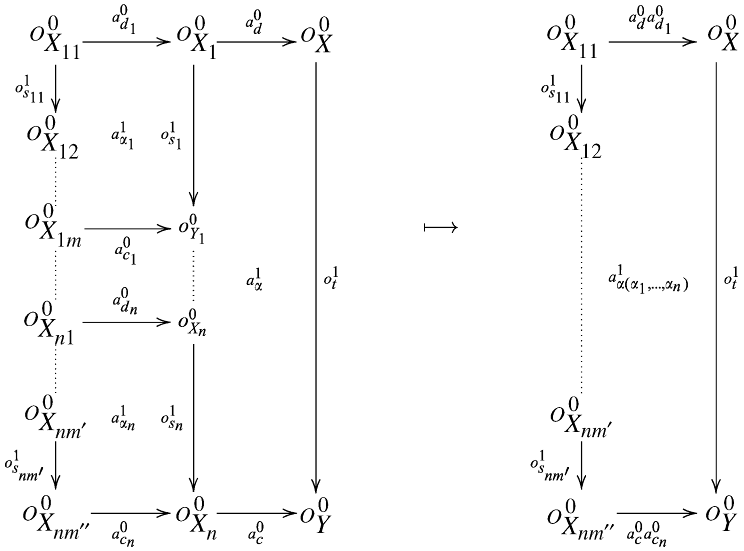
$$d_1(a^1_\alpha) = a^0_d,$$

$$c_1(a^1_\alpha) = a^0_c,$$

$$s_1(a^1_\alpha) = \langle o^1_{s_1}, \ldots, o^1_{s_n} \rangle,$$

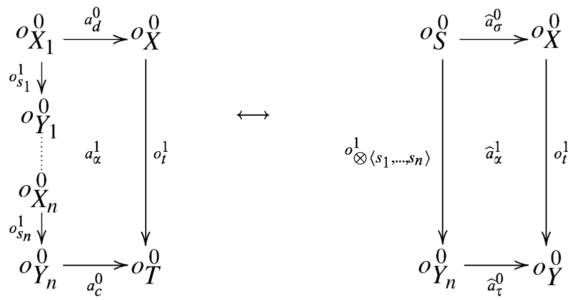
$$t_1(a^1_\alpha) = o^1_t,$$

and the further evident information about 0-domain and codomains. These cells are equipped with a composition operation



associative and unitary as expected. A similar 'multicategory' structure is provided for the third span above, this time with the more complex pasting composition, which we will use below to account for interchange.

Let us now consider *representability* for these multicategories in 2-gph. A cell $a^1_\pi : \langle o^1_{s_1}, \ldots, o^1_{s_n} \rangle \Rightarrow o^1_{\otimes\langle s_1,\ldots,s_n\rangle}$ in $A_1$ is *universal* if precomposition with it sets up a bijection




natural in $o^1_t$. Furthermore, such universal cells should be closed under composition and (quite importantly) $d_i, c_i : A_i \to A_{i-1}$ should preserve universal cells. To see how the interchange isomorphism arises with this reformulation, consider again for simplicity that the 0-level is trivial and look once again at the situation we considered before. We can realise the relevant tensor composites by universal cells with the appropriate sources. Both ways around, the mutlicategory composite cells are universal (since these are closed under such composition) in $A_2$ for the same source element in $PDO_2$, hence their targets are canonically isomorphic.

Thus, a *multicategory in* $\omega$-gph consists of a multigraph in $\omega$-gph so that the source of a $n$-cell (an element in $A_n$) is a pasting diagram of elements of $O_n$. Such a multigraph is equipped with unitary and associative multicategory composites as explained above. It is *representable* when for every pasting diagram $p$ in $O_n$ there is a universal $n$-cell whose source is $p$, and such universal cells are closed under composition and preserved by $d_i, c_i : A_i \to A_{i-1}$ for all $i > 1$. The morphisms between such are the evident ones, preserving the combinatorial source-target information, composites and identities and (for representables) preserving universal cells.

By Theorems 5.6 and 8.2 we have

**10.1. Corollary.** (Universal version of monoidal globular categories)**.**

- $\boxed{\mathcal{R}ep\mathcal{M}ulticat_s(\omega\text{-gph}) \equiv T_\omega\text{-alg} \equiv \textit{Strict-Monoidal-Globular-Cat}}$ *where* $\mathcal{R}ep\mathcal{M}ulticat_s(\omega\text{-gph})$ *denotes the* 2*-category of strict representable multicategories and strict morphisms of such.*
- $\boxed{\mathcal{R}ep\mathcal{M}ulticat(\omega\text{-gph}) \equiv \text{Ps-}T_\omega\text{-alg} \equiv \textit{Monoidal-Globular-Cat}}$

As a further consequence of our general theory we can establish the following coherence results:

**10.2. Corollary.** (Coherence for monoidal globular categories).
- *The inclusion*

$$\mathcal{R}ep\mathcal{M}ulticat_s(\omega\text{-gph}) \hookrightarrow \mathcal{R}ep\mathcal{M}ulticat(\omega\text{-gph})$$

*has a left biadjoint whose unit is a (pseudo-natural) equivalence.*
- *The inclusion*

$$StrictMonoidalGlobularCat \hookrightarrow MonoidalGlobularCat$$

*has a left biadjoint whose unit is a (pseudo-natural) equivalence.*

We have therefore recovered the coherence result for monoidal globular categories by methods altogether different to those in [1, Section 4]. Notice that Theorem 4.1 in [1] follows from the above and the fact that a strong morphism out of a free monoidal globular category can be strictified (cf. [11, Proposition 10.3]).

**10.3. Remark.** A multicategory in $\omega$-gph with object of objects the terminal $\omega$-graph **1** amounts to an *operad* in *Span* in the sense [1].

### *10.3. Globular monoids and their classifier*

The terminal monoidal globular category **1** has underlying $\omega$-graph the terminal such, that is, the one with only one cell at every dimension (denoted $U_\omega$ in [1]). A lax morphism from **1** into a monoidal globular category $\mathbb{C}$ amounts to what Batanin and Street call a *globular monoid* in $\mathbb{C}$ (cf. [7]). In elementary terms, a globular monoid $(M, \iota, \mu)$ in $\mathbb{C}$ is given by:
- objects $M_n$ of $\mathbb{C}_n$ for every $n$, compatible with domain and codomain in $\mathbb{C}$ ($d_i(M_i) = c_i(M_i) = M_{i-1}$),
- morphisms $\iota_n : I_n(M_n) \to M_n$ and $\mu_n : M_n \otimes_n M_n \to M_n$ which make $(M_n, \iota_n, \mu_n)$ a monoid in the monoidal category $(\mathbb{C}_n, I_n(M_n), \otimes_n)$,
- the monoid structure commutes with the interchange isomorphisms: for every $i < j < n$,

$$\mu_j \circ (\mu_i \otimes_j \mu_i) \circ \gamma^{M_n,M_n}_{M_n,M_n} = \mu_j \circ (\mu_j \otimes_i \mu_j),$$

$$\mu_i \circ (\iota_j \otimes_i \iota_j) = \lambda : I_j(M_n) \otimes_i I_j(M_n) \xrightarrow{\sim} I_j(M_n).$$

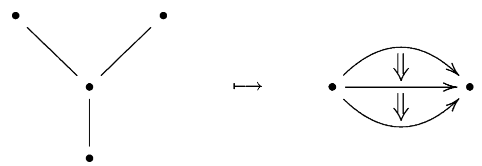
According to our classification of lax morphisms of Section 6, or rather its more explicit version in terms of multicategories in Section 9.1, such a globular monoid corresponds to a (strict) morphism of (strict) monoidal globular categories $FR(\mathbf{1}) \to \mathbb{C}$.

**10.4. Corollary.** (Classification of globular monoids). *Given a (strict) monoidal globular category $\mathbb{C}$, there is an (isomorphism) equivalence of categories*

$$\mathsf{GlobularMonoid}(\mathbb{C}) \equiv (\mathsf{Ps}-)StrictMonoidalGlobularCat(FR(\mathbf{1}), \mathbb{C})$$

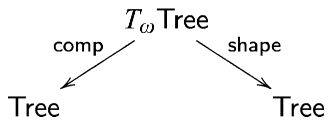
*realised* (*from right to left*) *by precomposition with the* generic monoid $G : \mathbf{1} \to FR(\mathbf{1})$.

To obtain an explicit description of the globular monoid classifier $FR(\mathbf{1})$ we need to know the free monoidal globular category on $R\mathbf{1}$. To describe $R\mathbf{1}$ in turn we must know $T_\omega(\mathbf{1})$, the free $\omega$-category on the terminal $\omega$-graph. This object has been described by Batanin [1,7] in terms of trees. The $n$-cells $\omega$-category Tree are finite trees of height $n$ (such a tree is formally described as a functor $\tau : [n]^{\mathrm{op}} \to \Delta$ such that $\tau(0) = [0]$ in *ibid*) which give the relevant combinatorial information of the pasting diagrams they represent



There is in fact an explicit construction associating to a tree $\tau$ the $\omega$-graph (globular set) $||\tau||$ it represents, cf. [1,24].

The (strict) monoidal globular category $FR(\mathbf{1})$ has underlying graph




where the cells of $T_\omega$Tree are trees labelled by trees. A *labelling* of a tree is formally defined via its associated $\omega$-graph: a labelling of $\tau$ is a morphism in $l : ||\tau|| \to$ Tree in $\omega$-gph. The composition operation comp : $T_\omega$Tree $\to$ Tree amounts to grafting the trees in the labelling according to the shape of the labelling tree, or equivalently performing the pasting composite of the associated $\omega$-graphs. We thus obtain the monoidal globular category $\Omega$ of [7], together with its generic globular monoid $U_\omega$ (with its unique globular monoid structure).

## 11. Pseudo-functors

Given a small category $\mathbb{C}$ with set of objects $C$, consider the functor 2-category $\mathscr{K} = [C, \mathscr{C}at]$. Clearly $\mathscr{K}$ admits a calculus of bimodules and Kleisli objects for them, with all the relevant structure given pointwise. The category $\mathbb{C}$ acts on $\mathscr{K}$ defining a 2-monad $\mathbb{C} \star \_ : \mathscr{K} \to \mathscr{K}$ which can be described explicitly as follows:

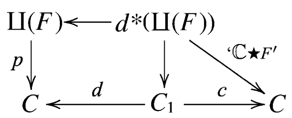
*Functor*: given a functor $F : C \to \mathscr{C}at$, the functor $\mathbb{C} \star F : C \to \mathscr{C}at$ has action

$$\mathbb{C} \star F(x) = \coprod_{f : y \to x} F(y)$$

and we write $\langle f, \varphi \rangle$ for an object of $\mathbb{C} \star F(x)$, with $\varphi \in F(\mathsf{dom} f)$.

*Unit*: $\eta_F : F \Rightarrow \mathbb{C} \star F$ takes the object $\varphi \in Fx$ to $\langle id_x, \varphi \rangle$.
*Multiplication*: $\mu_F : \mathbb{C}\star\mathbb{C}\star F(x) \Rightarrow \mathbb{C}\star F(x)$ takes the object $\langle f, \langle g, \varphi \rangle\rangle$ to $\langle f \circ g, \varphi \rangle$.

**11.1. Remark.** The action of $\mathbb{C}$ on $\mathscr{K}$ can be grasped more easily by identifying $\mathscr{K}$ with $\mathscr{C}at/C$, by regarding $F : C \to \mathscr{C}at$ as the family $\coprod(F) \to C : (x, \varphi) \mapsto x$. Given the small category $\mathbb{C}$: $C \xleftarrow{d} C_1 \xrightarrow{c} C$ the family corresponding to $\mathbb{C} \star F$ is obtained by pullback against $d$, followed by composition with $c$:




With this point of view we can easily show that $\mathbb{C}\star$ is cartesian, along the lines of [4].

$\mathbb{C}\star$*-algebras and pseudo-algebras*: Let $\alpha : \mathbb{C}\star F \to F$ be a $\mathbb{C}\star$-algebra. This amounts to a $C$-collection of functors $\alpha_x : \coprod_{f : y \to x} F(y) \to F(x)$. It is clear that such $\alpha_x$'s satisfying the algebra axioms, give the action on morphisms to endow $F$ with the structure of a functor $F : \mathbb{C} \to \mathscr{C}at$, namely

$$F(f : y \to x) = \alpha_x(\langle f, \_ \rangle) : F(y) \to F(x)$$

and a further easy identification of 1- and 2-cells yields

$$\boxed{\mathbb{C} \star \text{-alg} \equiv [\mathbb{C}, \mathscr{C}at]}$$

and similarly

$$\boxed{\text{Ps-}\mathbb{C} \star \text{-alg} \equiv \text{Ps-}[\mathbb{C}, \mathscr{C}at]}$$

where Ps-$[\mathbb{C}, \mathscr{C}at]$ consists of pseudo- functors, pseudo-natural transformations and modifications.

*Lax* Bimod($\mathbb{C}\star$)-*algebras*: To obtain the corresponding objects over which pseudo-functors become properties, we must analyse the bicategory $\text{Bimod}_{\mathbb{C}\star}(\mathscr{C}at)$. A bimodule $M : \mathbb{C} \star F \nrightarrow F$ amounts to a $C$-indexed collection of bimodules in $\mathscr{C}at$, $M_x : \mathbb{C} \star F(x) \nrightarrow F(x)$. Such $M_x$ corresponds in turn to a family of bimodules $\langle M^f : F(y) \nrightarrow$

$F(x)\rangle_{f\,:\,y\to x}$ according to the following chain of identifications:

$$\frac{\dfrac{\dfrac{\displaystyle\coprod_{f\,:\,y\to x} F(y) \not\to F(x)}{\left(\displaystyle\coprod_{f\,:\,y\to x} F(y)\right)^{op} \times F(x) \to \mathcal{S}et}}{\displaystyle\prod_{f\,:\,y\to x} [F(y)^{op} \times F(x) \to \mathcal{S}et]}}{\langle F(y) \not\to F(x)\rangle_{f\,:\,y\to x}}$$

A normal monad (or equivalently a normal lax $\mathsf{Bimod}(\mathbb{C}\bigstar)$-algebra) on such $M$ consists of

*Unit*: $\iota : \eta_F^* \Rightarrow M$ which induces the identity

$$\mathsf{Hom}_{Fx} = M^{id_x} : Fx \to Fx.$$

*Multiplication*: $m : (\mathbb{C} \bigstar M) \bullet M \Rightarrow (\mu_F)_\# \bullet M$ amounts to a collection of bimodule morphisms

$$\langle m^{f,g} : M^f \bullet M^g \Rightarrow M^{f\circ g}\rangle_{f:y\to x, g:z\to y}$$

subject to the associativity and unit axioms which make $M$ a *lax functor* $M : \mathbb{C} \to \mathsf{Bimod}(\mathscr{C}at)$ such that $Mx = Fx$ on objects. A morphism $h$ between two such (normal) monads $M : \mathbb{C} \bigstar F \nrightarrow F$ and $N : \mathbb{C} \bigstar G \nrightarrow G$ amounts to a $C$-collection of functors $h_x : Fx \to Gx$ and bimodule morphisms

$$\theta_f : M^f \bullet (h_x)_\# \Rightarrow (h_y)_\# \bullet N^f$$

for each $f : y \to x$ in $\mathbb{C}$; the compatibility with the unit and multiplication of the monads makes such collection into a lax transformation $h_\# : M \Rightarrow N : \mathbb{C} \to \mathsf{Bimod}(\mathscr{C}at)$. Similarly, a 2-cell $\alpha : h \Rightarrow k$ between two such morphisms corresponds to a modification $\alpha_\# : h_\# \Rightarrow k_\#$. Hence,

$$\boxed{\mathsf{Lax\text{-}Bimod}(\mathbb{C}\bigstar)\text{-}\mathsf{alg} \equiv \mathsf{Lax}_{\mathsf{rep}}[\mathbb{C}, \mathsf{Bimod}(\mathscr{C}at)]}$$

where $\mathsf{Lax}_{\mathsf{rep}}[\mathbb{C}, \mathsf{Bimod}(\mathscr{C}at)]$ denotes the 2-category of normal lax functors, representable lax transformations (those whose component bimodules are representable) and modifications.

**11.2. Remark.** Here we could proceed further from the general theory and identify $\mathsf{Lax}_{\mathsf{rep}}[\mathbb{C}.\mathsf{Bimod}(\mathscr{C}at)]$ with $\mathscr{C}at/\mathbb{C}$ (a result usually credited to Bénabou). Now, adjoint pseudo-algebras on $\mathscr{C}at/\mathbb{C}$ are Grothendieck (co)fibrations, as expected:

$$\mathit{Strict}\mathsf{Lax}_{\mathrm{rep}}[\mathbb{C},\mathsf{Bimod}(\mathscr{C}at)] \simeq \mathit{SplitCofibrations}/\mathbb{C}$$

$$\mathit{Representable}\mathsf{Lax}_{\mathrm{rep}}[\mathbb{C},\mathsf{Bimod}(\mathscr{C}at)] \simeq \mathit{Cofibrations}/\mathbb{C}$$

**11.3. Remark.** In this situation it is fairly straightforward to see Theorem 5.4 at work: a lax functor $M:\mathbb{C}\to\mathsf{Bimod}(\mathscr{C}at)$ is representable iff each bimodule $M_f: Mx \nrightarrow My$ is so, thus $M_f = Ff_{\#}$ for some functor $Ff: Fx \to Fy$. The structural 2-cell $m$ is an isomorphism iff each $\langle m^{f,g}: M^g \bullet M^f \Rightarrow M^{f\circ g}\rangle_{f:y\to x, g:z\to y}$ is invertible. Clearly such isomorphisms give the required $m_{f,g}: Ff\circ Fg \xrightarrow{\sim} F(f\circ g)$ which make $F:\mathbb{C}\to\mathscr{C}at$ a pseudo-functor. So *Representable*-$\mathsf{Lax}_{\mathrm{rep}}[\mathbb{C},\mathsf{Bimod}(\mathscr{C}at)] \simeq \mathsf{Ps}\text{-}[\mathbb{C},\mathscr{C}at]$.

Once again Theorem 7.4 applies to obtain the standard coherence results.

**11.4. Corollary.** (Coherence for pseudo-functors).
- *The inclusion*

$$\mathit{Strict}\mathsf{Lax}_{\mathrm{rep}}[\mathbb{C},\mathsf{Bimod}(\mathscr{C}at)] \hookrightarrow \mathit{Representable}\mathsf{Lax}_{\mathrm{rep}}[\mathbb{C},\mathsf{Bimod}(\mathscr{C}at)]$$

*has a left biadjoint whose unit is a (pseudo-natural) equivalence.*
- *The inclusion*

$$[\mathbb{C},\mathscr{C}at] \hookrightarrow \mathsf{Ps}\text{-}[\mathbb{C},\mathscr{C}at]$$

*has a left biadjoint whose unit is a (pseudo-natural) equivalence.*

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