THE COMPLEXITY OF REGULAR SUBGRAPH RECOGNITION

F. CHEAH and D.G. CORNEIL

Department of Computer Science, University of Toronto, Toronto, Ont., Canada M5S 1A4

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Given a graph G we wish to determine whether it contains a partial k-regular subgraph. In this paper it is shown that this problem is NP-complete for graphs with maximum degree k + 1 when \( k \geq 3 \). (The k = 1, 2 cases are trivially solvable in polynomial time.) The k-regular subgraph problem remains NP-complete when G is restricted to Hamiltonian or Eulerian graphs. We also examine the cases where the k-regular subgraph is further restricted to being spanning and/or connected.

1. Introduction

In this paper we examine the complexity of the \( k \)-regular subgraph recognition problem \((k-R)\): "Given a graph G, does it contain a k-regular subgraph". (Unless otherwise stated, all subgraphs are assumed to be partial subgraphs.) Clearly the problem belongs to P for \( k = 1 \) or 2. Chvátal et al. [3] showed the NP-completeness for \( k = 3 \). Much of the interest in the problem is due to the following conjecture posed by Berge (see [1]): "Every 4-regular simple graph contains a 3 regular subgraph". Recently Tashkinov [9] proved the conjecture thereby showing that the 3-R problem belongs to P for 4-regular graphs G. It is interesting to note that Tashnikov's proof is not constructive and that the problem of finding such a 3-regular subgraph remains open.

In Section 2 of this paper, we show that the \( k \)-R problem is NP-complete for \( k \geq 3 \) and that it remains NP-complete for various restrictions on the given graph G including maximum degree of \( k + 1 \). Similar results are shown in Section 3 where we require the \( k \)-regular subgraph to be spanning.

Throughout this paper we will use the notion of an \( h \)-partition of the vertex set \( V \), namely subsets \( V_1, V_2, \ldots, V_h \) of \( V \) such that \( \bigcup_{i=1}^{h} V_i = V \) and \( V_i \cap V_j = \emptyset \) for \( i \neq j \). Thus the standard 3-colorable graph recognition (3 C) problem can be stated as:

Given an arbitrary graph \( G \), does there exist a tripartition on \( V(G) \) such that \( G[V_i] \) (the subgraph induced on \( V_i \)) is void for \( 1 \leq i \leq 3 \)?

2. The \( k \)-regular subgraph recognition problem

We now determine the complexity of the \( k \)-R problem for arbitrary graphs.
(Clearly the k-R problem belongs to P for \( k = 1 \) or \( k = 2 \).) After showing it NP-complete for \( k \geq 3 \) we examine the situation for various families of restricted input.

**Theorem 2.1.** The \( k \)-regular subgraph recognition problem is NP-complete for \( k \geq 3 \).

**Proof.** It is easy to see that the \( k \)-R problem belongs to NP. We will reduce the 3-C problem (shown to be NP-complete by Karp [5]) to the \( k \)-R problem. Let \( K'_k \) denote the complete graph on \( k \) vertices with one missing edge.

Given an arbitrary graph \( G \) as an input to the 3-C problem, \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( E(G) = \{e_1, e_2, \ldots, e_m\} \), construct a new graph \( G' \) as follows:

1. For each \( v_i \) in \( G \), construct 3 cycles \( C^i_1, C^i_2, C^i_3 \) in \( G' \), each of length \( 2d(v_i) + 1 \). Denote the vertices in \( C^i_h \) as \( c^h_{ij} \), \( 1 \leq i \leq n, 1 \leq j \leq 2d(v_i) + 1, 1 \leq h \leq 3 \).
2. For each \( e_j, 1 \leq j \leq m \), construct \( 3(k - 2) \) corresponding subgraphs \( D^h_{jp} \) in \( G' \), \( 1 \leq p \leq k - 2, 1 \leq h \leq 3 \), where each \( D^h_{jp} \) is a \( K_{k+1} \). Denote the two vertices in \( D^h_{jp} \) with degree \( k - 1 \) as \( x^h_{jp} \) and \( y^h_{jp} \).
3. Suppose \( e_j \) is an edge incident to \( v_s \) and \( v_t \) in \( G \). For each \( h, 1 \leq h \leq 3 \), let \( e^h_{st} \) and \( c^h_{ij} (c^h_{ij} \text{ and } c^h_{ij}) \) be two vertices in \( C^h_i \) (\( C^h_j \)) such that \( c^h_{ij} \) and \( c^h_{ij} \) still have degree two. For \( 1 \leq p \leq 3(k - 2) \), add the edges \( (c^h_{ij}, x^h_{jp}), (c^h_{ij}, y^h_{jp}), (c^h_{ij}, x^h_{jp}) \), and \( (c^h_{ij}, y^h_{jp}) \) to \( E(G') \).
4. All the edges in \( G \) have been considered in step (3), each cycle \( C^h_i (1 \leq i \leq n, 1 \leq h \leq 3) \) will contain exactly one vertex of degree 2. Name these vertices \( w^h_i \), \( 1 \leq i \leq n, 1 \leq h \leq 3 \).
5. Add a cycle \( C' \) of length \( (k - 1)n \) to \( G' \); this cycle is on the new vertices \( x^h_{ij}, y^h_{ij} \) and \( y^h_{ij}, w^h_i \).
6. As an example Fig. 1 shows \( G' \) for a simple graph \( G \) with \( k = 3 \). Clearly the graph \( G' \) can be constructed in polynomial time for any given \( G \).

We shall now show that the given graph \( G \) is 3-colorable iff the graph \( G' \) constructed contains a \( k \)-regular subgraph.

(\( \Rightarrow \)) Given \( G \) and any tripartition of \( V(G) \) into subsets \( V_1, V_2, \) and \( V_3, \) we can construct a corresponding \( H \), a subgraph of \( G' \) as follows:

1. All the vertices \( a_{ij}, 1 \leq i \leq k - 1, 1 \leq j \leq n, \) are in \( V(H) \).
2. All the vertices \( u_{ij}, 1 \leq i \leq n, 1 \leq j \leq k - 2 \) are in \( V(H) \).
3. If \( v_i \) in \( G \) is in subset \( V_c, 1 \leq c \leq 3 \), then the cycle \( C^c_i \) is in \( H \). If the subset \( V_c \) is not \( V_1 \), then add also the vertices in the \( k - 2 \) \( K'_{k+1} \)'s of the \( U_i \).
(4) If $e_j, 1 \leq j \leq m$, is adjacent to vertex $v_k$, which is in subset $V_c$, then the subgraphs $D_{ij}$ adjacent to $C_C$ are in $V(H)$ for $1 \leq p \leq k - 2$.

(5) The subgraph $H$ is defined to be $G'[V(H)]$.

It is trivial to check that the subgraph $H$ thus constructed exists for any tripartition of $G$. Furthermore, all the vertices in $H$ have degree $k$ except possibly those vertices $x_{ip}$ and $y_{ip}$ in $D_{ip}$ for $1 \leq j \leq m, 1 \leq p \leq k - 2$, and $1 \leq c \leq 3$. It is then easy to check that if the tripartition is a coloring, i.e., $G$ is 3-colorable, then the subgraph $H$ constructed is $k$-regular.
Assume that $G'$ contains a $k$-regular subgraph $H$, the following properties must be true concerning $H$:

1. All the vertices $a_{ij}$ and $u_{ij}$, $1 \leq i \leq k - 1$, $1 \leq j \leq n$, and $1 \leq l \leq k - 2$ are in $V(H)$.

This is because the subgraph $H$ cannot be $k$-regular without the inclusion of at least one $u_{ij}$ vertex. But the inclusion of one such vertex forces all the vertices in $C'$ and other $u_{ij}$ vertices to be included.

2. For each $i$, $1 \leq i \leq n$, exactly one of the cycle $C_i^h$, $1 \leq h \leq 3$, is in $V(H)$. If more than one of the $C_i^h$ is in $V(H)$, then the vertices $u_{il}$, $1 \leq l \leq k - 2$, will have degree $k + 1$ in $H$ contradicting the fact that $H$ is $k$-regular.

3. For each $i$, $1 \leq i \leq n$, if $C_i^h \subseteq H$, then $C_j^h \not\subseteq H$, for all $j$ such that $(i, j) \in E(G)$.

This is to ensure that all the vertices $x_{jp}^h$ and $y_{jp}^h$, $1 \leq j \leq m$, $1 \leq p \leq k - 2$, $1 \leq h \leq 3$, have degree $k$.

By property (2), it follows that subgraph $H$ in $G'$ corresponds to a tripartitioning for the vertices of $G$ such that vertex $u_i$ is in partition $c$ if the cycle $C_i^c \in H$. Property (3) ensures that adjacent vertices are in different partition, thereby showing that the tripartition corresponding to $H$ is in fact a coloring of $G$. Hence $G$ is 3-colorable if $G'$ contains a $k$-regular subgraph $H$. This completes the proof. $\square$

Since the graph $G'$ constructed in the proof of Theorem 2.1 has maximum degree $k + 1$, we have the following corollary:

**Corollary 2.2.** The $k$-R problem remains NP-complete even when the input is restricted to graphs with maximum degree $k + 1$.

When $k = 3$, the general $k$-R problem simplifies to the one proven by Chvátal et al. [3]. It should be noted however that their proof does not allow the maximum degree to be restricted to 4.

**Corollary 2.3** [3]. Given an arbitrary graph $G$, it is NP-complete to decide whether it contains a 3-regular subgraph.

Note that the Tashkinov’s proof of the Berge Conjecture implies that the $k$-R problem is polynomial for $k = 3$ and input restricted to regular graphs of degree 4. The complexity of the $k$-R problem restricted to regular graphs of degree $k + 1$ remains open for $k \geq 4$.

The next two corollaries show that the $k$-R problem remains NP-complete for two restricted classes of graphs.

**Corollary 2.4.** The $k$-R problem remains NP-complete when the input is restricted to Hamiltonian graphs with $\Delta(G) = k + 3$, for $k \geq 3$. 

**Proof.** We shall show that the k-R problem can be reduced to this problem. Given an arbitrary graph $G$ with $\Delta(G) = k + 1$ as an input to the k-R problem, where $|V(G)| = n$, construct $G'$ as follows:

$$V(G') = V(G) \cup \{u_1, u_2, \ldots, u_n\},$$

$$E(G') = E(G) \cup \{(1 \leq i \leq n - 1) \ (v_i, u_i), \ (v_i+1, u_i)\} \cup \{(v_n, u_n)\} \cup \{(v_1, u_1)\}.$$

Clearly $G'$ is Hamiltonian with the Hamiltonian circuit: $v_1, u_1, u_2, v_2, \ldots, v_n, u_n, v_1$. Furthermore, $\Delta(G') = k + 3$.

Since all the $u$'s have degree two in $G'$, they cannot be contained in a k-regular subgraph of $G'$ (if one exists) for $k \geq 3$. Hence a k-regular subgraph of $G'$ is essentially a k-regular subgraph of $G$, and vice versa. Since the k-R problem is NP-complete by Theorem 2.1, so is this problem.

**Corollary 2.5.** The k-R problem remains NP-complete when the input is restricted to Eulerian graphs with $\Delta(G) = k + 2$, for $k \geq 3$.

**Proof.** We shall again reduce the k-R problem to this one. Given an arbitrary graph $G$ with $\Delta(G) = k + 1$, and $|V(G)| = n$. Let the set of vertices with odd degrees be $D = \{d_1, d_2, \ldots, d_k\}$. Since the number of vertices with odd degrees has to be even, $k = 2j$ for some $j \in \mathbb{N}$. We can construct $G'$ as follows:

$$V(G') = V(G) \cup \{u_1, u_2, \ldots, u_j\},$$

$$E(G') = E(G) \cup \{(1 \leq i \leq j) \ (d_{i-1}, u_i)\} \cup \{(1 \leq i \leq j) \ (d_{i}, u_i)\}.$$

Since $G \setminus D$ has vertices of even degrees, and all vertices in $D$ have even degrees in $G'$ by the addition of one adjacent edge, and all new vertices $\{u_i\}$ have degree 2, it follows that $G'$ is Eulerian.

Furthermore, none of the new vertices $\{u_i\}$ can be contained in a k-regular subgraph of $G'$ (if one exists) since they only have degree 2. Hence $G$ has a k-regular subgraph iff $G'$ has one, and the proof is complete.

3. Extensions of the k-R problem to the k-regular spanning subgraph problems

We have already shown that the k-R problem is NP-complete even when $\Delta(G) = k + 1$. In this section we study the complexity of the k-R problem when various spanning and connectivity constraints are imposed on the k-regular subgraph.

3.1. The k-regular spanning subgraphs partitioning problem (the k-RSP problem)

Define the k-regular spanning subgraphs partitioning problem (in short: the k-RSP problem) as:
Given an arbitrary simple graph \( G \), does \( G \) have an \( h \)-partitioning \( V_1, V_2, \ldots, V_h \), for some \( h \), such that for each \( i \), \( |V_i| \geq 2k + 3 \) and \( G[V_i] \) contains a \( k \)-regular connected spanning subgraph?

In order to show that the \( k \)-RSP problem is NP-complete, we use the NP-completeness of the Hamiltonian subgraphs partitioning problem (in short: the HSP problem) defined as:

Given an arbitrary directed graph \( G \), does there exist an \( h \)-partitioning \( V_1, V_2, \ldots, V_h \) for some \( h \), such that for each \( i \), \( |V_i| \geq 3 \) and \( G[V_i] \) contains a Hamiltonian circuit?

The HSP problem was shown to be NP-complete by Valiant [11].

**Theorem 3.1.** The \( k \) RSP problem is NP complete for \( k \geq 2 \).

**Proof.** We shall reduce the HSP problem to the \( k \)-RSP problem. Given an arbitrary directed graph \( G \) as an input to the HSP problem, \( V(G) = \{v_1, v_2, \ldots, v_n\} \), construct \( G' \) as follows:

1. Transform each vertex \( v_j, 1 \leq j \leq n \), into two vertices \( v^*_j \) and \( v^\prime_j \), and add a new clique \( K^1_{k-1} \) with all edges between \( v^*_j \) and \( v^\prime_j \) and the vertices of the corresponding clique \( K^1_{k-1} \). Note that there is no edge between \( v^*_j \) and \( v^\prime_j \).

2. For each arc \((v_i, v_j)\) going from \( v_i \) to \( v_j \), add the edge \((v^*_i, v^*_j)\) to \( G' \).

If \( G \) contains partitions \( V_1, V_2, \ldots, V_h \) for some \( h \), such that for each \( i \), \( |V_i| \geq 3 \), and \( G[V_i] \) induces a subgraph \( G_i \) containing a Hamiltonian circuit, a \( k \)-regular spanning subgraph partitioning (as defined by the \( k \)-RSP problem) can be constructed for \( G' \) as follows:

Take the same set of partitions. For each \( v_j \in V_i \) in \( G \), put \( \{v^*_j, v^\prime_j\} \cup K^1_{k-1} \) in \( U_i \) in \( G' \). Since the vertices in \( V_i \) induce a Hamiltonian circuit, the corresponding vertices and edges of the Hamiltonian circuit will form a \( k \)-regular subgraph spanning \( U_i \) in \( G' \). Furthermore, since \( |V_i| \geq 3 \) for all \( i \), \( 1 \leq i \leq h \), it follows that \( |U_i| \geq 3k + 3 \) for all \( i \), \( 1 \leq i \leq h \). Certainly \( |U_i| \geq 2k + 3 \) for all \( i \), \( 1 \leq i \leq h \). Taking all the partitions \( U_i \) will give us a \( k \)-regular spanning subgraphs partitioning of \( G' \).

On the other hand, if \( G' \) has a \( k \)-regular spanning subgraphs partitioning, it must consist of \( h \) partitions, \( U_1, U_2, \ldots, U_h \) where:

1. If \( v^*_j \in U_i \), so do \( v^\prime_j \) and \( K^1_{k-1} \).
2. Since \( |U_i| \geq 2k + 3 \) for all \( i \), \( 1 \leq i \leq h \), and by (1), \( |U_i| = (k + 1)l \) for some \( l \geq 3 \), each \( U_i \) must contain at least \( 3k + 3 \) vertices.

We can now identify the vertices \( v^*_j \), \( v^\prime_j \) and \( K^1_{k-1} \), making the edge connected to \( v^*_j \) as an in-arc, and that connected to \( v^\prime_j \) as an out-arc. Thus we can obtain a partition containing a Hamiltonian circuit in \( G \). By observation (2), this partition has at least three vertices. \( \square \)
3.2. The $k$-regular connected spanning subgraph problem

Define the $k$-regular connected spanning subgraph problem (in short: the $k$-RCS problem) as:

Given an arbitrary graph $G$, does it contain a $k$-regular connected spanning subgraph?

**Theorem 3.2.** The $k$-RCS problem is NP-complete for $k \geq 2$.

**Proof.** We shall use the following notation in proving the theorem:

**Definition 3.3.** The $h$-expansion of a vertex $v$ is constructed as follows:

1. Construct a $K_{h+1}$ and remove $\lceil \frac{h}{2} \rceil - 1$ vertex-disjoint edges.
2. Make $v$ adjacent to those vertices which have had their degree decreased by one.

The $k$-RCS problem for $k = 2$ is just the Hamiltonian circuit problem shown to be NP-complete in [5]. For $k > 2$, we shall prove the NP-completeness by reduction from the Hamiltonian circuit problem.

Given an arbitrary graph $G$, $V(G) = \{v_1, v_2, \ldots, v_n\}$.

1. If $k$ is even, then construct the graph $G'$ by doing a $k$-expansion on every vertex $v_i$ in $G$, $1 \leq i \leq n$.
2. If $k$ is odd, then:
   - First to each $v_i \in V$ add new vertices $u_i^1, u_i^2, \ldots, u_i^{k-2}$, and make $v_i$ adjacent to each such $u_i^j$, $1 \leq j \leq k-2$, $1 \leq i \leq n$.
   - Form the new graph $G'$ by doing a $k$-expansion on each vertex $u_i^j$ for $1 \leq j \leq k-2$, $1 \leq i \leq n$.

Note that in both cases, the vertex $v_i$ has its degree increased by $k - 2$, while all the new vertices have degree $k$. Clearly if $G$ has a Hamiltonian circuit, that Hamiltonian circuit together with all the new vertices and expansions attached will be a $k$-regular connected spanning subgraph in $G'$.

Conversely, assume $G'$ has a $k$-regular connected spanning subgraph, say $H$. Then by taking away all the vertices in $H$ which are not in $G$ (i.e., those new vertices and expansions), we get a 2-regular connected spanning subgraph for $G$, which is a Hamiltonian circuit for $G$. \( \square \)

Since the undirected Hamiltonian circuit problem is NP-complete for graphs with maximum degree 3, it follows that the $k$-RSP problem is NP-complete even for graphs of maximum degree $k + 1$. This gives us the following corollary:

**Corollary 3.4.** The $k$-RCS problem is NP-complete for graphs of maximum degree $\Delta = k + 1$. 
We now examine the \(k\)-RS problem where the spanning subgraph is not required to be connected. The relaxation of the spanning subgraph problem leads to a polynomial time algorithm. Formally the \(k\)-RS problem is defined as:

**Definition 3.5.** The \(k\)-RS problem:

Given an arbitrary graph \(G\), does it contain a \(k\)-regular spanning subgraph?

The following theorem follows from Tutte's famous \(k\)-factor theorem [10].

**Theorem 3.6.** There exists a polynomial time algorithm for deciding the \(k\)-RS problem.

**Proof.** This is a modification of the proof given by Tutte in his \(k\)-factor theorem. Given an arbitrary graph \(G\), \(V(G) = \{v_1, v_2, \ldots, v_n\}\), we only have to consider the case where \(d_i = \deg(v_i) \geq k\) for all \(1 \leq i \leq n\).

Construct another graph \(G'\) as follows:

1. For each vertex \(v_i\), construct a complete bipartite subgraph with the partitions \(X_i\) of size \(d_i - k\) and \(Y_i\) of size \(d_i\). Denote the vertices in \(Y_i\) as \(y_{ij}\), for \(1 \leq j \leq d_i\).
2. Order the vertices in \(\Gamma(v_i) \setminus v_i\) from 1 to \(d_i\). For each edge \((v_i, v_k)\) where \(v_i\) is the \(j\)th vertex in \(\Gamma(v_k) \setminus v_k\) and \(v_k\) is the \(l\)th vertex in \(\Gamma(v_i) \setminus v_i\), add the edge \((y_{ij}, y_{kl})\) to \(G'\).

We will now prove that \(G\) contains a \(k\)-regular spanning subgraph iff \(G'\) is 1-factorable.

1. If \(G\) has a \(k\)-regular spanning subgraph \(H\), we can construct a 1-factor \(F\) for \(G'\) as follows:
   1. Take all the edges in \(G'\) which correspond to those in \(H\).
   2. After step (1), there are exactly \(d_i - k\) vertices in each subset \(Y_i\) which are left unpicked. These \(d_i - k\) vertices are completely connected to the \(d_i - k\) \(X_i\) vertices and the remaining edges of \(F\) can be picked easily.

2. If \(G'\) is 1-factorable with a 1-factor \(F\), we can construct a \(k\)-regular spanning subgraph \(H\) for \(G\) as follows:
   1. Let the subset of edges of \(F'\) be those edges in \(F\) with both end points being \(Y_i\) vertices.
   2. Let \(H\) be the edges in \(G\) corresponding to those edges in \(F'\) of \(G'\).

Since each vertex will have its degree reduced by \(d_i - k\) so that all the \(X_i\) vertices can be matched, the subgraph \(H\) thus constructed will then have to be \(k\)-regular and this completes the proof.

Since 1-factorability can be decided in polynomial time, hence the \(k\)-RS problem can also be solved in polynomial time. \(\square\)
4. Concluding remarks

Although some of the complexity problems related to regular subgraphs are answered in this paper, many others remain open.

One of these open questions is the $k$-$l$ problem: "Given integers $k$ and $l$, decide if an input $k$-regular graph contains an $l$-regular subgraph". The case where $l = k - 1$ has been studied by various people with much success. By the proof of the Berge Conjecture [9], we know that all $k$-regular graphs contain a $(k - 1)$-regular subgraph for $k \leq 4$. For $k \geq 6$, Koh and Saucr [6] showed that there are families of $k$ regular graphs which do not contain a $(k - 1)$-regular subgraph. The $k$-$(k - 1)$ conjecture remains open for $k = 5$. For the case where $k$ varies and $l = 3$, Tashkinov [8] has shown that all $k$-regular graphs contain a 3-regular subgraph by using Petersen’s Theorem [7], Tutte’s $k$-factor theorem [10] and the Berge Conjecture [9]. The problem has not been settled for $l > 3$.

It is known that not all 4-regular multigraphs contain a 3-regular subgraph. An obvious counterexample is an odd cycle with all the edges being multiple edges. However, it is conceivable that only some families of 4-regular multigraphs do not contain a 3-regular subgraph, and they may be recognized in polynomial time. This complexity question still remains open.

The results presented in this paper are reported in Cheah’s master thesis [2]. A detailed study of other extensions of the Berge Conjecture is also given in that thesis.

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