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Convergence analysis of a finite element method based on different moduli in tension and compression

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ABSTRACT

When analyzing materials that exhibit different mechanical behaviors in tension and compression, an iterative approach is required due to material nonlinearities. Because of this iterative strategy, numerical instabilities may occur in the computational procedure. In this paper, we analyze the reason why iterative computation sometimes does not converge. We also present a method to accelerate convergence. This method is the introduction of a new pattern of shear modulus that was strictly derived according to the constitutive model based on the bimodular elasticity theory presented by Ambartsumyan. We test this procedure with a numerical example concerning a plane stress problem. Results obtained from this example show that the proposed method reduces the cost of computation and accelerates the convergence of the solution.

1. Introduction

Classical elasticity theory assumes that materials have the same elastic properties in tension and compression, but this is only a simplified interpretation, and does not account for material nonlinearities. Many studies have indicated that most materials, including concrete, ceramics, graphite, and some composites, exhibit different tensile and compressive strains given the same stress applied in tension or compression. Thus, materials exhibit different elastic moduli in tension and compression. These materials are known as bimodular materials (Jones, 1977). Overall, there are two basic material models widely used in theoretical analysis within the engineering profession. One of these models is the criterion of positive–negative signs in the longitudinal strain of fibers put forward by Bert (1977). This model is mainly applicable to orthotropic materials, and is therefore widely used for research on laminated composites (Bruno et al., 1994; Tseng and Lee, 1995; Tseng and Jiang, 1998; Zinno and Greco, 2001). Another model is the criterion of positive–negative signs of principal stress put forward by Ambartsumyan (1986). This model is mainly applicable to isotropic materials. In civil engineering, the stress state in a principal direction is a key point in the analysis of some components like beams and columns. However, shear stresses and the resulting diagonal tension must also be carefully considered in the design of reinforced concrete. This paper will focus on discussions of the latter model based on principal direction.

The elasticity theory of different tension–compression moduli presented by Ambartsumyan (1986) asserts that Young's modulus depends not only on material properties, but also on the stress state of the point in question. Therefore, the elastic modulus is related to the material, shape, boundary conditions, and external loads of the structure, and hence has nonlinear characteristics. This bimodular theory assumes small deformations and follows the common rules of elastic continuum mechanics. In this model, the differential equations of equilibrium and the geometrical equations are the same as those of classical materials theory, with the exception of the physical equations.

Ambartsumyan (1986) linearized the nonlinear model, the second material model mentioned above, into two straight lines whose tangents at the origin are discontinuous, as shown in Fig. 1. This bimodular theory defines its constitutive model based on principal directions, and therefore inevitably neglects a description of the shear modulus. This model also lacks the ability to describe experimental results of elastic coefficients in complex states of stress. Analytical solutions are available in a few cases, although they only concern beams and columns (Yao and Ye, 2004a,b; He et al., 2007a,b; He et al., 2008). In some complex problems, it is necessary to resort to finite element method (FEM) based on an iterative technique (Zhang and Wang, 1989; Ye et al., 2004; He and Chen, 2005). Because the stress state of the point in question is unknown in advance, we have to begin with a single modulus problem, thus gaining the initial stress state to form a corresponding elasticity

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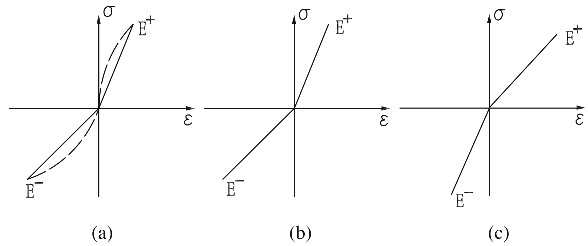

Fig. 1. Constitutive model of bimodular materials: (a) nonlinear model from actual state; (b) bilinear model when $E^+ > E^-$ and (c) bilinear model when $E^+ < E^-$.

matrix for each element. Generally, direct iterative methods based on an incrementally evolving stiffness have been adopted by many researchers. The convergence problem is, however, hard to solve.

Zhang and Wang (1989) presented FEM for solving problems with different moduli, pointed out that the elasticity matrix $[\overline{D}]$ is essentially different from the counterpart in classical theory, and introduced the basic idea of accelerative convergence. Yang et al. (1992) transformed bimodular problems into initial stress problems to allow for their solution. Ye (1997) put forward a new algorithm in which the elasticity matrix is modified while the Poisson ratio is assumed to be constant. Based on the idea of accelerative convergence presented by Zhang and Wang (1989), Gao and Liu (1998) analyzed a bending plate with different tension–compression moduli. Based on the analysis of shear modulus, Liu and Zhang (2000) put forward a shear modulus pattern used for numerical computation. Yang and Zhu (2006) presented a new algorithm based on the smooth function technique. Although the initial stress method (Yang et al., 1992) and smooth function technique (Yang and Zhu, 2006) avoided inconveniences introduced by shear stiffness, the computational effort and the iterative convergence rate depend greatly on the selection of initial values and parameters. Therefore, determination of the shear modulus in numerical computations is still an important problem. However, all current shear modulus patterns are empirical and lack strict theoretical backing. To eliminate the randomness in determining the shear modulus, we derive a new pattern of shear modulus in this paper. Numerical examples indicate that the new pattern is effective for accelerating convergence, and that the symmetric features of elastic structures will change due to the introduction of a bimodulus.

2. Regression of the bimodular elasticity matrix

Studies indicate that bimodular elasticity theory is consistent with classical theory, i.e., formulas and basic equations may return to their counterparts in classical theory when $E^+ = E^-$, $\mu^+ = \mu^-$, where E^+ and E^- are the tensile and compressive elasticity moduli, and μ^+ and μ^- are the tensile and compressive Poisson ratios, respectively.

In bimodular elasticity theory, given a principal stress and a principal strain $\{\sigma_I\}$ and $\{\varepsilon_I\}$, the constitutive relation built on the principal direction may be written as

$$\{\varepsilon_I\} = [a]\{\sigma_I\}, \quad \{\sigma_I\} = [D]\{\varepsilon_I\}, \tag{1}$$

where, $[a]$ is a matrix of flexibility coefficients determined by the signs of the principal stresses, and $[D]$ is the elasticity matrix in the principal direction. These matrices satisfy the following relation

$$[D] = [a]^{-1}. \tag{2}$$

The elasticity matrix $[\overline{D}]$, mapped onto general coordinates via conversion, may be written as

$$[\overline{D}] = [L]^T[D][L], \tag{3}$$

where $[L]$ is the converting matrix and $[L]^T$ is its transpose.

In a plane stress problem, the formulations of $[L]$, $[a]$, and $[D]$ are, respectively,

$$[L] = \begin{bmatrix} l_1^2 & m_1^2 & l_1m_1 \\ l_2^2 & m_2^2 & l_2m_2 \end{bmatrix}, \quad [a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad [D] = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \tag{4}$$

where, l_1, m_1 are directional cosines of one principal direction, α, with respect to the initial coordinates x, y, and l_2, m_2 are the directional cosines of another principal direction, β. According to basic assumptions from bimodular elasticity theory, the elements of $[a]$ and $[D]$ are as follows:

① For $\sigma_\alpha > 0, \sigma_\beta > 0$, the constitutive model in the principal direction is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \end{Bmatrix} = \begin{bmatrix} \frac{1}{E^+} & -\frac{\mu^+}{E^+} \\ -\frac{\mu^+}{E^+} & \frac{1}{E^+} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \end{Bmatrix}. \tag{5}$$

So $a_{11} = a_{22} = \frac{1}{E^+}$, $a_{12} = a_{21} = -\frac{\mu^+}{E^+}$ and $d_{11} = d_{22} = \frac{E^+}{1-(\mu^+)^2}$, $d_{12} = d_{21} = \frac{\mu^+E^+}{1-(\mu^+)^2}$.

② For $\sigma_\alpha < 0, \sigma_\beta < 0$, the constitutive model in the principal direction is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \end{Bmatrix} = \begin{bmatrix} \frac{1}{E^-} & -\frac{\mu^-}{E^-} \\ -\frac{\mu^-}{E^-} & \frac{1}{E^-} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \end{Bmatrix}. \tag{6}$$

So $a_{11} = a_{22} = \frac{1}{E^-}$, $a_{12} = a_{21} = -\frac{\mu^-}{E^-}$ and $d_{11} = d_{22} = \frac{E^-}{1-(\mu^-)^2}$, $d_{12} = d_{21} = \frac{\mu^-E^-}{1-(\mu^-)^2}$.

③ For $\sigma_\alpha > 0, \sigma_\beta < 0$, the constitutive model in the principal direction is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \end{Bmatrix} = \begin{bmatrix} \frac{1}{E^+} & -\frac{\mu^-}{E^-} \\ -\frac{\mu^+}{E^+} & \frac{1}{E^-} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \end{Bmatrix}. \tag{7}$$

So $a_{11} = \frac{1}{E^+}$, $a_{22} = \frac{1}{E^-}$, $a_{12} = -\frac{\mu^-}{E^-}$, $a_{21} = -\frac{\mu^+}{E^+}$ and $d_{11} = \frac{E^+}{1-\mu^+\mu^-}$, $d_{12} = \frac{\mu^+E^-}{1-\mu^+\mu^-}$, $d_{21} = \frac{\mu^-E^+}{1-\mu^+\mu^-}$, $d_{22} = \frac{E^-}{1-\mu^+\mu^-}$.

If the point in question is in case ① or ②, the problem is the same as classical theory. If the point in question is in case ③, a new characteristic concerning different moduli in tension and compression is inevitably introduced here. We will therefore focus our discussion on this latter case.

Under the assumption that $\mu^+/E^+ = \mu^-/E^-$ (Ambartsumyan, 1986), substituting Eq. (4) into Eqs. (2) and (3), we obtain a symmetrical matrix $[\overline{D}]$. While $E^+ = E^- = E$, and $\mu^+ = \mu^- = \mu$, we have $d_{11} = d_{22} = E/(1-\mu^2)$ and $d_{12} = d_{21} = \mu E/(1-\mu^2)$. Hence, the regressive elasticity matrix, $[\overline{D}]'$, is

$$[\overline{D}]' = \frac{E}{1-\mu^2}\begin{bmatrix} \overline{D}'_{11} & \overline{D}'_{12} & \overline{D}'_{13} \\ \overline{D}'_{21} & \overline{D}'_{22} & \overline{D}'_{23} \\ \overline{D}'_{31} & \overline{D}'_{32} & \overline{D}'_{33} \end{bmatrix}, \tag{8}$$

where,

$$\begin{cases} \overline{D}'_{11} = l_1^4 + 2l_1^2l_2^2\mu + l_2^4 \\ \overline{D}'_{12} = l_1^2m_1^2 + (l_2^2m_1^2 + l_1^2m_2^2)\mu + l_2^2m_2^2 = \overline{D}'_{21} \\ \overline{D}'_{13} = l_1^3m_1 + (l_1l_2^2m_1 + l_1^2l_2m_2)\mu + l_2^3m_2 = \overline{D}'_{31} \\ \overline{D}'_{22} = m_1^4 + 2m_1^2m_2^2\mu + m_2^4 \\ \overline{D}'_{23} = l_1m_1^3 + (l_1m_1m_2^2 + l_2m_1^2m_2)\mu + l_2m_2^3 = \overline{D}'_{32} \\ \overline{D}'_{33} = l_1^2m_1^2 + 2l_1l_2m_1m_2\mu + l_2^2m_2^2 \end{cases} \tag{9}$$

Based on the consistency of the two elasticity theories, when $E^+ = E^- = E$ and $\mu^+ = \mu^- = \mu$, Eq. (8) should return to its counterpart from classical theory, i.e., it should return to the familiar matrix

$$[\overline{D}]' = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix}. \quad (10)$$

However, Eq. (8) fails this test.

This unsatisfied regression shows that it is not preferable to directly derive the elasticity matrix from the initial model. With regard to the FEM process, this may be the fundamental reason why convergence is always slow.

3. Basic ideas of accelerative convergence

To satisfy the regression and accelerate convergence, we can let the matrices have an integral feature. Shear stress and shear strain are set equal to zero to formulate physical Eq. (1), so the principal stress and the principal strain (in a plane problem) may be written as

$$\{\sigma_I\} = [\sigma_\alpha \ \sigma_\beta \ \tau_{\alpha\beta}]^T, \quad \{\varepsilon_I\} = [\varepsilon_\alpha \ \varepsilon_\beta \ \varepsilon_{\alpha\beta}]^T. \quad (11)$$

In Eq. (11), $\tau_{\alpha\beta} = \varepsilon_{\alpha\beta} = 0$. The relation between stress and strain in the principal direction and the corresponding elasticity matrix are, respectively,

$$\{\sigma_I\} = [D]\{\varepsilon_I\}, \quad [D] = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}. \quad (12)$$

The strain energy is formulated as

$$U = \frac{1}{2}\{\varepsilon_I\}^T[D]\{\varepsilon_I\}. \quad (13)$$

The corresponding converting matrix $[L]$ should be written as

$$[L] = \begin{bmatrix} l_1^2 & m_1^2 & l_1m_1 \\ l_2^2 & m_2^2 & l_2m_2 \\ 2l_1l_2 & 2m_1m_2 & l_1m_2 + l_2m_1 \end{bmatrix}. \quad (14)$$

The elements of the elasticity matrix on general coordinates are computed as follows

$$\begin{cases} \overline{D}_{11} = l_1^4 d_{11} + 2l_1^2 l_2^2 d_{12} + l_2^4 d_{22} + 4l_1^2 l_2^2 d_{33} \\ \overline{D}_{12} = l_1^2 m_1^2 d_{11} + (l_2^2 m_1^2 + l_1^2 m_2^2) d_{12} + l_2^2 m_2^2 d_{22} \\ \qquad + 4l_1 l_2 m_1 m_2 d_{33} = \overline{D}_{21} \\ \overline{D}_{13} = l_1^3 m_1 d_{11} + (l_1 l_2^2 m_1 + l_1^2 l_2 m_2) d_{12} + l_2^3 m_2 d_{22} \\ \qquad + 2l_1 l_2 (l_1 m_2 + l_2 m_1) d_{33} = \overline{D}_{31} \\ \overline{D}_{22} = m_1^4 d_{11} + 2m_1^2 m_2^2 d_{12} + m_2^4 d_{22} + 4m_1^2 m_2^2 d_{33} \\ \overline{D}_{23} = l_1 m_1^3 d_{11} + (l_1 m_1 m_2^2 + l_2 m_1^2 m_2) d_{12} + l_2 m_2^3 d_{22} \\ \qquad + 2m_1 m_2 (l_1 m_2 + l_2 m_1) d_{33} = \overline{D}_{32} \\ \overline{D}_{33} = l_1^2 m_1^2 d_{11} + 2l_1 l_2 m_1 m_2 d_{12} + l_2^2 m_2^2 d_{22} + (l_1 m_2 + l_2 m_1)^2 d_{33} \end{cases} . \quad (15)$$

The elasticity matrix is also a symmetrical matrix under $\mu^+/E^+ = \mu^-/E^-$. The elements in Eq. (15) are supplemented by item d_{33}, which is essentially the shear modulus. For $E^+ = E^- = E$, $\mu^+ = \mu^- = \mu$, we have

$$d_{11} = d_{22} = \frac{E}{1-\mu^2}, \ d_{12} = \frac{\mu E}{1-\mu^2}, \ d_{33} = \frac{E}{2(1+\mu)}. \quad (16)$$

The directional cosines l_1, m_1, l_2, m_2 satisfy the following relations

$$\begin{aligned} &l_1^2 + l_2^2 = 1, \ m_1^2 + m_2^2 = 1, \ l_1^2 + m_1^2 = 1, \ l_2^2 + m_2^2 = 1, \\ &l_1 l_2 + m_1 m_2 = 0, \ l_1 m_1 + l_2 m_2 = 0, \ l_1 m_2 - l_2 m_1 = 1. \end{aligned} \quad (17)$$

Substituting Eqs. (16) and (17) into (15), we find that the elements of the elasticity matrix return to Eq. (10). This fact indicates that, in the constitutive model defined in the principal direction, the regression of the elasticity matrix is easily satisfied once we consider influences introduced by the shear modulus.

4. Constitutional conditions of the shear modulus pattern

In bimodular elasticity theory, the constitutive relation of each point depends on the stress state of that point. The shear modulus, G, of each point is a nonlinear function involving principal stresses σ_α, σ_β, σ_γ, the tensile and compressive moduli of elasticity E^+, E^-, and the tensile and compressive Poisson ratios μ^+, μ^-. The shear modulus may, therefore, be written as

$$G = f(\sigma_\alpha, \sigma_\beta, \sigma_\gamma, E^+, E^-, \mu^+, \mu^-). \quad (18)$$

When the point in question belongs to the first class discussed above, all principal stresses are uniformly positive $(G^+ = E^+/2(1+\mu^+))$ or uniformly negative $(G^- = E^-/2(1+\mu^-))$. The two shear moduli are essentially the same as those of classical theory, and they are thus easily obtained. However, when the point in question belongs to the second class discussed above, the signs of the three principal stresses are different. Because of this, we will inevitably encounter difficulties when determining the shear modulus pattern. In past computations, G has been taken as an average over the tensile–compressive elasticity moduli and the tensile–compressive Poisson ratios, i.e.,

$$\begin{aligned} G &= \frac{(E^+ + E^-)/2}{2[1 + (\mu^+ + \mu^-)/2]} = \frac{E^+ + E^-}{2(2 + \mu^+ + \mu^-)} \\ &= \frac{E^+ + E^-}{2(1+\mu^+) + 2(1+\mu^-)}. \end{aligned} \quad (19)$$

Eq. (19) neglects the influences brought about by the stress state of the point in question, so it is not an optimal solution. Based on the idea that G should be weighted according to the ratio of tensile or compressive principal stresses to the sum of all principal stresses in absolute value, the following pattern was proposed (Liu and Zhang, 2000)

$$G_{xy} = \frac{\eta E^+ + (1-\eta)E^-}{2\eta(1+\mu^+) + 2(1-\eta)(1+\mu^-)}, \quad (20)$$

where, η is a factor for accelerating convergence, and its value is the ratio of positive principal stress to the sum of the three principal stresses in absolute value, such that $0 \leqslant \eta \leqslant 1$. There are four cases, as follows:

Case 1: while $\sigma_\alpha > 0, \sigma_\beta > 0, \sigma_\gamma < 0$, $\eta = \frac{\sigma_\alpha + \sigma_\beta}{\sigma_\alpha + \sigma_\beta + |\sigma_\gamma|}$;
Case 2: while $\sigma_\alpha > 0, \sigma_\beta < 0, \sigma_\gamma < 0, \eta = \frac{\sigma_\alpha}{\sigma_\alpha + |\sigma_\beta| + |\sigma_\gamma|}$;
Case 3: while $\sigma_\alpha > 0, \sigma_\beta > 0, \sigma_\gamma > 0, \eta = 1$;
Case 4: while $\sigma_\alpha < 0, \sigma_\beta < 0, \sigma_\gamma < 0, \eta = 0$.

By multiplying the items E^+ and $2(1+\mu^+)$ by η, and multiplying the items E^- and $2(1+\mu^-)$ by $1-\eta$, we easily obtain Eq. (20) from Eq. (19). Consequently, a strict derivation in theory is necessary in order to eliminate the need for *a priori* assumptions.

5. Theoretical derivation of shear modulus patterns

In a spatial problem, let the stress and strain components in general coordinates x, y, z be, respectively,

$$\{\sigma\} = (\sigma_x \ \sigma_y \ \sigma_z \ \tau_{yz} \ \tau_{zx} \ \tau_{xy})^T, \quad (21)$$

and

$$\{\varepsilon\} = (\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \varepsilon_{yz} \ \varepsilon_{zx} \ \varepsilon_{xy})^T. \quad (22)$$

Let the stress and strain components in the principal coordinates α, β, γ be, respectively,

$$\{\sigma_I\} = (\sigma_\alpha\ \sigma_\beta\ \sigma_\gamma)^T, \tag{23}$$

and

$$\{\varepsilon_I\} = (\varepsilon_\alpha\ \varepsilon_\beta\ \varepsilon_\gamma)^T. \tag{24}$$

The constitutive model of bimodular materials presented by Ambartsumyan is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix}, \tag{25}$$

where, $a_{ij}(i, j = 1,2,3)$ denotes the flexibility coefficients determined by the polarity of the signs of the principal stress. For instance, if $\sigma_\alpha > 0$, $\sigma_\beta < 0$, $\sigma_\gamma > 0$, the physical equation should be

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} \frac{1}{E^+} & -\frac{\mu^-}{E^-} & -\frac{\mu^+}{E^+} \\ -\frac{\mu^+}{E^+} & \frac{1}{E^-} & -\frac{\mu^+}{E^+} \\ -\frac{\mu^+}{E^+} & -\frac{\mu^-}{E^-} & \frac{1}{E^+} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix}. \tag{26}$$

The rest of the physical equations may be deduced analogously. Due to the fact that $\mu^+/E^+ = \mu^-/E^-$, the symmetry of the flexibility matrix is assured. Therefore, $a_{12} = a_{21} = a_{13} = a_{31} = a_{23} = a_{32}$. Eq. (25) may be rewritten as

$$\begin{cases} \varepsilon_\alpha = a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{12}\sigma_\gamma \\ \varepsilon_\beta = a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{12}\sigma_\gamma\,, \\ \varepsilon_\gamma = a_{12}\sigma_\alpha + a_{12}\sigma_\beta + a_{33}\sigma_\gamma \end{cases} \tag{27}$$

where only four flexibility coefficients, a_{11}, a_{22}, a_{33}, a_{12}, are independent. The directional cosines relating the principal coordinates, α, β, γ, to the general coordinates x, y, z are shown in Table 1. (Note that the definitions of the directional cosines here are different from those in Sections 2 and 3.)

The three shear stresses in general coordinates may be formulated using principal stresses and directional cosines as

$$\begin{cases} \tau_{yz} = l_2 l_3\sigma_\alpha + m_2 m_3\sigma_\beta + n_2 n_3\sigma_\gamma \\ \tau_{zx} = l_1 l_3\sigma_\alpha + m_1 m_3\sigma_\beta + n_1 n_3\sigma_\gamma\,. \\ \tau_{xy} = l_1 l_2\sigma_\alpha + m_1 m_2\sigma_\beta + n_1 n_2\sigma_\gamma \end{cases} \tag{28}$$

Similarly, the shear strains may be formulated in terms of principal strains and directional cosines as

$$\begin{cases} \varepsilon_{yz} = 2(l_2 l_3\varepsilon_\alpha + m_2 m_3\varepsilon_\beta + n_2 n_3\varepsilon_\gamma) \\ \varepsilon_{zx} = 2(l_1 l_3\varepsilon_\alpha + m_1 m_3\varepsilon_\beta + n_1 n_3\varepsilon_\gamma)\,. \\ \varepsilon_{xy} = 2(l_1 l_2\varepsilon_\alpha + m_1 m_2\varepsilon_\beta + n_1 n_2\varepsilon_\gamma) \end{cases} \tag{29}$$

Now, let us derive the shear modulus. Substituting Eq. (27) into the first formula in Eq. (29), we have the following computational process

Table 1
Direction cosines (in Section 6).

Coordinates	α	β	γ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

$$\begin{aligned} \varepsilon_{yz} &= 2l_2l_3(a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{12}\sigma_\gamma) + 2m_2m_3(a_{22}\sigma_\beta + a_{12}\sigma_\alpha + a_{12}\sigma_\gamma) \\ &\quad + 2n_2n_3(a_{33}\sigma_\gamma + a_{12}\sigma_\beta + a_{12}\sigma_\alpha) \\ &= 2a_{11}(l_2l_3\sigma_\alpha + m_2m_3\sigma_\beta + n_2n_3\sigma_\gamma) + 2(a_{22} - a_{11})m_2m_3\sigma_\beta \\ &\quad + 2(a_{33} - a_{11})n_2n_3\sigma_\gamma + 2a_{12}\left[(l_2l_3 + m_2m_3 + n_2n_3)(\sigma_\alpha + \sigma_\beta + \sigma_\gamma)\right. \\ &\quad \left. -(l_2l_3\sigma_\alpha + m_2m_3\sigma_\beta + n_2n_3\sigma_\gamma)\right] \end{aligned} \tag{30}$$

Considering $l_2l_3 + m_2m_3 + n_2n_3 = 0$ and the first expression in Eq. (28), we have

$$\varepsilon_{yz} = 2(a_{11} - a_{12})\tau_{yz} + 2(a_{22} - a_{11})m_2m_3\sigma_\beta + 2(a_{33} - a_{11})n_2n_3\sigma_\gamma. \tag{31}$$

Similarly, ε_{zx} and ε_{xy} may be derived by analogy to give

$$\begin{cases} \varepsilon_{yz} = 2(a_{11} - a_{12})\tau_{yz} + 2(a_{22} - a_{11})m_2m_3\sigma_\beta + 2(a_{33} - a_{11})n_2n_3\sigma_\gamma \\ \varepsilon_{zx} = 2(a_{11} - a_{12})\tau_{zx} + 2(a_{22} - a_{11})m_1m_3\sigma_\beta + 2(a_{33} - a_{11})n_2n_3\sigma_\gamma\,. \\ \varepsilon_{xy} = 2(a_{11} - a_{12})\tau_{xy} + 2(a_{22} - a_{11})m_1m_2\sigma_\beta + 2(a_{33} - a_{11})n_2n_3\sigma_\gamma \end{cases} \tag{32}$$

In the state of general stress, the sign of a certain principal stress can be different from the sign of the other two principal stresses. In three-dimensional spatial problems, Ambartsumyan (1986) assumes that the sign of the principal stress σ_β is different from the signs of σ_α and σ_γ, i.e., if $\sigma_\beta > 0$ then $\sigma_\alpha < 0, \sigma_\gamma < 0$, alternatively, if $\sigma_\beta < 0$ then $\sigma_\alpha > 0, \sigma_\gamma > 0$. In this case, $a_{33} - a_{11} = 0$, so Eq. (32) may be simplified as

$$\begin{cases} \varepsilon_{yz} = 2(a_{11} - a_{12})\tau_{yz} + 2(a_{22} - a_{11})m_2m_3\sigma_\beta \\ \varepsilon_{zx} = 2(a_{11} - a_{12})\tau_{zx} + 2(a_{22} - a_{11})m_1m_3\sigma_\beta\,. \\ \varepsilon_{xy} = 2(a_{11} - a_{12})\tau_{xy} + 2(a_{22} - a_{11})m_1m_2\sigma_\beta \end{cases} \tag{33}$$

Substituting Eq. (28) into (33), we obtain

$$\begin{cases} \varepsilon_{yz} = 2\left[(a_{11} - a_{12})l_2l_3\sigma_\alpha + (a_{11} - a_{12})n_2n_3\sigma_\gamma + (a_{22} - a_{12})m_2m_3\sigma_\beta\right] \\ \varepsilon_{zx} = 2\left[(a_{11} - a_{12})l_1l_3\sigma_\alpha + (a_{11} - a_{12})n_1n_3\sigma_\gamma + (a_{22} - a_{12})m_1m_3\sigma_\beta\right]\,. \\ \varepsilon_{xy} = 2\left[(a_{11} - a_{12})l_1l_2\sigma_\alpha + (a_{11} - a_{12})n_1n_2\sigma_\gamma + (a_{22} - a_{12})m_1m_2\sigma_\beta\right] \end{cases} \tag{34}$$

The right ends of Eq. (34) are multiplied by the corresponding shear stress components τ_{yz}, τ_{zx}, τ_{xy} and then divided by the same components as in Eq. (28). After these operations, we obtain

$$\begin{cases} \varepsilon_{yz} = \frac{2[(a_{11}-a_{12})l_2l_3\sigma_\alpha + (a_{11}-a_{12})n_2n_3\sigma_\gamma + (a_{22}-a_{12})m_2m_3\sigma_\beta]}{l_2l_3\sigma_\alpha + m_2m_3\sigma_\beta + n_2n_3\sigma_\gamma}\tau_{yz} \\ \varepsilon_{zx} = \frac{2[(a_{11}-a_{12})l_1l_3\sigma_\alpha + (a_{11}-a_{12})n_1n_3\sigma_\gamma + (a_{22}-a_{12})m_1m_3\sigma_\beta]}{l_1l_3\sigma_\alpha + m_1m_3\sigma_\beta + n_1n_3\sigma_\gamma}\tau_{zx}\,. \\ \varepsilon_{xy} = \frac{2[(a_{11}-a_{12})l_1l_2\sigma_\alpha + (a_{11}-a_{12})n_1n_2\sigma_\gamma + (a_{22}-a_{12})m_1m_2\sigma_\beta]}{l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma}\tau_{xy} \end{cases} \tag{35}$$

If $\varepsilon_{ik} = G_{ik}^{-1}\tau_{ik}$, the shear modulus G_{ik} should depend mainly on the state of the principal stress. The resulting formulas in the planes *yoz*, *zox*, *xoy* are as follows.

$$\begin{cases} G_{yz} = \frac{l_2l_3\sigma_\alpha + m_2m_3\sigma_\beta + n_2n_3\sigma_\gamma}{2\left[(a_{11}-a_{12})l_2l_3\sigma_\alpha + (a_{11}-a_{12})n_2n_3\sigma_\gamma + (a_{22}-a_{12})m_2m_3\sigma_\beta\right]} \\ G_{zx} = \frac{l_1l_3\sigma_\alpha + m_1m_3\sigma_\beta + n_1n_3\sigma_\gamma}{2\left[(a_{11}-a_{12})l_1l_3\sigma_\alpha + (a_{11}-a_{12})n_1n_3\sigma_\gamma + (a_{22}-a_{12})m_1m_3\sigma_\beta\right]}\,. \\ G_{xy} = \frac{l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma}{2\left[(a_{11}-a_{12})l_1l_2\sigma_\alpha + (a_{11}-a_{12})n_1n_2\sigma_\gamma + (a_{22}-a_{12})m_1m_2\sigma_\beta\right]} \end{cases} \tag{36}$$

In a state of plane stress, $\sigma_\gamma = \sigma_z = \tau_{xz} = \tau_{yz} = 0$. Only stresses and strains in plane *xoy* are non-zero when the principal direction γ is coincident with axis z. Obviously, the directional cosines should satisfy the following

$$\begin{cases} n_1 = n_2 = l_3 = m_3 = 0, n_3 = 1, \\ l_1^2 + m_1^2 = 1, l_2^2 + m_2^2 = 1, l_1m_1 + l_2m_2 = 0, \\ l_1^2 + l_2^2 = 1, m_1^2 + m_2^2 = 1, l_1l_2 + m_1m_2 = 0. \end{cases} \tag{37}$$

Therefore, from the third formula in Eq. (35), we obtain

$$\varepsilon_{xy} = 2\left[(a_{11} - a_{12})\frac{\sigma_\alpha}{\sigma_\alpha - \sigma_\beta} - (a_{22} - a_{12})\frac{\sigma_\beta}{\sigma_\alpha - \sigma_\beta}\right]\tau_{xy}. \tag{38}$$

The shear modulus G_{xy} in the plane xoy is

$$\frac{1}{G_{xy}} = 2\left[(a_{11} - a_{12})\frac{\sigma_\alpha}{\sigma_\alpha - \sigma_\beta} - (a_{22} - a_{12})\frac{\sigma_\beta}{\sigma_\alpha - \sigma_\beta}\right]. \tag{39}$$

In order to determine the flexibility coefficients, we may assume $\sigma_\alpha > 0$ and $\sigma_\beta < 0$ or $\sigma_\alpha < 0$ and $\sigma_\beta > 0$. If we firstly assume $\sigma_\alpha > 0$ and $\sigma_\beta < 0$, the flexibility coefficients may be determined uniquely as follows

$$a_{11} = \frac{1}{E^+}, \quad a_{22} = \frac{1}{E^-}, \quad a_{12} = -\frac{\mu^+}{E^+} = -\frac{\mu^-}{E^-}. \tag{40}$$

Substituting Eq. (40) into (39), we have

$$G_{xy} = \frac{E^+E^-(\sigma_\alpha - \sigma_\beta)}{2(1+\mu^+)E^-\sigma_\alpha - 2(1+\mu^-)E^+\sigma_\beta}. \tag{41}$$

Due to the fact that $\sigma_\alpha > 0$, $\sigma_\beta < 0$, σ_α, σ_β may be rewritten as $\sigma_\alpha = |\sigma_\alpha|$ and $\sigma_\beta = -|\sigma_\beta|$. Substituting the formulas above into Eq. (41) and dividing the numerator and denominator by $|\sigma_\alpha| + |\sigma_\beta|$, we obtain

$$G_{xy} = \frac{\frac{|\sigma_\alpha|}{|\sigma_\alpha|+|\sigma_\beta|}E^+E^- + \frac{|\sigma_\beta|}{|\sigma_\alpha|+|\sigma_\beta|}E^+E^-}{2\frac{|\sigma_\alpha|}{|\sigma_\alpha|+|\sigma_\beta|}(1+\mu^+)E^- + 2\frac{|\sigma_\beta|}{|\sigma_\alpha|+|\sigma_\beta|}(1+\mu^-)E^+}. \tag{42}$$

Introducing the factor η, we have

$$\frac{|\sigma_\alpha|}{|\sigma_\alpha| + |\sigma_\beta|} = \eta, \quad \frac{|\sigma_\beta|}{|\sigma_\alpha| + |\sigma_\beta|} = 1 - \eta. \tag{43}$$

Therefore, Eq. (42) may be rewritten as

$$G_{xy} = \frac{\eta E^+E^- + (1-\eta)E^+E^-}{2\eta(1+\mu^+)E^- + 2(1-\eta)(1+\mu^-)E^+}. \tag{44}$$

The numerator and denominator of Eq. (44) are both divided by E^+E^-, giving

$$G_{xy} = \frac{1}{\eta\frac{2(1+\mu^+)}{E^+} + (1-\eta)\frac{2(1+\mu^-)}{E^-}} = \frac{1}{\frac{\eta}{G^+} + \frac{1-\eta}{G^-}}, \tag{45}$$

where, $G^+ = E^+/2(1+\mu^+)$ and $G^- = E^-/2(1+\mu^-)$ may be interpreted as the tensile and compressive shear moduli, respectively. Note that if we assume $\sigma_\alpha < 0$ and $\sigma_\beta > 0$, the same results can also be obtained by repeating the derivative steps from Eq. (40) to (45). Eq. (44) or (45) is a new pattern of shear modulus, and is strictly derived from the bimodular material model.

Eqs. (44) and (45) satisfy the following two conditions: ① The shear modulus depends on not only the magnitude and direction of the resulting principal stresses, but also on their signs. This condition satisfies the fundamental assumption that the constitutive relation is determined by the stress state of the point in question; ② The shear modulus can satisfy the regression characteristic while $E^+ = E^-$, and can satisfy the requirement of consistency between the bimodular theory and classical theory. Therefore, Eqs. (44) and (45) eliminate the use of *a priori* assumptions. Compared to Eq. (20), Eq. (45) is concise, as it may be expressed using only η and G^+,G^-.

When $\eta = 0$ or $\eta = 1$, Eqs. (45) and (20) are the same, but, when $0 < \eta < 1$, they are different. This difference is especially pronounced when the point in question is in a state of pure shear, $\sigma_\alpha = \tau, \sigma_\beta = -\tau$ and $\eta = 0.5$. From Eq. (45), we obtain

$$G_{xy} = \frac{E^+E^-}{(1+\mu^+)E^- + (1+\mu^-)E^+} = \frac{2G^+G^-}{G^+ + G^-}, \tag{46}$$

while from Eq. (20), we obtain

$$G_{xy} = \frac{E^+ + E^-}{2(1+\mu^+) + 2(1+\mu^-)} = \frac{G^+}{1 + \frac{1+\mu^-}{1+\mu^+}} + \frac{G^-}{1 + \frac{1+\mu^+}{1+\mu^-}}. \tag{47}$$

Obviously, Eqs. (46) and (47) are different.

6. Numerical examples

6.1. Comparisons of convergence

In this paper, we adopted a computational example (Zhang and Wang, 1989) concerning a bimodular plane stress problem to demonstrate the efficiency of our new shear modulus pattern at accelerating convergence. This problem is illustrated in Fig. 2. The object studied is made of bimodular materials, for example, organic glass and its top is subjected to a horizontal force, $P = 10$KN. The values of the mechanical parameters are $E^+ = 2.2$ GPa, $E^- = 3.22$ GPa, $\mu^+ = 0.22$, $\mu^- = 0.322$, respectively. For convenience of analysis and comparison, the elastic problem with different moduli is called problem *D*, while the classical problem with a single modulus is called problem *S*. In problem *S*, $E = (E^+ + E^-)/2 = 2.71$ GPa and $\mu = 0.22$. A FEM program based on different moduli in tension and compression was worked out to compute problem *D*. To examine the effect on accelerating convergence, both the pattern presented by Liu and Zhang (2000) and the new pattern derived in this paper were considered. The computational results are listed in Tables 2 and 3, and the iterative numbers are listed in Table 4.

From Tables 2 and 3, we see that the inclusion of a bimodulus greatly influences the results: The sign of the horizontal displacement, u, of node 2 is changed. The vertical displacements, v, of nodes 4 and 5 are changed greatly. The vertical displacements, v, of nodes 2 and 6 are changed from zero to 2.0106 and 3.9223 μm, respectively. The maximum stress change is in the principal stress σ_1 of element ③. The maximum displacement change reaches (8.7208-6.1157)/6.1157=42.6%, which happens in the vertical displacements, v, of node 4. We conclude that the introduction of a bimodulus has a significant influence on the calculated stiffness of an elastic structure.

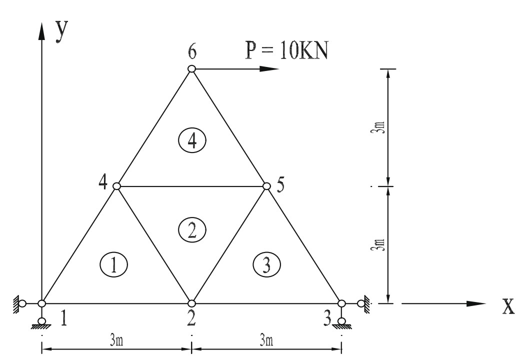

Fig. 2. Elastic plane stress problem with bimodulus (antisymmetry).

Table 2
Nodal displacements u and v (a rightward load).

Nodes	Horizontal displacements u (μm)	Vertical displacements v (μm)
Problem S		
1	0.319949E−19	0.109732E−18
2	0.194151E+00	−0.367601E−06
3	0.319949E−19	−0.109732E−18
4	0.102090E+02	0.611571E+01
5	0.102090E+02	−0.611571E+01
6	0.404478E+02	−0.200846E−06
Problem D		
1	0.281691E−19	0.133413E−18
2	−0.247341E−01	0.201055E+01
3	0.342126E−19	−0.946459E−19
4	0.101218E+02	0.872081E+01
5	0.109312E+02	−0.408509E+01
6	0.425152E+02	0.392227E+01

Table 3
Normal stress σ_x, σ_y, shear stress τ_{xy}, principal stress σ_1, σ_2 and principal direction of σ_1 to axis x (a rightward load).

Elements	σ_x (KPa)	σ_y (KPa)	τ_{xy} (KPa)	σ_1 (KPa)	σ_2 (KPa)	α (rad)
Problem S						
①	0.1462E+01	0.5846E+01	0.3744E+01	0.7992E+01	−0.6845E+00	60.1766
②	0.2698E−07	0.1226E−06	−0.8206E+00	0.8206E+00	−0.8206E+00	−45.0000
③	−0.1462E+01	−0.5846E+01	0.3744E+01	0.6845E+00	−0.7992E+01	29.8234
④	0.7847E−08	0.3567E−07	0.6667E+01	0.6667E+01	−0.6667E+01	45.0000
Problem D						
①	0.9974E+00	0.5799E+01	0.3767E+01	0.7865E+01	−0.1069E+01	61.2554
②	0.6314E+00	0.3153E+00	−0.7098E+00	0.1201E+01	−0.2538E+00	−38.7234
③	−0.1981E+01	−0.6115E+01	0.3609E+01	0.1115E+00	−0.8207E+01	30.1021
④	−0.5526E+00	−0.1711E−06	0.6667E+01	0.6396E+01	−0.6949E+01	46.1867

Table 4
Comparison of iterative numbers.

Optional stop	Iterative numbers	
$\lambda(\mu$m)	Shear modulus pattern in Eq. (20)	Shear modulus pattern in Eq. (45)
1E−1	4	3
5E−2	4	3
1E−2	6	3
5E−3	6	5
1E−3	14	12
9E−4	15	12
8E−4	15	12
7E−4	>100000	12
6E−4	>100000	12
5E−4	>100000	12
4E−4	>100000	>100000

If we take no any methods, the convergence speed is too slow to obtain stable results. The convergent effect will be quite obvious once the elastic matrix is modified. However, the form of the shear modulus introduced here will also directly influence convergence speed. In the iterative computation, the optional stop is defined as $|u_{i+1} - u_i| < \lambda$, i.e., the difference of the continuous two computations is less than a given small value λ (μm). From Table 4, we see that if we take Eq. (20) as the shear modulus, the convergence speed becomes slow at λ = 7E−4 μm. If we take Eq. (45) as the shear modulus, the convergence speed will become slow at λ = 4E−4 μm. These results show that the convergence of shear modulus using the equations derived in this paper may be superior to convergence using conventional equations.

6.2. Symmetry and antisymmetry

It is also interesting to study the symmetry of elastic structures possessing bimodulus. An example of this, as shown in Fig. 3, is easily obtained by simply modifying the direction of load P in Fig. 2. Thus, the original antisymmetric problem is converted into a symmetric one. The computational results of the displacements are listed in Table 5 and the computational results of the stresses are listed in Table 6.

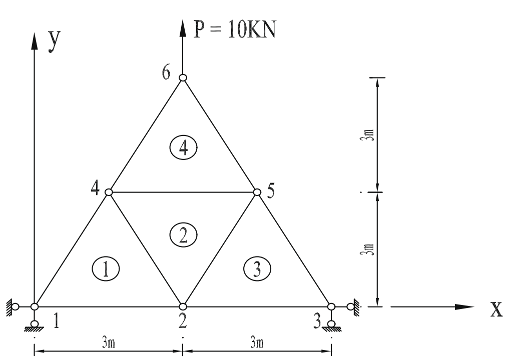

Fig. 3. Elastic plane stress problem with bimodulus (symmetry).

In classical elasticity, if the shape and boundary conditions of an elastic structure are symmetric to an axis, and the applied external loads are also symmetric or antisymmetric to the same axis, the stress and displacement components will satisfy the following relations

① Symmetric problem
On the symmetric axis,

$$u = 0, \tau_{xy} = 0. \tag{48}$$

In areas off the symmetric axis,

$$\begin{cases} u(x,y) = -u(-x,y), v(x,y) = v(-x,y), \\ \sigma_x(x,y) = \sigma_x(-x,y), \sigma_y(x,y) = \sigma_y(-x,y), \\ \tau_{xy}(x,y) = -\tau_{xy}(-x,y). \end{cases} \tag{49}$$

② Antisymmetric problem
On the antisymmetric axis,

$$v = 0, \sigma_x = 0(\sigma_y = 0). \tag{50}$$

In areas off the antisymmetric axis,

$$\begin{cases} u(x,y) = u(-x,y), v(x,y) = -v(-x,y), \\ \sigma_x(x,y) = -\sigma_x(-x,y), \sigma_y(x,y) = -\sigma_y(-x,y), \\ \tau_{xy}(x,y) = \tau_{xy}(-x,y). \end{cases} \tag{51}$$

By comparing the results from Tables 2, 3, 5 and 6, we find that, in problem S, symmetry and antisymmetry hold. In problem D, symmetry of the elastic structure still holds but antisymmetry does not. Consequently, the superposition theorem will fail. This

Table 5
Nodal displacements u and v (an upward load).

Nodes	Horizontal displacements u (μm)	Vertical displacements v (μm)
Problem S		
1	0.151320E−19	0.548662E−19
2	−0.413010E−07	0.500520E+01
3	−0.151320E−19	0.548662E−19
4	−0.607919E−01	0.566884E+01
5	0.607917E−01	0.566884E+01
6	−0.200846E−06	0.126650E+02
Problem D		
1	0.143167E−19	0.667137E−19
2	0.823375E−07	0.605853E+01
3	−0.143167E−19	0.667137E−19
4	−0.755744E−01	0.696329E+01
5	0.755754E−01	0.696329E+01
6	0.111548E−05	0.155809E+02

Table 6
Normal stress σ_x, σ_y, shear stress τ_{xy}, principal stress σ_1, σ_2 and principal direction of σ_1 to axis x (an upward load).

Elements	σ_x (KPa)	σ_y (KPa)	τ_{xy} (KPa)	σ_1 (KPa)	σ_2 (KPa)	α (rad)
Problem S						
①	0.6612E+00	0.3006E+01	0.1831E+01	0.4007E+01	−0.3402E+00	61.3171
②	0.2540E+00	0.6554E+00	−0.1950E−06	0.6554E+00	0.2540E+00	90.0000
③	0.6612E+00	0.3006E+01	−0.1831E+01	0.4007E+01	−0.3402E+00	−61.3171
④	0.1576E+01	0.6667E+01	−0.2171E−06	0.6667E+01	0.1576E+01	90.0000
Problem D						
①	0.5425E+00	0.2972E+01	0.1847E+01	0.3968E+01	−0.4538E+00	61.6632
②	0.2699E+00	0.7229E+00	0.1242E−06	0.7229E+00	0.2699E+00	90.0000
③	0.5425E+00	0.2972E+01	−0.1847E+01	0.3968E+01	−0.4538E+00	−61.6632
④	0.1578E+01	0.6667E+01	0.1863E−06	0.6667E+01	0.1578E+01	90.0000

phenomenon is caused by the introduction of different moduli in tension and compression. Thus, we should devote attention to the analysis of bimodular elastic structures.

7. Conclusions

In this paper, we analyzed the convergence of a finite element method based on different moduli in tension and compression. The new pattern of shear modulus derived in this paper is able to improve the convergence of a finite element computation for a 2-D continuum exhibiting a bimodular behavior. Several important conclusions are summarized in the following.

(1) The introduction of the shear modulus can accelerate convergence speed, and the shear modulus should be determined theoretically in order to eliminate the need for *a priori* assumptions.
(2) The theoretical derivation to shear modulus in this paper may be regarded as a supplement to the bimodular elasticity theory.
(3) From the results of the numerical example in this paper, the convergence of shear modulus using the equations derived in this paper is superior to convergence using conventional equations.

It should be point out that, because this paper is devoted to the theoretical derivation of the shear modulus, no more computational examples are given. The results of accelerating convergence are limited to a single case; therefore, future works will be done in order to check the efficiency of the proposed methodology in other circumstances.

This work will be helpful for predicting the mechanical behaviors of bimodular materials. In particular, these results may be useful to analyze concrete-like materials and fiber-reinforced composite materials that contain cracks and undergoing contact, whose macroscopic constitutive behavior depends on the direction of the macroscopic strain, similarly to the case of the bimodular materials (Leguillon and Sanchez-Palencia, 1982; Bisegna and Luciano, 1998; Greco, 2009).

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