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GERGONNE'S 1815 PAPER ON THE DESIGN AND ANALYSIS OF POLYNOMIAL REGRESSION EXPERIMENTS

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SUMMARIES

Brief mention is made of early work in the design of experiments by Galen, Avicenna, and Laplace, after which a seemingly unknown 1815 paper of Gergonne's on polynomial regression is introduced and discussed, and a translation of the paper presented.

Ce mémoire discute quelques oeuvres par Galen, Avicenne, et Laplace sur le plan d'expériences, et ensuite une mémoire de Gergonne de 1815 sur la régression polynôme est introduite et une traduction en anglais est presentée.

1. *Introduction*

The design of experiments may well be oldest of all the fields of statistics. Examples of planned, controlled experiments considerably predate the first attempts at formal analysis of random data, going back at least to the Old Testament, perhaps further: one such example can be found in the first chapter of the Book of Daniel, where Daniel proved his diet superior to Nebuchadnezzar's, the king's servants acting (appropriately) as controls. [2] For other early examples of decision making, see Rabinovitch [1973] and Sheynin [1973].

Some thoughts on sample size can be found in work of Galen the Physician dating from 150 A.D. [Galen 150, 96-119], where Galen presents a debate between an empiricist and a dogmatist on the relative merits of experience and theory in medical research. An important question in the debate is how one committed to an experimental approach can decide that sufficient evidence is at hand: if forty-nine cases fail to convince, yet fifty are conclusive, has not one been convinced by but a single case?

In the eleventh century, many modern principles of design were spelled out by the famous Arabic doctor, scientist, and philosopher Avicenna in the second volume of his *Canon of Medicine*, the leading medical text for nearly eight centuries. Avicenna listed seven rules for medical experimentation, stressing the need for controls and replication, the danger of confounding effects, the necessity of varying one factor at a time, and the wisdom of observing the effects for many differing factor levels. To quote A.C. Crombie's summary of Avicenna's seven rules [Crombie 1952, 89-90]:

1. *The drug must be free from any extraneous quality; for example, we must not test the effect of water when it is heated, but wait until it has cooled down.*

2. *The experimentation must be done with a simple and not a composite disease, for in the second case it would be impossible to infer from the cure what was the curing cause of the drug.*

3. *The drug must be tested with two contrary types of disease, because sometimes a drug cured one disease by its essential qualities and another by its accidental ones. It could not be inferred, simply from the fact that it cured a certain type of disease, that a drug necessarily had a given quality.*

4. *The quality of the drug must correspond to the strength of the disease. For example, there were some drugs whose "heat" was less than the "coldness" of certain diseases, so that they would have no effect on them. The experiment should therefore be done first with a weaker type of disease, then with diseases of gradually increasing strength.*

5. *The time of action must be observed, so that essence and accident are not confused. For example, heated water might temporarily have a heating effect because of an acquired extraneous accident, but after a time it would return to its cold nature.*

6. *The effect of the drug must be seen to occur constantly or in many cases, for if this did not happen it was an accidental effect.*

7. *The experimentation must be done with the human body, for testing a drug on a lion or a horse might not prove anything about its effect on man.*

With some principles of planned experimentation over 800 years old, it is not surprising that work on the design of *regression* experiments has a long history, and in fact preceded the introduction of the method of least squares. One early example of this can be found in the first volume [1798] of Laplace's *Mécanique céleste* (sections 28-29, chapter 4, book 2), where it is recommended that if random errors are present, a polynomial regression function will be best estimated at a point by spreading the observations in a wide interval about that point, rather than by clustering them closely near that point as one would do if errors were not anticipated. The purpose of this present note is to introduce a paper which enlarges upon Laplace's recommendation, in the context of later work of Legendre.

The paper we present was written by Joseph-Diez Gergonne of the University of Montpellier, editor and founder of the journal *Annales de mathématiques pures et appliquées*, and it appeared in that journal in 1815 under the title "Application de la méthode des moindre quarrés a l'interpolation des suites." It appeared in the midst of a twenty-year period in which it might be claimed that mathematical statistics advanced further than in any similar period in history.

In 1805 the first publication of the method of least squares appeared, with Legendre suggesting the method as a means of solving inconsistent linear equations arising from astronomical observations, but with no explicit mention of any probabalistic considerations [Legendre 1805]. In 1809 Gauss discussed this method, claimed priority of use while acknowledging Legendre's prior publication, and for the first time linked it with the theory of probability by a somewhat circular argument based upon the principle of the arithmetic mean [Gauss 1809, Plackett 1972, Eisenhart 1964]. Least squares gained a firmer theoretical foundation with the appearance of Laplace's *Théorie analytique des probabilités* in 1812, as Laplace proved the first general central limit theorem and showed how it followed from this theorem that least squares yielded the best (minimum expected absolute error) estimates if the number of observations was large [Laplace 1812, Todhunter 1865]. In 1821, Gauss returned to this subject to build upon Laplace's work and prove that least squares estimates had minimum variance among all unbiased estimates, for any number of observations [Eisenhart 1968]. By the time Gergonne's paper appeared, the method of least squares was just becoming widely known and used. It does not, however, appear to have been used for the fitting of polynomials as a means of estimating derivatives prior to Gergonne's paper of 1815.

Gergonne's paper was not a landmark of this era; indeed it seems to have completely escaped the attention of all bibliographers of the statistical literature of that time [3], although Gergonne's journal was widely circulated and was likely to have been read by many European mathematicians. Although the paper itself contains no really startling results, it is an extremely interesting document in the history of statistics, both as one of the earliest attempts to discuss problems of design and analysis that are inspiring so much research today, and for the insight it gives us into the spread and development of statistical thought in the early years of the nineteenth century.

2. *Gergonne's Paper*

The problem Gergonne considered is one we would now describe as follows: given an observational situation in which a response depends upon a single independent variable, and in which one wishes to estimate the value of the response function and its derivatives at a single point, how should one select the values of the independent variable at which the experiment will be performed, when random errors in the observed responses are expected. Gergonne's treatment of this problem is interesting but not profound. He began with a general discussion of the problem of interpolation, viewed both geometrically (in terms of points and curves) and algebraically (in terms of variables and functions). He observed that even with no errors present, the problem is somewhat indeterminate, but that with sufficiently many observations this would not cause serious difficulty, and one could conveniently fit a simple polynomial model to the data.

The first method of fitting he discussed is the one which was most prevalent at the time: fit a polynomial consisting of as many terms as there are data points. Gergonne was aware of the difficulties this method presented when the number of observations was large, but he went on to extend an argument of Legendre's analyzing the effect an error in a single observation would have on the derivatives of the interpolating polynomial, concluding that Laplace's advice was sound: within the class of equally spaced designs, accuracy increases with increasing spread and more distant spacing.

Gergonne then noted that the only way a widely spaced experiment could be achieved over a narrow range would be by discarding (or declining to take) observations, and suggested that a much more sensible procedure would be to use least squares. He developed the normal equations for polynomial regression, discussed the numerical simplification which came with an equally spaced design, and showed how any design may, by the appropriate transformation on the independent variable, be transformed to an equal spacing design to simplify calculations. The paper closes by posing a problem that cannot be said to be well solved today: "we know that a number of points, however many, are located *near* a parabolic curve of unknown fixed degree, and we wish to know the most likely ['plus probablement'] value of the degree of this curve."

In many respects, the paper belongs more to data analysis than statistics. By not explicitly introducing any probability structure, Gergonne was following the example of Legendre rather than that of Gauss or Laplace, thus illustrating that true scientific innovation is often very slow in catching on: the *technique* of least squares was not in principle greatly different from many of the techniques which preceded it, and it was its computational simplicity coupled with the authority of Gauss and Laplace which led to its early widespread adoption. The truly innovative work of Gauss and Laplace, incorporating probability models as a foundation and justification for the adoption of this technique, was not well understood by Gergonne and many others at this time, but spread only very slowly as the work was extended and improved over the following century.

Gergonne's paper was, however, novel in a number of respects. It presents what may be the first explicit application of the principle of least squares to a general polynomial regression model. The next such paper of which I am aware was read in 1823 in St. Petersburg by C.F. Degen, Professor at the University of Copenhagen, and it dealt almost exclusively with the algebra of fitting parabolas by the method of least squares [Degen 1831]. (Degen made no reference to Gergonne.) More significantly, Gergonne's paper is one of the earliest attempts to deal mathematically with a design problem in a regression framework, showing that the planning of experiments was already being considered in mathematical terms in 1815. The paper also describes the use of coding as a device for simplifying computations, and it displays a surprisingly modern feel for the problems of statistical analysis and model fitting, including a realization that polynomial models are ill-suited for extrapolation and an understanding that no single method of analysis gives a uniquely best answer.

It is likely that Gergonne's paper, written in the south of France by an educated man who followed work in all the major intellectual centers of Europe, is closer to the general level of statistical thought in Europe than is the work of giants such as Laplace and Gauss. Although the paper seems not to have been cited in the statistical literature, it would be a mistake to conclude that it must then necessarily have had no influence on the statistical practice of the time. To see how this may be, we turn to Gergonne's journal and its role in the development of applied mathematics.

3. *Gergonne and His Journal*

Joseph-Diez Gergonne (1771-1859) is best known as the co-founder and editor of the *Annales de mathématiques pures et appliquées*. Gergonne founded his *Annales* in 1810, at which time it was the first and only journal devoted entirely to mathematics and its applications. It remained the only such journal until the first appearance of Crelle's journal in 1826.

Gergonne's *Annales* was a remarkably lively and broad journal. By the time he became rector of the University of Montpellier and ceased publication of the journal in 1831, he had published articles on nearly every branch of pure mathematics, and on a wide range of applications including optics, circulation of the blood, sundials, economics, political science, celestial mechanics, gambling, and law. The list of contributers includes some of the foremost mathematicians of the time: Cauchy, Poisson, Ampère, Abel, Poncelet, and Galois. Gergonne himself contributed over 200 papers, a majority in geometry, the field in which Gergonne was most interested and widely recognized. Many of his papers were published anonymously, attributed to "un abboné" ("a subscriber"); these included his only other effort in statistics, a paper on the estimation of means (1821).

Of Gergonne's own work, the only portion which receives recognition in most histories of mathematics is his work in geometry, where he became embroiled in a bitter priority fight with Poncelet over the discovery of reciprocal polars and the principle of duality. In many respects his achievements as editor were greater than those as author; his journal was widely read and had a lasting influence on the development of mathematics far beyond that of the individual articles.

Bibliographical Note

The best accessible treatment of Gergonne's life and work is the article by D.J. Struik in the *Dictionary of Scientific Biography* [Struik 1972] (with references), although Struik has overlooked a number of important sources, including Bouisson [1859] and Henry [1881], making his account incomplete and incorrect in some minor respects, such as the date of Gergonne's death and the spelling of his middle name. An account of Gergonne's life by Guggenbuhl [1959] contains a portrait.

NOTES

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2. This early comparison experiment was brought to my attention by Professor William Kruskal, to whom I am indebted for other comments as well.
3. It is not mentioned by Merriman (1877), Gore (1902), or Kendall and Doig (1968), nor is Gergonne listed in Lancaster (1968). The paper *is* listed in the Royal Society of London's *Catalogue of Scientific Papers 1800-1900, Subject Index* (Vol. I, Pure Mathematics) under "Interpolation." Other than bibliographies of Gergonne's work, I know of no other citation. The paper came to my attention in the course of a systematic inspection of Gergonne's journal, in connection with another study.

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A contemporary portrait, from a photograph in the collection of Prof. G.A. Oravus of McMaster University

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THE APPLICATION OF THE METHOD OF LEAST SQUARES TO THE INTERPOLATION OF SEQUENCES

By J.D. Gergonne

Translated by Ralph St. John, Bowling Green State University and S.M. Stigler, University of Wisconsin

Translators' Note: *An effort has been made not to introduce any modern* statistical *terminology and to reflect Gergonne's thinking accurately. To ease the way for modern readers, however, some of the* mathematical *terminology has been updated (examples: "polynomial function" for "fonction complète, rationelle et entière" and "derivatives" for "coefficiens différentiels." All italics, including those in the quotation from Laplace, are Gergonne's, as are the footnotes unless otherwise indicated. Some readers may be unfamiliar with the osculating circle, a geometric measure*