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Rapid methods for the conformal mapping of multiply connected regions

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Abstract: We present fast methods for the conformal mapping of simply, doubly and multiply connected regions onto certain canonical regions in the plane. Our mapping procedure consists of two parts. First we solve an integral equation on the boundary of the region we wish to map. The solution of this integral equation is needed to determine the boundary correspondence. We have chosen to use the integral equation formulation of Mikhlin. Although it is not widely used, this formulation has the advantage that it leads to integral equations of the second kind with unique solutions and bounded kernels. The solutions are also periodic, allowing for effective use of the trapezoid rule. Once we have solved the integral equation we use a rapid method we have previously developed to determine the mapping function in the interior of the region. This method makes use of fast Poisson solvers, and thereby circumvents the difficulties associated with computing integrals at points near the boundary of the region, and avoids the expense of computing many integrals. We also provide results of numerical experiments.

Keywords: Numerical conformal mapping, multiply connected regions.

Introduction

In this paper we describe an efficient method we have implemented for computing the conformal mapping of simply and multiply connected regions onto certain canonical regions. In particular, we have used the method to map simply connected regions onto discs, to map doubly connected regions onto annuli, and to map regions of higher connectivity onto slit discs.

The mapping procedure consists of two parts. First we solve an integral equation on the boundary of the region. The solution of this integral equation provides an analytic formula for values of the mapping function on the boundary. The integral equation, due to Mikhlin [10], is not new, but has not been widely used. Once we have solved the integral equation we use a rapid method that we have previously developed [9] to find the mapping function in the interior of the region.

The method we use to find the mapping in the interior of the region makes use of fast Poisson solvers, and can be used to evaluate the derivatives of the mapping as well as the mapping to second order accuracy, even near the boundary of the region. It is particularly attractive since the cost of evaluating the mapping is essentially just the cost of applying a fast Poisson solver twice.

We note that the question of finding the mapping in the interior of the region is often neglected. Many methods for conformal mapping are only methods for deriving and solving an integral equation for the boundary correspondence. The problem of finding the mapping elsewhere is thereby reduced to the problem of evaluating certain integrals in the interior of the region. There are, however, several difficulties encountered in computing these integrals directly. First, it is clearly very expensive to evaluate integrals at many points. This is particularly true since the kernels of the integrals are often expensive to evaluate. An even more serious problem is that it is difficult to compute the mapping accurately where it is often needed the most—near the boundary of the region. This is because the kernels of the integrals are always unbounded as the the point at which one computes them nears the boundary.

Our method overcomes these two difficulties. In fact, in our method we only compute the mapping by evaluating any integrals at the edge of a rectangular region in which we embed the the region we wish to map. Furthermore, we can compute the mapping to second order accuracy at all points of the region.

The idea of the method is the following. The conformal mapping is an analytic function. We use the solution of the integral equation to define another analytic function in the rest of the embedding region. The extension is discontinuous, but the discontinuities between the mapping and its extension can be expressed in terms of the solution of the integral equation. Once we know the discontinuities we use them to compute approximations to the discrete Laplacians of the real and imaginary parts of the extended analytic function. Then we use fast Poisson solvers to find the mapping itself.

We note that our method of computing the conformal mapping at points inside the region can be combined with other integral equation formulations of the problem. This is because in many methods the mapping can be expressed in terms of single or double layer density functions, which can be rapidly computed by our method. For example, our method can be combined with Symm's method [2].

We have chosen to use the integral equation formulation of Mikhlin. When the region is simply connected this formulation reduces the problem to solving a Dirichlet problem, while for regions of higher connectivity it reduces the problem to solving a modified Dirichlet problem. A modified Dirichlet problem is an elliptic boundary value problem on a multiply connected region in which the the boundary values of the function are specified only up to an additive constant on all but one of the boundary curves. The constants are determined by the requirement that the conjugate of the solution of the problem be single valued in the region.

This integral equation formulation has several advantages. In particular, all the integral equations that arise are Fredholm integral equations of the second kind with bounded kernels. This is in contrast to the commonly used method of Symm, where the integral equations are integral equations of the first kind [2,6] and the kernels have a logarithmic singularity. We note that integral equations of the second kind do not suffer from the problem of being ill-conditioned, and reliable error estimates are available [1]. Of course, when we use an integral equation of the second kind, the singularity of the kernel at points near but not on the boundary is worse. This is a major reason why engineers often prefer to formulate problems using integral equations of the first kind. However, since we do not compute the mapping in the interior by evaluating any integrals this singularity problem is of no importance. This observation is central for our solution technique, and is what makes the method work so well.

In addition, the solution of the integral equations is always unique. This is not the case, for example, in the method of Gershgorin [4]. Moreover, in his integral equation formulation and others the solution of the integral equation is not periodic. In Mikhlin's formulation it is. This allows for effective use of the trapezoid rule, which is highly accurate on smooth closed contours. Also, the kernels are inexpensive to evaluate. From the solution of the integral equation it is possible to determine constants such as the radius of the inner boundary when we map onto an annulus, and the location of the slits when we map onto the slit disc. We also note that a similar integral equation formulation can be used to map exterior as well as interior regions. Finally, we mention that since we map the given region onto the canonical region, the mapping problem is linear. Other recently developed methods [3,5,13] provide the mapping onto the given region, and therefore require the solution of a nonlinear system of equations.

We have chosen to solve the integral equation by a direct method. We note, however, that it may be more efficient to solve it iteratively. For example, we note that Young et al. [15] have solved similar systems using a conjugate gradient method applied to the normal equations.

In the first section of this paper we present the mathematical formulation of the problem, in the second and third we present the integral equation formulation and method used to solve the integral equations, and in the fourth we give the method used to evaluate the mapping in the interior of the region. In the last section we present results of numerical experiments.

1. Mathematical formulation

1.1. Simply connected regions

Assume we have a simply connected region D with smooth boundary L, and we wish to map it onto the unit disc $|w| < 1$. Let $z = \alpha$ be the point that gets mapped onto the center of the disc, $w = 0$. Following Mikhlin [10], the mapping function $w(z)$ can be written in the form

$$w(z) = (z - \alpha)g(z), \tag{1.1}$$

where $g(z)$ is analytic and nonzero in D. It follows that the function

$$\phi(z) = \log g(z)$$

is also analytic in D. We can determine the boundary values of the real part of ϕ by noting that if t is a point on the boundary of D, then

$$|w(t)| = |t - \alpha|\,|g(t)| = 1,$$

and so

$$\operatorname{Re} \phi(t) = \log|g(t)| = -\log|t - \alpha|.$$

Therefore, in order to find ϕ we can first solve Laplace's equation with Dirichlet boundary values $-\log|t - \alpha|$ to determine $\operatorname{Re} \phi$, and then determine the conjugate harmonic function. Having done this we use that fact that

$$g(z) = e^{\phi(z)}$$

to determine $g(z)$, and thereby $w(z)$. We note that this formulation is the same as the one used in Symm's method [2].

1.2. Doubly connected regions

We now consider the problem of mapping a doubly connected region D in the z plane with smooth boundary onto an annulus $R_1 < |w| < 1$. Suppose D is bounded on the outside by the curve L_0, and on the inside by the curve L_1. We assume that L_0 is mapped onto $|w| = 1$, and L_1 is mapped onto $|w| = R_1$.

For simplicity we also assume that the origin of the coordinate system in the z plane is inside the region bounded by L_1. Since the mapping function $w(z)$ is bounded and nonzero in D, the function $\log w(z)$ is nonsingular in D. Define

$$\phi(z) = \log w(z)/z = \log w(z) - \log z.$$

Upon traversing L_0 or any curve homotopic to it in D in the counterclockwise direction, both Arg $w(z)$ and Arg z increase by 2π. It follows that $\phi(z)$ is single valued and regular in D. Since we know that both boundary curves are mapped onto circles we can easily find the boundary values of Re ϕ.

For t in L_0, $|w(t)| = 1$, and for t in L_1, $|w(t)| = R_1$. Hence

$$\operatorname{Re} \phi(t) = \begin{cases} -\log|t|, & t \in L_0, \\ -\log|t| + \log R_1, & t \in L_1. \end{cases} \tag{1.2}$$

The value of R_1 is determined by the condition that the conjugate function Re ϕ be single valued. The problem of finding Re ϕ therefore reduces to solving a modified Dirichlet problem with boundary values $-\log|t|$.

It is possible to show that the value of R_1 so determined is always less than 1, and that L_0 and L_1 are in one-to-one correspondence with the circles $|w| = 1$ and $|w| = R_1$. See [10]. It follows that the function $\omega(z) = z \exp \phi(z)$ maps D onto the annulus.

1.3. The mapping of multiconnected regions onto the slit disc

We now assume that our region D is $(n + 1)$-connected where $n \geqslant 2$. We wish to map it onto the the unit disc in the w plane with concentric cuts.

Suppose D is bounded on the outside by a smooth curve L_0, and on the inside by smooth curves $L_1, L_2, \ldots, L_n$. We assume that the point $z = \alpha$ which gets mapped to the origin is inside D.

We again use the fact that the map $w(z)$ can be written

$$w(z) = (z - \alpha)g(z),$$

where $g(z)$ is analytic and nonzero in D. As in the case of a simply connected region we have

$$\operatorname{Re} \phi(t) = -\log|t - \alpha|, \quad t \in L_0.$$

Suppose that the curve L_i gets mapped onto a portion of the circle $|w| = R_i$, i.e. $|w(z)| = R_i$ for t in L_i. Then

$$\operatorname{Re} \phi(t) = -\log|t - \alpha|, \quad t \in L_0.$$

Suppose that the curve L_i gets mapped onto a portion of the circle $|w| = R_i$, i.e. $|w(z)| = R_i$ for t in L_i. Then

$$\operatorname{Re} \phi(t) = -\log|t - \alpha| + \log R_i \quad \text{for } t \in L_i, \quad 2 < i < n. \tag{1.3}$$

Once again the problem of finding the real part of ϕ reduces to solving a modified Dirichlet problem. Also, as before, once we know Re ϕ we can determine Im ϕ and thereby $g(z)$ and $w(z)$.

2. Integral equations

We solve both the Dirichlet and modified Dirichlet problems described in Section 1 by using integral equation formulations. In both cases we assume that the solution can be written as the integral of a double layer density function μ:

$$u(x, y) = \frac{1}{2\pi i} \int_{\partial D} \mu(s) \frac{\partial \log r(x, y, \tilde{x}(s), \tilde{y}(s))}{\partial n_s} \mathrm{d}s$$

where

$$r^2 = (x - \tilde{x}(s))^2 + (y - \tilde{y}(s))^2,$$

or, equivalently, as the real part of the Cauchy integral with density μ:

$$u(x, y) = \mathrm{Re} \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(z)}{z - \zeta} \mathrm{d}z.$$

If the region D is simply connected, then we have an ordinary Dirichlet problem with boundary data

$$u(t) = -\log|t - \alpha|.$$

It can be shown [10] that the density function $\mu(s)$ is the solution of the following integral equation on the boundary of D

$$\mu(t) + \frac{1}{\pi} \int_{\partial D} \mu(s) \frac{\partial \log r(s, t)}{\partial n_s} \mathrm{d}s = -2 \log|t - \alpha|.$$

When the region is not simply connected we solve a modified Dirichlet problem. We must determine not only the density function $\mu(s)$, but also the constants $\log R_i$ which appear on the right-hand sides of equations (1.2) and (1.3). Following [10, p. 146], let $\mu(t)$ denote the solution of the integral equation

$$\mu(t) + \frac{1}{\pi} \int_{\partial D} \mu(s) \left[\frac{\partial \log r(s, t)}{\partial n_s} - a(s, t) \right] \mathrm{d}s = -2 \log|t - \alpha|$$

where

$$a(s, t) = \begin{cases} 1 & \text{if } s, t \text{ lie on the same curve,} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$R_i = \frac{1}{\pi} \int_{L_i} \mu(s) \mathrm{d}s.$$

Then it is easy to verify that $\mu(s)$ is the correct density function and that $\{R_i\}$ are the required constants.

3. Solution of the integral equations

The integral equations we solve are all Fredholm integral equations of the second kind, that is, equations of the form

$$g(t)-\int_a^b G(t, s)g(s)\mathrm{d}s = d(t) \quad \text{for } a<t<b,$$

where $g(t)$, $h(t)$, and $G(s, t)$ are continuous for $a<s,t<b$, and

$$\int_a^b\int_a^b |G(s, t)|^2 \,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Moreover, the kernels are all bounded. (When $s=t$ on ∂D we have

$$\frac{\partial \log r(s, t)}{\partial n_s} = \tfrac{1}{2}\kappa(s),$$

where $\kappa(s)$ is the curvature at s.)

We solve the integral equation for most regions by using a Nyström method with the trapezoid rule as the quadrature formula:

$$\mu_n(t_i) + \frac{1}{\pi}\sum_j\left[\frac{\partial \log r(t_i, t_j)}{\partial n_{t_j}} + a(t_i, t_j)\right]\mu_n(t_j)h = d(t_i)$$

where h is the mesh width. In our experiments we chose the mesh points equally spaced with respect to some boundary parameter. (The points used as nodes for solving the integral equation are independent of the mesh points used for the fast solver.) We note that the accuracy of the solution of the integral is the same as the accuracy of the quadrature formula [1]. Using the Euler–Maclaurin formula it is easy to see that the trapezoid rule is highly accurate on periodic regions. We can therefore usually expect very high accuracy using this method.

For some regions, such as an oval of Cassini [12, p. 255], where two different parts of the boundary are close to each other, and therefore the kernel is very large, this method is not sufficiently accurate. In such cases we integrate the kernel exactly:

$$\int_{t_{i-1}}^{t_{i+1}} \frac{\partial \log r(s, t_j)}{\partial n_s}\mathrm{d}s = \int_{t_{i-1}}^{t_{i+1}} \frac{\partial \theta(s, t_j)}{\partial s}\mathrm{d}s = \theta(t_{i+1}, t_j) - \theta(t_{i-1}, t_j).$$

We thereby obtain the system of equations:

$$\mu(t_i) + \frac{1}{2\pi}\sum_j \Delta\theta + ha(t_i, t_j)\mu(t_j) = d(t_i)$$

where $\Delta\theta = \arctan(t_{i+1}-t_j)/(t_{i-1}-t_j)$ is the angle between the lines joining the points t_{i+1} and t_{i-1} to the point t_j.

For our numerical experiments we chose to solve these systems directly using Gaussian elimination. It is also possible to solve them iteratively [14, 15]. (We note that although the integral equation is in general nonsymmetric it nevertheless has positive real eigenvalues [8].)

4. Solution evaluation in the interior

After we compute the density function $\mu(t)$, it remains to evaluate the Cauchy integral $\phi(t)=(1/2\pi)\int\mu(t)/(z-t)\mathrm{d}z$ at points in the interior of the regions. We use the method described in [9].

We first show how to compute the real part of $\phi(t)$, that is

$$\frac{1}{2\pi}\int\mu(t)\frac{\partial \log r(s,t)}{\partial n_s}\mathrm{d}s. \tag{4.1}$$

In order to do this we first embed D in some larger region R such a rectangle with a uniform mesh in the x and y directions for which there exists a fast Poisson solver. We note that (4.1) defines another function $\tilde{u}$ at points outside D. The function $\tilde{u}$ is harmonic, but it is a discontinuous extension of u. Denote by $U(t)$ the combined function, that is the function that is equal to u at points inside D, and equal to $\tilde{u}$ at points of $R-D$. Our objective is to compute an approximation to the discrete Laplacian of U at all the mesh points of R. Since u and $\tilde{u}$ are harmonic, at mesh points which have all four of their nearest neighbors on the same side of the boundary we set the discrete Laplacian equal to 0. It remains to show how to compute an approximation at the other (irregular) mesh points.

The idea is the following. It is well known that u and $\tilde{u}$ are continuous in the normal direction, but have a jump in the tangential direction equal to the density μ. Consequently, it is easy to find expressions for the jumps in the derivatives of u and $\tilde{u}$ in the coordinate directions. For example

$$u_x-\tilde{u}_x=\mu'(s)\frac{x'(s)}{x'(s)^2+y'(s)^2}$$

and

$$u_y-\tilde{u}_y=\mu'(s)\frac{y'(s)}{x'(s)^2+y'(s)^2}.$$

Such expressions are composed entirely of evaluations of μ and its derivatives and derivatives of the boundary curve. We use these jumps to compute an approximation to the discrete difference operators at the irregular mesh points.

Suppose, for example, that a point p is inside the region D, and its neighbor to the left, p_{E}, is not. Let p' be the point where the line between p and p_{E} intersects the boundary, and let h_2 be the distance between p' and p_{E}. By manipulating the Taylor series at p and p_{E}, we can derive the following expression for $\tilde{u}(p)-u(p_{\mathrm{E}})$:

$$\begin{aligned}u(p)-\tilde{u}(p_{\mathrm{E}})&=(\tilde{u}(p')-u(p'))+h_2(\tilde{u}_x(p')-u_x(p'))\\&\quad+\tfrac{1}{2}h_2{}^2(\tilde{u}_{xx}(p')-u_{xx}(p'))+hu_x(p)+\tfrac{1}{2}h^2u_{xx}(p)+\mathrm{O}(h^3).\end{aligned}$$

The first three terms can be expressed in terms of the solution of the integral equation and the boundary data. The other terms are the usual Taylor series terms. We obtain the same type of formula for the difference between U at p and U at its other neighbors, except that there may not be any boundary terms. Therefore, if we can solve the integral equation we can compute an approximation to the discrete Laplacian of U which is the sum of the four difference operators, at all points of the grid. As for boundary values of U, we need only compute an approximation to the integral (4.1) at mesh points at the edge of the embedding region.

We can also compute the conjugate function Im $g = v$ at small additional cost. That is because by using the Cauchy–Riemann equations we can express the discontinuities in v in terms of the discontinuities in u. We can thereby also compute the discrete Laplacian of v. Once we know the discrete Laplacians of u and v we need only apply a fast Poisson solver twice to obtain their values at the mesh points of the grid. The solution will be second-order accurate in h, where h is the mesh width in R. If greater accuracy is required then it is possible to obtain greater accuracy by computing higher order accurate approximations to the discrete Laplacians and using higher order accurate Poisson solvers. See [9] for details.

5. Results of numerical experiments

We ran experiments on simply, doubly and triply connected regions.

The first simply connected regions we tested were ellipses. Figures 1 and 2 are the graphs of the images of the segments of the coordinate lines $x = ih$ and $y = jh$ inside two ellipses. For Fig. 1 the semiaxes of the ellipse were 0.2 and 0.35, and for Fig. 2 they were 0.35 and 0.3. In both examples $h = \frac{1}{32}$. We used the trapezoid rule with 30 mesh points to discretize the integral equation. In this and all subsequent examples the embedding region was the unit square.

The second simply connected region we tested was an oval of Cassini:

$$x(\theta) = R(\theta)\cos(\theta), \qquad y(\theta) = R(\theta)\sin(\theta),$$

$$R(\theta) = \sqrt{c^2 \cos 2\theta + \sqrt{a^4 - c^4 \sin 2\theta}} .$$

In our experiments we chose $c = 0.10$ and $a = 0.43$. To discretize the integral equation we used

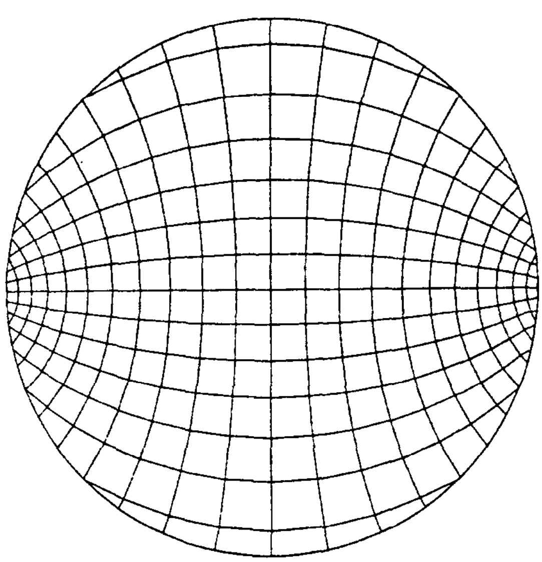

Fig. 1.

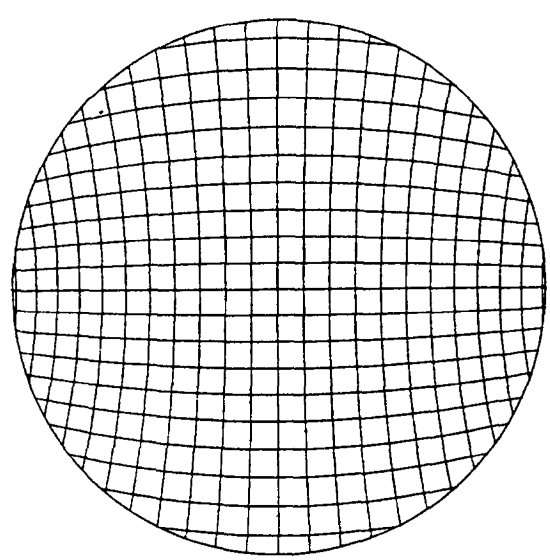

Fig. 2.

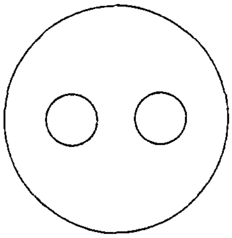

Fig. 3.

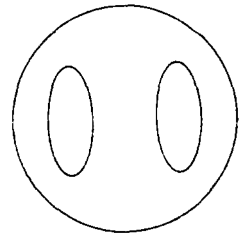

Fig. 4.

the second method described in Section 3 (that is, we integrated the kernel exactly), and we used 90 mesh points equally spaced with respect to θ. The exact mapping function is known [12, p. 256]. When we embedded the region in the unit square with mesh witdth $h = \frac{1}{32}$ the maximum error in the interior was 0.54 E−2, and when we used $h = \frac{1}{64}$ the maximum error was 0.24 E−2.

The doubly connected region we tested was the region bounded by the two eccentric circles, $|z| = 0.35$, and $|z - 0.08| = 0.14$. The exact solution is a bilinear transformation and is known exactly [11, p. 174]. Moreover, the solution of the integral equation is also known exactly. We found that when we used a total of 70 points, the error we made in solving the integral equation was 0.78 E−6. From this, other numerical experiments [9], and theoretical results [1], we believe

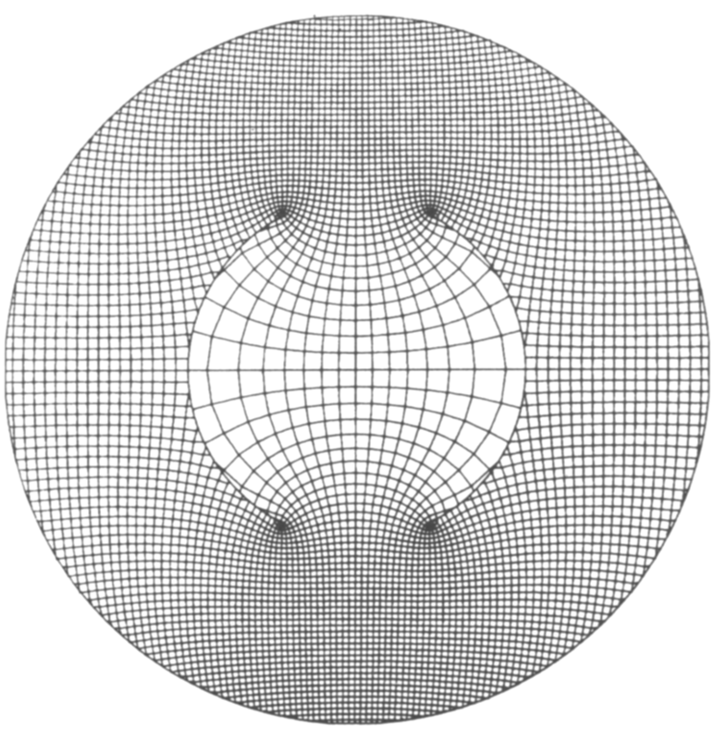

Fig. 5.

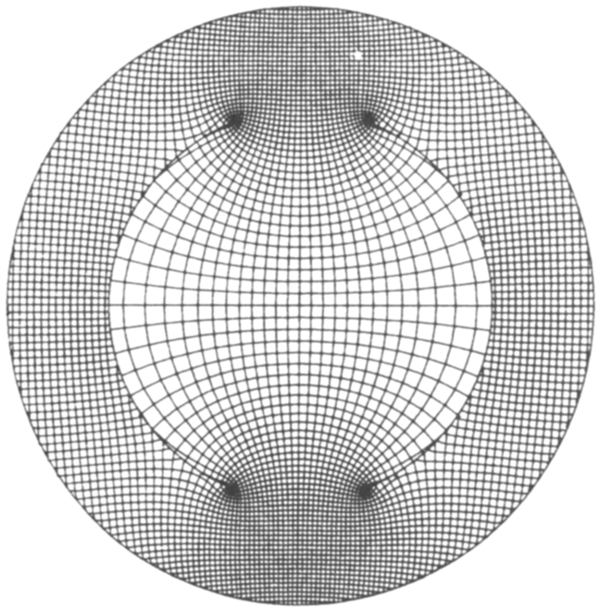

Fig. 6.

that we solve the integral equation very accurately. However, since the fast Poisson solver we use to compute the map in the interior is only second order accurate, we lose accuracy. To be specific, when we used mesh $h = \frac{1}{32}$ on the unit square the error was 0.19 E−2, and when we used mesh width $h = \frac{1}{64}$ the error was 0.51 E−3. We believe that, since this accuracy was achieved at points very close to the boundary, the results are quite good. However, as we mentioned before, if greater accuracy is required then one can use a higher order accurate Poisson solver.

The last regions we tested were triply connected. In both cases the outside boundary curve was the circle $|z| = 0.35$. In the first example (Fig. 3), the inner boundary curves were circles of radius 0.08 centered at $x = -0.14$ and $x = 0.14$, and in the second (Fig. 4), the inner curves were elllipses with semiaxes 0.06 and 0.13 centered at $x = 0.17$ and $x = -0.17$. Figures 5 and 6 are the images of the mesh lines inside these regions. The mesh width on the square was $h = \frac{1}{128}$, and a total of 180 mesh points were used to solve the integral equation in both examples. For clarity we have also plotted the unit circle and the slits.

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