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Holomorphic geometric models for representations of C^*-algebras

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Abstract

Representations of Banach–Lie groups are realized on Hilbert spaces formed by sections of holomorphic homogeneous vector bundles. These sections are obtained by means of a new notion of reproducing kernel, which is suitable for dealing with involutive diffeomorphisms defined on the base spaces of the bundles. The theory involves considering complexifications of homogeneous spaces acted on by groups of unitaries, and applies in particular to representations of C^*-algebras endowed with conditional expectations. In this way, we present holomorphic geometric models for the Stinespring dilations of completely positive maps. The general results are further illustrated by a discussion of several specific topics, including similarity orbits of representations of amenable Banach algebras, similarity orbits of conditional expectations, geometric models of representations of Cuntz algebras, the relationship to endomorphisms of $\mathcal{B}(\mathcal{H})$, and non-commutative stochastic analysis.
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1. Introduction

Originally, the interest in the study of representations of algebras and groups of operators on infinite-dimensional Hilbert or Banach spaces is to be found, as one of the main motivations, in problems arising from Quantum Physics. In this setting, unitary groups of operators can be interpreted as symmetry groups while the self-adjoint operators are thought of as observable objects, hence the direct approach to such questions leads naturally to representations both involving algebras generated by commutative or non-commutative canonical relations, and groups of unitaries on Hilbert spaces; see for instance [31,59], or [58]. Over the years, there have been important developments of this initial approach, in papers devoted to analyze or classify a wide variety of representations, and yet many questions remain open in the subject. It is certainly desirable to transfer to this field methods, or at least ideas, of the rich representation theory of finite-dimensional Lie groups.

In this respect, recall that geometric representation theory is a classical topic in finite dimensions. Its purpose is to shed light on certain classes of representations by means of their geometric realizations (see for instance [45]). Thus the construction of geometric models of representations lies at the heart of that topic. One of the classical results obtained in this direction is the Bott–Borel–Weil theorem concerning realizations of irreducible representations of compact Lie groups, in spaces of sections (or higher cohomology groups) of holomorphic vector bundles over flag manifolds [15]. Section spaces of vector bundles also appear in methods of induction of representations of Lie groups [29,35].

In special situations, or for particular aims, these ideas have been applied in the setting of infinite-dimensional Lie groups; see for example [16,35,47]. However, several difficult points are encountered when one tries to extend these methods in general, and perhaps the most important one is related to the lack of an algebraic structure theory for representations of these groups. Also, it is not a minor question the fact that, in infinite dimensions, there is no sufficiently well-suited theory of integration. The most reasonable way to deal with these problems seems to be to restrict both the class of groups and the class of representations one is working with. Moreover, one is led quite frequently to employ methods of operator algebras. See for example [61], where the study of factor representations of the group $\mathrm{U}(\infty)$ and AF-algebras is undertaken. There, a key role is played by the Gelfand–Naimark–Segal (or GNS, for short) representations constructed out of states of suitable maximal abelian self-adjoint subalgebras.

The importance of GNS representations as well as that of the geometric properties of state spaces in operator theory are well known. In [11], geometric realizations of restrictions of GNS representations to groups of unitaries in C^*-algebras are investigated, by considering suitable versions of reproducing kernels on vector bundles, in order to build representation spaces formed by sections. (This technique has already a well-established place in representation theory of finite-dimensional Lie groups; see for instance the monograph [45].) In some more detail, let $B \subseteq A$ be unital C^*-algebras such that there exists a conditional expectation $E : A \to B$. Let U_A and U_B be the unitary groups of A and B, respectively, and φ a state of A such that $\varphi \circ E = \varphi$. A reproducing kernel Hilbert space $\mathcal{H}_{\varphi,E}$ can be constructed out of φ and E, consisting of $\mathcal{C}^\infty$ sections of a certain Hermitian vector bundle with base the homogeneous space $\mathrm{U}_A/\mathrm{U}_B$, and the restriction to U_A of the GNS representation associated with φ can be realized as the natural multiplication of U_A on $\mathcal{H}_{\varphi,E}$; see [11, Theorem 5.4]. This theorem relates the GNS representations to the geometric representation theory, in the spirit of the Bott–Borel–Weil theorem. In view of this result and of the powerful method of induction developed in [16], it is most natural to ask about similar results for more general representations of infinite-dimensional Lie groups.

The above circle of ideas is naturally connected with holomorphy. Recall that this is the classical setting of the Bott–Borel–Weil theorem of [15] involving the flag manifolds, and it reinforces the strength of applications. The idea of *complexification* plays a central role in this area, inasmuch as one of the ways to describe the complex structure of the flag manifolds is to view the latter as homogeneous spaces of complexifications of compact Lie groups. (See [13,14,40,62] for recent advances in understanding the differential geometric flavor of the process of complexification.) In some cases involving finiteness properties of spectra and traces of elements in a C^*-algebra, it is possible to prove that the aforementioned homogeneous space $\mathrm{U}_A/\mathrm{U}_B$ is a *complex* manifold as well and the Hilbert space $\mathcal{H}_{\varphi,E}$ is formed by *holomorphic* sections (see [11, Theorem 5.8]). Similar results about complex structures hold in the special case of tautological bundles over Grassmann manifolds associated with involutive algebras, where they are related to constructions of almost hypercomplex structures; see [10].

Apart from the above two examples, the holomorphic character of the manifolds $\mathrm{U}_A/\mathrm{U}_B$ (and associated bundles) is far from being clear in general. Thus, since the tautological bundles considered in [10] are universal among manifolds of type $\mathrm{U}_A/\mathrm{U}_B$, it sounds sensible to investigate complexifications of the manifolds $\mathrm{U}_A/\mathrm{U}_B$ in general. On the other hand, the aforementioned conditional expectation $E: A \to B$ has a geometric meaning as a connection form defining a reductive structure in the homogeneous space $\mathrm{G}_A/\mathrm{G}_B$; see [2,21]. Since X is the Lie algebra of the complex Banach–Lie group G_X for $X = A$ and $X = B$, it is desirable to incorporate full groups of invertibles to the framework established in [11]. Note also that G_X is the universal complexification of U_X, according to the discussion of [46].

The above considerations suggest to find out the character of spaces $\mathrm{G}_A/\mathrm{G}_B$ as natural candidates to complexifications of spaces $\mathrm{U}_A/\mathrm{U}_B$, as well as their relationship with vector bundles and kernels.

Brief description of the present paper. One of our aims in the present paper is to extend the geometric representation theory of unitary groups of operator algebras to the complex setting of full groups of invertible elements. For this purpose we need a method to realize the representation spaces as Hilbert spaces of sections in holomorphic vector bundles. If one tries to mimic the arguments of [11] then one runs into troubles very soon (regarding the construction of appropriate reproducing kernels), due to the fact that general invertible elements of a C^*-algebra lack, when considered in an inner product, helpful cancellative properties (that unitaries have). This can be overcome by using certain involutions $z \mapsto z^{-*}$ (that come from the involutions of C^*-algebras) on the bases of the bundles, but then the problem is that our bundles lose their Hermitian character.

So we are naturally led toward developing a special theory of reproducing kernels on vector bundles. Section 2 includes a discussion of a version of Hermitian vector bundles suitable for our purposes. We call them *like-Hermitian*. The bases of such vector bundles are equipped with involutive diffeomorphisms $z \mapsto z^{-*}$, so that we need to find out a class of reproducing kernels, compatible in a suitable sense with the corresponding diffeomorphisms, which we call here reproducing $(-*)$-kernels. The very basic elements for the theory of reproducing $(-*)$-kernels are presented in Section 3 (it is our intention to develop such a theory more sistematically in forthcoming papers). In Section 4 we discuss examples of the above notions which arise in relation to homogeneous manifolds obtained by (smooth) actions of complex Banach–Lie groups (see Definition 2.10). These examples play a critical role for our main constructions of geometric models of representations; see Theorems 4.2 and 4.4. In particular, Theorem 4.4 provides the holomorphic versions of such realizations. In order to include the homogeneous spaces of unitary groups

$\mathrm{U}_A/\mathrm{U}_B$ in the theory and to avoid the fact that they are not necessarily complex manifolds, we had to view them as embedded into their natural complexifications $\mathrm{G}_A/\mathrm{G}_B$.

The special case where G_A is the group of invertibles in a C^*-algebra is treated in Section 5. Using a significant polar decomposition of G_A found by Porta and Recht, relative to a prescribed conditional expectation (see [53]), it is possible to interpret the manifold $\mathrm{G}_A/\mathrm{G}_B$ as (diffeomorphic to) the tangent bundle of $\mathrm{U}_A/\mathrm{U}_B$, see Theorems 5.1 and 5.5 below. These properties resemble very much similar properties enjoyed by complexifications of manifolds of compact type in finite dimensions. This may well mean that the homogeneous spaces $\mathrm{U}_A/\mathrm{U}_B$ and $\mathrm{G}_A/\mathrm{G}_B$ are suitable substitutes for compact homogeneous spaces in the infinite-dimensional setting.

The set of ideas previously exposed can be used to investigate geometric models for representations which arise as Stinespring dilations of completely positive maps on C^*-algebras A. In this way we shall actually end up with a geometric dilation theory of completely positive maps. This in particular enables us to get more examples of representations of Banach–Lie groups (namely, U_A, G_A) which admit geometric realizations in the sense of [11]. At this point, it is noteworthy that, just by differentiating, it is possible to recover the whole dilation on A and not only its restriction to U_A or G_A, see Theorem 6.10. So this provides a geometric interpretation of the classical methods of extension and induction of representations of C^*-algebras (see [25,54]). We should point out here that there exist earlier approaches in which completely positive maps have been considered under geometric perspectives—see for instance [3,6,52], or [44]—however they are different from the present line of investigation.

The last section of the paper, Section 7, is devoted to showing, by means of several specific examples, that the theory established here has interesting links with quite a number of different subjects in operator theory and related areas.

For the sake of better explanation, we conclude this introduction by a summary of the main points considered in the paper. These are:

– a theory of reproducing kernels on vector bundles that takes into account prescribed involutions of the bundle bases (Section 3);
– in the case of homogeneous vector bundles we investigate a circle of ideas centered on the relationship between reproducing kernels and complexifications of homogeneous spaces (Theorems 4.4 and 5.1);
– by using the previous items we model the representation spaces of Stinespring dilations as spaces of holomorphic sections in certain homogeneous vector bundles; thereby we set forth a rich panel of differential geometric structures accompanying the dilations of completely positive maps (Section 6); for one thing, we provide a geometric perspective on induced representations of C^*-algebras (cf. [54]);
– as an illustration of our results we describe in Section 7 a number of geometric properties of orbits of representations of nuclear C^*-algebras and injective von Neumann algebras (Corollary 7.2), similarity orbits of conditional expectations, and some relationships with representations of Cuntz algebras and endomorphisms of $\mathcal{B}(\mathcal{H})$, as well as with non-commutative stochastic analysis.

2. Like-Hermitian structures

We are going to introduce a variation of the notion of Hermitian vector bundle, which will turn out to provide the appropriate setting for the geometric representation theory of involutive Banach–Lie groups as developed in Section 4.

Definition 2.1. Assume that Z is a real Banach manifold equipped with a diffeomorphism $z \mapsto z^{-*}$, $Z \to Z$, which is involutive in the sense that $(z^{-*})^{-*} = z$ for all $z \in Z$. Denote by $p_1, p_2 : Z \times Z \to Z$ the natural projection maps. Let $\Pi : D \to Z$ be a smooth vector bundle whose fibers are complex Banach spaces (see for instance [1] or [38] for details on infinite-dimensional vector bundles).

We define a *like-Hermitian structure* on the bundle Π (with typical fiber the Banach space $\mathcal{E}$) as a family $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z \in Z}$ with the following properties:

(a) For every $z \in Z$, $(\cdot \mid \cdot)_{z,z^{-*}} : D_z \times D_{z^{-*}} \to \mathbb{C}$ is a sesquilinear strong duality pairing.
(b) For all $z \in Z$, $\xi \in D_z$, and $\eta \in D_{z^{-*}}$ we have $\overline{(\xi \mid \eta)_{z,z^{-*}}} = (\eta \mid \xi)_{z^{-*},z}$.
(c) If V is an arbitrary open subset of Z, and $\Psi_0 : V \times \mathcal{E} \to \Pi^{-1}(V)$ and $\Psi_1 : V^{-*} \times \mathcal{E} \to \Pi^{-1}(V^{-*})$ are trivializations of the vector bundle Π over V and V^{-*} $(:= \{z^{-*} \mid z \in V\})$, respectively, then the function $(z, x, y) \mapsto (\Psi_0(z, x) \mid \Psi_1(z^{-*}, y))_{z,z^{-*}}$, $V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$, is smooth.

We call *like-Hermitian vector bundle* any vector bundle equipped with a like-Hermitian structure.

Remark 2.2. Here we explain the meaning of condition (a) in Definition 2.1. To this end let $\mathcal{X}$ and $\mathcal{Y}$ be two complex Banach spaces. A functional $(\cdot \mid \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ is said to be a *sesquilinear strong duality pairing* if it is continuous, is linear in the first variable and antilinear in the second variable, and both the mappings

$$x \mapsto (x \mid \cdot), \quad \mathcal{X} \to (\overline{\mathcal{Y}})^*, \quad \text{and} \quad y \mapsto (\cdot \mid y), \quad \overline{\mathcal{Y}} \to \mathcal{X}^*,$$

are (not necessarily isometric) isomorphisms of complex Banach spaces.

Here we denote, for any complex Banach space $\mathcal{Z}$, by $\mathcal{Z}^*$ its dual Banach space (i.e., the space of all continuous linear functionals $\mathcal{Z} \to \mathbb{C}$) and by $\overline{\mathcal{Z}}$ the complex-conjugate Banach space. That is, the *real* Banach spaces underlying $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ coincide, and for any z in the corresponding real Banach space and $\lambda \in \mathbb{C}$ we have $\lambda \cdot z$ (in $\overline{\mathcal{Z}}$) $= \bar{\lambda} \cdot z$ (in $\mathcal{Z}$).

Remark 2.3. For later use we now record the following fact. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces over $\mathbb{C}$, and let $(\cdot \mid \cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ be a sesquilinear strong duality pairing. Now let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let $T : \mathcal{H} \to \mathcal{X}$ be a continuous linear operator. Then there exists a unique linear operator $S : \mathcal{Y} \to \mathcal{H}$ such that

$$(\forall h \in \mathcal{H}, y \in \mathcal{Y}) \quad (Th \mid y) = (h \mid Sy)_{\mathcal{H}}. \tag{2.1}$$

Conversely, for every bounded linear operator $S : \mathcal{Y} \to \mathcal{H}$ there exists a unique bounded linear operator $T : \mathcal{H} \to \mathcal{X}$ satisfying (2.1), and we denote $S^{-*} := T$ and $T^{-*} := S$.

Remark 2.4. In Definition 2.1 if $z^{-*} = z$ and $(\xi \mid \xi)_{z,z} \geqslant 0$ for all $z \in Z$ and $\xi \in D_z$, then we shall speak simply about *Hermitian* structures and bundles, since this is just the usual notion of Hermitian structure on a vector bundle. See for instance Definition 1.1 in [67, Chapter III] for the classical case of finite-dimensional Hermitian vector bundles.

Example 2.5. Let $\Pi : D \to Z$ be a smooth vector bundle whose fibers are complex Banach spaces. Assume that there exist a complex Hilbert space $\mathcal{H}$ and a smooth map $\Theta : D \to \mathcal{H}$ with the property that $\Theta|_{D_z}: D_z \to \mathcal{H}$ is a bounded linear operator for all $z \in Z$. Then Θ determines a family of continuous sesquilinear functionals

$$(\cdot \mid \cdot)_{z,z^{-*}}: D_z \times D_{z^{-*}} \to \mathbb{C}, \quad (\eta_1 \mid \eta_2)_{z,z^{-*}} = \big(\Theta(\eta_1) \mid \Theta(\eta_2)\big)_{\mathcal{H}}.$$

If in addition $\Theta|_{D_z}: D_z \to \mathcal{H}$ is injective and has closed range, and the scalar product of $\mathcal{H}$ determines a sesquilinear strong duality pairing between the subspaces $\Theta(D_z)$ and $\Theta(D_{z^{-*}})$ whenever $z \in Z$, then it is easy to see that we get a like-Hermitian structure on the vector bundle Π.

Definition 2.6. An *involutive* Banach–Lie group is a (real or complex) Banach–Lie group G equipped with a diffeomorphism $u \mapsto u^*$ satisfying $(uv)^* = v^*u^*$ and $(u^*)^* = u$ for all $u, v \in G$. In this case we denote

$$(\forall u \in G) \quad u^{-*} := \big(u^{-1}\big)^*$$

and

$$G^+ := \{u^*u \mid u \in G\}$$

and the elements of G^+ are called the *positive* elements of G.

If in addition H is a Banach–Lie subgroup of G, then we say that H is an *involutive* Banach–Lie subgroup if $u^* \in H$ whenever $u \in H$.

Remark 2.7. If G is an involutive Banach–Lie group then for every $u \in G$ we have $(u^{-1})^* = (u^*)^{-1}$ and moreover $\mathbf{1}^* = \mathbf{1}$. To see this, just note that the mapping $u \mapsto (u^*)^{-1}$ is an automorphism of G, hence it commutes with the inversion mapping and leaves $\mathbf{1}$ fixed.

Example 2.8. Every Banach–Lie group G has a trivial structure of involutive Banach–Lie group defined by $u^* := u^{-1}$ for all $u \in G$. In this case the set of positive elements is $G^+ = \{\mathbf{1}\}$.

Example 2.9. Let A be a unital C^*-algebra with the group of invertible elements denoted by G_A. Then G_A has a natural structure of involutive complex Banach–Lie group defined by the involution of A. If B is any C^*-subalgebra of A such that there exists a conditional expectation $E: A \to B$, then G_B is an involutive complex Banach–Lie subgroup of G_A.

The following definition provides us with an important example of like-Hermitian vector bundles, associated with a given representation, which plays a central role through the paper.

Definition 2.10. Assume that we have the following data:

- G_A is an involutive real (respectively, complex) Banach–Lie group and G_B is an involutive real (respectively, complex) Banach–Lie subgroup of G_A.
- For $X = A$ or $X = B$, assume $\mathcal{H}_X$ is a complex Hilbert space with $\mathcal{H}_B$ closed subspace in $\mathcal{H}_A$, and $\pi_X: G_X \to \mathcal{B}(\mathcal{H}_X)$ is a uniformly continuous (respectively, holomorphic) $*$-representation such that $\pi_B(u) = \pi_A(u)|_{\mathcal{H}_B}$ for all $u \in G_B$. By $*$-representation we mean that $\pi_A(u^*) = \pi_A(u)^*$ for all $u \in G_A$.
- We denote by $P: \mathcal{H}_A \to \mathcal{H}_B$ the orthogonal projection.

We define an equivalence relation on $G_A \times \mathcal{H}_B$ by

$$(u, f) \sim (u', f') \quad \text{whenever there exists } w \in G_B \text{ such that } u' = uw \text{ and } f' = \pi_B\big(w^{-1}\big)f.$$

For every pair $(u, f) \in G_A \times \mathcal{H}_B$ we define its equivalence class by $[(u, f)]$ and let $D = G_A \times_{G_B} \mathcal{H}_B$ denote the corresponding set of equivalence classes. Then there exists a natural onto map

$$\Pi : \big[(u, f)\big] \mapsto s := uG_B, \quad D \to G_A/G_B.$$

For $s \in G_A/G_B$, let $D_s := \Pi^{-1}(s)$ denote the *fiber* on s. Note that $(u, f) \sim (u', f')$ implies that $\pi_A(u)f = \pi_A(u')f'$ so that the correspondence $[(u, f)] \mapsto \pi_A(u)f$, $D_s \to \pi_A(u)\mathcal{H}_B$, gives rise to a complex linear structure on D_s. Moreover,

$$\big\|\big[(u, f)\big]\big\|_{D_s} := \big\|\pi_A(u)f\big\|_{\mathcal{H}_A}$$

where $[(u, f)] \in D_s$, defines on D_s a Hilbertian norm.

Clearly, this structure does not depend on the choice of u. Nevertheless, note that the natural bijection from $\mathcal{H}_B$ onto the fiber $\Pi^{-1}(s)$ defined by

$$\Theta_u : f \mapsto \big[(u, f)\big], \quad \mathcal{H}_B \to \Pi^{-1}(s),$$

is a topological isomorphism but it *need not be* an isometry. In other words, the fiberwise maps

$$\Theta_v\Theta_u^{-1} : \big[(u, f)\big] \mapsto f \mapsto \big[(v, f)\big], \quad D_s \to \mathcal{H}_B \to D_t,$$

where $s = uG_B$, $t = vG_B$ and $f \in \mathcal{H}_B$, are topological isomorphisms but they are not unitary transformations in general. As a complex Hilbert space, D_s has so many realizations of the topological dual or predual. We next consider the following ones. For $\xi = [(u, f)]$, $\eta = [(v, g)]$ in D, and $s = uG_B$, $t = vG_B$, we set as in Example 2.5,

$$(\xi \mid \eta)_D \equiv (\xi \mid \eta)_{s,t} := \big(\pi_A(u)f \mid \pi_A(v)g\big)_{\mathcal{H}_A},$$

where $(\cdot \mid \cdot)_{\mathcal{H}_A}$ is the inner product which defines the complex Hilbert structure on $\mathcal{H}_A$ and, by restriction, on $\mathcal{H}_B$. This is a well-defined, non-negative sesquilinear form on D. In particular $(\cdot \mid \cdot)_{s,t} = \overline{(\cdot \mid \cdot)_{t,s}}$. We are mainly interested in forms $(\cdot \mid \cdot)_{s,t}$ with $t = s^{-*} \in G_A/G_B$. In this case

$$\begin{aligned}\big(\big[(u, f)\big] \mid \big[\big(u^{-*}, g\big)\big]\big)_{s,s^{-*}} &= \big(\pi_A(u)f \mid \pi_A\big(u^{-*}\big)g\big)_{\mathcal{H}_A} = \big(\pi_A\big(u^{-1}\big)\pi_A(u)f \mid g\big)_{\mathcal{H}_A} \\ &= (f \mid g)_{\mathcal{H}_B},\end{aligned} \tag{2.2}$$

whenever $[(u, f)] \in D_s$ and $[(u^{-*}, g)] \in D_{s^{-*}}$. Thus Example 2.5 shows that the *basic mapping*

$$\Theta : \big[(u, f)\big] \mapsto \pi_A(u)f, \quad D \to \mathcal{H}_A,$$

gives rise to a like-Hermitian structure on the vector bundle Π.

We shall say that $\Pi : D \to G_A/G_B$ is the (holomorphic) *homogeneous like-Hermitian vector bundle* associated with the data (π_A, π_B, P).

Remark 2.11. Let us see that Definition 2.10 is correct, that is, condition (a) of Definition 2.1 is satisfied. In fact, let $u \in G_A$ arbitrary, denote $z = uG_B \in G_A/G_B$, and let φ be any bounded linear functional on $\overline{D}_{z^{-*}}$. Then the mapping $\tilde{\varphi}: g \mapsto [(u^{-*}, g)] \mapsto \varphi([(u^{-*}, g)])$, $\mathcal{H}_B \to \overline{D}_{z^{-*}} \to \mathbb{C}$, is antilinear and bounded. By the Riesz' theorem there exists $f \in \mathcal{H}_B$ such that

$$\varphi\big(\big[(u^{-*}, g)\big]\big) = \tilde{\varphi}(g) = (f \mid g)_{\mathcal{H}_B} \overset{(2.2)}{=} \big(\big[(u, f)\big] \mid \big[(u^{-*}, g)\big]\big)_{z, z^{-*}}$$

and so $(\cdot \mid \cdot)_{z,z^{-*}}$ is a sesquilinear strong duality pairing between D_z and $D_{z^{-*}}$.

3. Reproducing $(-*)$-kernels

Definition 3.1. Let $\Pi : D \to Z$ be a like-Hermitian bundle, with involution $-*$ in Z. A *reproducing* $(-*)$*-kernel* on Π is a section

$$K \in \Gamma\big(Z \times Z, \operatorname{Hom}\big(p_2^*\Pi, p_1^*\Pi\big)\big)$$

(whence $K(s,t): D_t \to D_s$ is a bounded linear operator for all $s, t \in Z$) which is $(-*)$-positive definite in the following sense: for every $n \geqslant 1$ and $t_j \in Z$, $\eta_j^{-*} \in D_{t_j^{-*}}$ $(j = 1, \dots, n)$,

$$\sum_{j,l=1}^{n} \big(\eta_j^{-*} \mid K\big(t_j, t_l^{-*}\big)\eta_l^{-*}\big)_{t_j^{-*}, t_j} = \sum_{j,l=1}^{n} \big(K\big(t_l, t_j^{-*}\big)\eta_j^{-*} \mid \eta_l^{-*}\big)_{t_l, t_l^{-*}} \geqslant 0.$$

Here $p_1, p_2 : Z \times Z \to Z$ are the natural projection mappings. If in addition $\Pi : D \to Z$ is a holomorphic like-Hermitian vector bundle and $K(\cdot, t)\eta \in \mathcal{O}(Z, D)$ for all $\eta \in D_t$ and $t \in Z$, then we say that K is a *holomorphic* reproducing $(-*)$-kernel.

Remark 3.2. In Definition 3.1, the symbol η_j^{-*} is just a way to refer to elements of $D_{t_j^{-*}}$, that is, η_j^{-*} is not associated to any element η_j of D_{t_j} necessarily. From the definition we have that $K(s, s^{-*}) \geqslant 0$ in the sense that $(K(s, s^{-*})\xi^{-*} \mid \xi^{-*})_{s,s^{-*}} \geqslant 0$ for all $\xi^{-*} \in D_{s^{-*}}$.

The following results are related to the extension of Theorem 4.2 in [11] to reproducing kernels on like-Hermitian vector bundles.

Proposition 3.3. *Let $\Pi : D \to Z$ be a smooth like-Hermitian vector bundle and, as usually, denote by $p_1, p_2 : Z \times Z \to Z$ the projections. Next consider a section $K \in \Gamma(Z \times Z,$ $\operatorname{Hom}(p_2^*\Pi, p_1^*\Pi))$ and for all $s \in Z$ and $\xi \in D_s$ denote $K_\xi = K(\cdot, s)\xi \in \Gamma(Z, D)$. Also denote*

$$\mathcal{H}_0^K := \operatorname{span}\{K_\xi \mid \xi \in D\} \subseteq \Gamma(Z, D).$$

Then K is a reproducing $(-*)$-kernel on Π if and only if there exists a complex Hilbert space $\mathcal{H}$ such that $\mathcal{H}_0^K$ is a dense linear subspace of $\mathcal{H}$ and

$$(K_\eta \mid K_\xi)_{\mathcal{H}} = \big(K(s^{-*}, t)\eta \mid \xi\big)_{s^{-*}, s} \tag{3.1}$$

whenever $s, t \in Z$, $\xi \in D_s$, and $\eta \in D_t$.

The proof is straightforward, just by adapting Definition 3.1 to the arguments employed in [11, Theorem 4.2], and we omit it.

Definition 3.4. Let $\Pi : D \to Z$ be a smooth like-Hermitian vector bundle, $p_1, p_2 : Z \times Z \to Z$ the projections, and let $K \in \Gamma(Z \times Z, \mathrm{Hom}(p_2^*\Pi, p_1^*\Pi))$ be a reproducing $(-*)$-kernel. As above, for all $s \in Z$ and $\xi \in D_s$, put $K_\xi = K(\cdot, s)\xi \in \Gamma(Z, D)$. It is clear that the Hilbert space $\mathcal{H}$ given by Proposition 3.3 is uniquely determined. We shall denote it by $\mathcal{H}^K$ and we shall call it the *reproducing* $(-*)$*-kernel Hilbert space* associated with K.

In the same framework we also define the mapping

$$\widehat{K} : D \to \mathcal{H}^K, \quad \widehat{K}(\xi) = K_\xi. \tag{3.2}$$

It follows by Lemma 3.5 below that for every $s \in Z$ there exists a bounded linear operator $\theta_s : \mathcal{H}^K \to D_{s^{-*}}$ such that

$$\big(\forall \xi \in D_s, h \in \mathcal{H}^K\big) \quad \big(\widehat{K}(\xi) \mid h\big)_{\mathcal{H}^K} = (\xi \mid \theta_s h)_{s,s^{-*}}. \tag{3.3}$$

Note that the operator θ_s is uniquely determined since $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z\in Z}$ is a like-Hermitian structure, and in the notation of Remark 2.3 we have

$$(\theta_s)^{-*} = \widehat{K}|_{D_{s^{-*}}}. \tag{3.4}$$

Lemma 3.5. *Assume the setting of Definition* 3.4. *Then for every $s \in Z$ the operator $\widehat{K}|_{D_s} : D_s \to \mathcal{H}^K$ is bounded, linear and adjointable, in the sense that there exists a bounded linear operator $\theta_s : \mathcal{H}^K \to D_{s^{-*}}$ such that* (3.3) *is satisfied.*

Proof. Since at every point of Z we have a sesquilinear strong duality pairing, it will be enough to show that for arbitrary $s \in Z$ the linear operator $\widehat{K}|_{D_s} : D_s \to \mathcal{H}^K$ is continuous. (See Remark 2.3.) To this end, let us denote by $\|\cdot\|_{D_s}$ any norm that defines the topology of the fiber D_s. Then for every $\xi \in D_s$ we have $\|\widehat{K}(\xi)\|_{\mathcal{H}^K} = \|K_\xi\|_{\mathcal{H}^K} = (K_\xi \mid K_\xi)^{1/2}_{\mathcal{H}^K} \overset{(3.1)}{=} (K(s^{-*}, s)\xi \mid \xi)^{1/2}_{s^{-*},s} \leqslant M_s^{1/2}\|\xi\|_{D_s}$, where $M_s > 0$ denotes the norm of the continuous sesquilinear functional $D_s \times D_s \to \mathbb{C}$ defined by $(\xi, \eta) \mapsto (K(s^{-*}, s)\xi \mid \eta)_{s^{-*},s}$. So the operator $\widehat{K}|_{D_s} : D_s \to \mathcal{H}^K$ is indeed bounded and $\|\widehat{K}|_{D_s}\| \leqslant M_s^{1/2}$. □

Example 3.6. Every reproducing kernel on a Hermitian vector bundle (see e.g., [11, Section 4]) provides an illustration for Definition 3.4. In fact, this follows since every Hermitian vector bundle is like-Hermitian.

Proposition 3.7. *Let $\Pi : D \to Z$ be a like-Hermitian bundle, and denote by $p_1, p_2 : Z \times Z \to Z$ the natural projections. Then for every reproducing $(-*)$-kernel $K \in \Gamma(Z \times Z, \mathrm{Hom}(p_2^*\Pi, p_1^*\Pi))$ there exists a unique linear mapping $\iota : \mathcal{H}^K \to \Gamma(Z, D)$ with the following properties*:

(a) *The restriction of ι to the dense subspace $\mathcal{H}_0^K$ is the identity mapping.*
(b) *The mapping ι is injective.*

(c) *The evaluation operator* $\mathrm{ev}^{\iota}_s : h \mapsto (\iota(h))(s)$, $\mathcal{H}^K \to D_s$, *is continuous linear for arbitrary* $s \in Z$, *and we have*

$$(\forall s, t \in Z) \quad K(s, t^{-*}) = \mathrm{ev}^{\iota}_s \circ \left(\mathrm{ev}^{\iota}_t\right)^{-*}.$$

Definition 3.8. In the setting of Proposition 3.7 we shall say that ι is the *realization operator* associated with the reproducing $(-*)$-kernel K.

Proof of Proposition 3.7. The uniqueness of ι is clear. To prove the existence of ι, note that for every $s \in Z$ there exists a bounded linear operator $\theta_s : \mathcal{H}^K \to D_{s^{-*}}$ such that

$$\left(\forall \xi \in D_s, h \in \mathcal{H}^K\right) \quad (K_\xi \mid h)_{\mathcal{H}^K} = (\xi \mid \theta_s h)_{s,s^{-*}} \tag{3.5}$$

(see Lemma 3.5). We shall define the wished-for mapping ι by

$$\iota : \mathcal{H}^K \to \Gamma(Z, D), \quad \big(\iota(h)\big)(s) := \theta_{s^{-*}} h \tag{3.6}$$

whenever $h \in \mathcal{H}^K$ and $s \in Z$. In particular we have

$$(\forall s \in Z) \quad \mathrm{ev}^{\iota}_s = \theta_{s^{-*}}, \tag{3.7}$$

and in addition Eq. (3.4) holds.

It is also clear that the mapping ι defined by (3.6) is linear. To prove that it is injective, let $h \in \mathcal{H}^K$ with $\iota(h) = 0$. Then $(\iota(h))(s^{-*}) = 0$ for all $s \in Z$, so that $\theta_s h = 0$ for all $s \in Z$, according to (3.6). Now (3.5) shows that $(K_\xi \mid h)_{\mathcal{H}^K} = 0$ for all $\xi \in D$, whence $h \perp \mathcal{H}^K_0$ in $\mathcal{H}^K$. Since $\mathcal{H}^K_0$ is a dense subspace of $\mathcal{H}^K$, it then follows that $h = 0$.

We shall check that the restriction of ι to $\mathcal{H}^K_0$ is the identity mapping. To this end it will be enough to see that for all $t \in Z$ and $\eta \in D_t$ we have $\iota(K_\eta) = K_\eta$. In fact, at any point $s \in Z$ we have $(\iota(K_\eta))(s) = \theta_{s^{-*}}(K_\eta)$ by (3.6). Hence for all $\xi \in D_{s^{-*}}$ we get

$$\begin{aligned}\left(\xi \mid \big(\iota(K_\eta)\big)(s)\right)_{s^{-*},s} &= \left(\xi \mid \theta_{s^{-*}}(K_\eta)\right)_{s^{-*},s} \overset{(3.5)}{=} (K_\xi \mid K_\eta)_{\mathcal{H}^K} \overset{(3.1)}{=} \left(K\left(t^{-*}, s^{-*}\right)\xi \mid \eta\right)_{t^{-*},t} \\ &= \left(\xi \mid K(s,t)\eta\right)_{s^{-*},s} = \left(\xi \mid K_\eta(s)\right)_{s^{-*},s}.\end{aligned}$$

Since $\xi \in D_{s^{-*}}$ is arbitrary and $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z \in Z}$ is a like-Hermitian structure, it then follows that $(\iota(K_\eta))(s) = K_\eta(s)$ for all $s \in Z$, whence $\iota(K_\eta) = K_\eta$, as desired.

Next we shall prove that ι has the asserted property (c). To this end, let $s, t \in Z$, $\eta \in D_{t^{-*}}$, and $\xi \in D_{s^{-*}}$ arbitrary. Then

$$\begin{aligned}\left(\left(\mathrm{ev}^{\iota}_s \circ \left(\mathrm{ev}^{\iota}_t\right)^{-*}\right)\eta \mid \xi\right)_{s,s^{-*}} &\overset{(3.7)}{=} \left(\left(\theta_{s^{-*}} \circ (\theta_{t^{-*}})^{-*}\right)\eta \mid \xi\right)_{s,s^{-*}} \\ &\overset{(3.3)}{=} \left(\left(\left(\theta_{t^{-*}}\right)^{-*}\right)\eta \mid K_\xi\right)_{\mathcal{H}^K} \overset{(3.4)}{=} (K_\eta \mid K_\xi)_{\mathcal{H}^K} \\ &\overset{(3.1)}{=} \left(K\left(s, t^{-*}\right)\eta \mid \xi\right)_{s,s^{-*}}.\end{aligned}$$

Since $\eta \in D_{t^{-*}}$ and $\xi \in D_{s^{-*}}$ are arbitrary and $\{(\cdot \mid \cdot)_{z,z^{-*}}\}_{z \in Z}$ is a like-Hermitian structure, it follows that $\mathrm{ev}^{\iota}_s \circ (\mathrm{ev}^{\iota}_t)^{-*} = K(s, t^{-*})$ for arbitrary $s, t \in Z$, as desired. $\square$

We now extend to our framework some basic properties of the classical reproducing kernels (see for instance the first chapter of [45]).

Proposition 3.9. *Assume that $\Pi : D \to Z$ is a like-Hermitian vector bundle, and K is a continuous reproducing $(-*)$-kernel on Π with the realization operator $\iota : \mathcal{H}^K \to \Gamma(Z, D)$. Then the following assertions hold*:

(a) *We have* $\operatorname{Ran} \iota \subseteq \mathcal{C}(Z, D)$ *and the mapping ι is continuous with respect to the topology of $\mathcal{C}(Z, D)$ defined by the uniform convergence on the compact subsets of Z.*
(b) *If Π is a holomorphic bundle and K is a holomorphic reproducing $(-*)$-kernel then we have* $\operatorname{Ran} \iota \subseteq \mathcal{O}(Z, D)$.

Proof. The proof has two stages.

1° At this stage we prove that every $s \in Z$ has an open neighborhood V_s such that for every sequence $\{h_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}^K$ convergent to some $h \in \mathcal{H}^K$ we have $\lim_{n\in\mathbb{N}}(\iota(h_n))(z) = (\iota(h))(z)$ uniformly for $z \in V_s$.

In fact, since the vector bundle Π is locally trivial, there exists an open neighborhood V of s such that Π is trivial over both V and $V^{-*} := \{z^{-*} \mid z \in V\}$. Let $\Psi_0 : V \times \mathcal{E} \to \Pi^{-1}(V)$ and $\Psi_1 : V^{-*} \times \mathcal{E} \to \Pi^{-1}(V^{-*})$ be trivializations of the vector bundle Π over V and V^{-*} respectively, where the Banach space $\mathcal{E}$ is the typical fiber of Π. In particular, these trivializations allow us to endow each fiber D_z with a norm (constructed out of the norm of $\mathcal{E}$) for $z \in V \cup V^{-*}$. On the other hand, property (c) in Definition 2.1 shows that the function

$$B : (z, x, y) \mapsto \big(\Psi_0(z, x) \mid \Psi_1(z^{-*}, y)\big)_{z,z^{-*}}, \quad V \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$$

is smooth. Then by property (c) in Definition 2.1 we get a well-defined mapping

$$\widetilde{B} : V \to \operatorname{Iso}\big(\mathcal{E}, \bar{\mathcal{E}}^*\big), \quad \widetilde{B}(z)x := B(z, x, \cdot) \text{ for } z \in V \text{ and } x \in \mathcal{E},$$

and it is straightforward to prove that $\widetilde{B}$ is continuous since B is so. Here $\operatorname{Iso}(\mathcal{E}, \bar{\mathcal{E}}^*)$ stands for the set of all topological isomorphisms $\mathcal{E} \to \bar{\mathcal{E}}^*$, which is an open subset of the complex Banach space $\mathcal{B}(\mathcal{E}, \bar{\mathcal{E}}^*)$. Then by shrinking the open neighborhood V of s we may assume that there exists $M > 0$ such that $\max\{\|\widetilde{B}(z)\|, \|\widetilde{B}(z)^{-1}\|\} < M$ whenever $z \in V$. In particular, for such z and $x \in \mathcal{E}$ we have $\|x\| < M\|\widetilde{B}(z)x\|$, and then the definition of the norm of $\widetilde{B}(z)x \in \bar{\mathcal{E}}^*$ implies the following fact:

$$(\forall z \in V)(\forall x \in \mathcal{E})\big(\exists y \in \mathcal{E},\ \|y\| = 1\big) \quad \|x\| \leqslant M\big|B(z, x, y)\big|.$$

In view of the fact that the norms of the fibers D_z and $D_{z^{-*}}$ are defined such that the operators $\Psi_0(z, \cdot) : \mathcal{E} \to D_z$ and $\Psi_1(z^{-*}, \cdot) : \mathcal{E} \to D_{z^{-*}}$ are isometries whenever $z \in V$, it then follows that

$$(\forall z \in V)(\forall \eta \in D_z)\big(\exists \xi \in D_{z^{-*}},\ \|\xi\| = 1\big) \quad \|\eta\| \leqslant M\big|(\xi \mid \eta)_{z^{-*},z}\big|. \tag{3.8}$$

On the other hand, it follows by (3.6) that $\|(\iota(h))(z)\|_{D_z} = \|\theta_{z^{-*}}(h)\|_{D_z}$ for arbitrary $z \in V$ and $h \in \mathcal{H}^K$. Then by (3.8) there exists $\xi \in D_{z^{-*}}$ such that $\|\xi\| = 1$ and

$$\big\|\big(\iota(h)\big)(z)\big\|_{D_z} \leqslant M\big|\big(\xi \mid \theta_{z^{-*}}(h)\big)_{z^{-*},z}\big| \overset{(3.5)}{=} M\big|(K_\xi \mid h)_{\mathcal{H}^K}\big| \leqslant M\|K_\xi\|_{\mathcal{H}^K}\|h\|_{\mathcal{H}^K}.$$

On the other hand, since $K : Z \times Z \to \mathrm{Hom}(p_2^*\Pi, p_1^*\Pi)$ is continuous, it follows that after shrinking again the neighborhood V of s we may suppose that $m := \sup_{z \in V} M_z < \infty$, where M_z denotes the norm of the bounded sesquilinear functional $D_z \times D_z \to \mathbb{C}$ defined by $(\eta_1, \eta_2) \mapsto (K(z^{-*}, z)\eta_1 \mid \eta_2)_{z^{-*},z}$ whenever $z \in V$. Then the computation from the proof of Lemma 3.5 shows that $\|K_\xi\|_{\mathcal{H}^K} \leqslant m^{1/2}\|\xi\|_{D_z} = m^{1/2}$. It then follows by the above inequalities that we end up with an open neighborhood V of s with the following property:

$$\big(\forall h \in \mathcal{H}^K\big)(\forall z \in V) \quad \big\|\big(\iota(h)\big)(z)\big\|_{D_z} \leqslant m^{1/2} M \|h\|_{\mathcal{H}^K},$$

which clearly implies the claim from the beginning of the present stage of the proof.

2° At this stage we come back to the proof of the assertions (a) and (b). Assertion (a) follows by means of a straightforward compactness reasoning and by what we proved at stage 1°, since $K_\xi \in \mathcal{C}(Z, D)$ whenever $\xi \in D$ and $\mathrm{span}\{K_\xi \mid \xi \in D\} = \mathcal{H}_0^K$. Finally, assertion (b) follows by the assertion (a) in a similar manner, since $\mathcal{O}(Z, D)$ is a closed subspace of $\mathcal{C}(Z, D)$ with respect to the topology of uniform convergence on the compact subsets of Z (see [65, Corollary 1.14]). □

Remark 3.10. It follows by Proposition 3.9(a) that every reproducing $(-*)$-kernel Hilbert space $\mathcal{H}^K$ is a Hilbert subspace of $\mathcal{C}(Z, D)$ in the sense of [57]. Thus the theory of reproducing $(-*)$-kernels developed in the present section provides a new class of examples of reproducing kernels in the sense of Laurent Schwartz.

4. Homogeneous like-Hermitian vector bundles and kernels

We develop here some aspects of the theory of kernels introduced in the previous section, when the manifold Z is assumed to be a homogeneous manifold arising from the (smooth) action of a Banach–Lie group. Specifically, we shall construct realizations of $*$-representations, of Banach–Lie groups, on spaces of analytic sections in like-Hermitian vector bundles. A critical role in this connection will be played by the class of examples derived from Definition 2.10.

Let G_A be an involutive real (respectively, complex) Banach–Lie group and G_B an involutive real (respectively, complex) Banach–Lie subgroup of G_A. For $X = A$ or $X = B$, let $\mathcal{H}_X$ be a complex Hilbert space with $\mathcal{H}_B$ closed subspace in $\mathcal{H}_A$ and $P : \mathcal{H}_A \to \mathcal{H}_B$ the corresponding orthogonal projection, and let $\pi_X : G_X \to \mathcal{B}(\mathcal{H}_X)$ be a uniformly continuous (respectively, holomorphic) $*$-representations such that $\pi_B(u) = \pi_A(u)|_{\mathcal{H}_B}$ for all $u \in G_B$. In addition, denote by $\Pi : D = G_A \times_{G_B} \mathcal{H}_B \to G_A/G_B$ the homogeneous like-Hermitian vector bundle associated with the data (π_A, π_B, P), and let $p_1, p_2 : G_A/G_B \times G_A/G_B \to G_A/G_B$ be the natural projections. Set

$$K(s,t)\eta = \big[\big(u, P\big(\pi_A\big(u^{-1}\big)\pi_A(v) f\big)\big)\big]$$

for $s, t \in G_A/G_B$, $s = uG_B$, $t = vG_B$, and $\eta = [(v, f)] \in D_t \subset D$.

Proposition 4.1. *In the above setting, K is a reproducing $(-*)$-kernel, for which the corresponding reproducing $(-*)$-kernel Hilbert space $\mathcal{H}^K \subset \mathcal{C}^\infty(G_A/G_B, D)$ (respectively $\mathcal{H}^K \subset \mathcal{O}(G_A/G_B, D)$) consists of sections of the form $F_h := [(\cdot, P(\pi_A(\cdot)^{-1}h))]$, $h \in \overline{\mathrm{span}}(\pi_A(G_A)\mathcal{H}_B)$ in $\mathcal{H}_A$.*

Proof. Let $s_j = u_j G_B \in G_A/G_B$ and $\xi_j = [(u_j^{-*}, f_j)] \in D_{s_j^{-*}}$ for $j = 1, \ldots, n$. We have

$$\begin{aligned}\sum_{j,l=1}^{n} \big(K\big(s_l, s_j^{-*}\big)\xi_j \mid \xi_l\big)_{s_l, s_l^{-*}} &= \sum_{j,l=1}^{n} \big(P\big(\pi_A\big(u_l^{-1}\big)\pi_A\big(u_j^{-*}\big) f_j\big) \mid f_l\big)_{\mathcal{H}_B} \\ &= \sum_{j,l=1}^{n} \big(\pi_A\big(u_l^{-1}\big)\pi_A\big(u_j^{-*}\big) f_j \mid f_l\big)_{\mathcal{H}_A} \\ &= \left(\sum_{j=1}^{n} \pi_A\big(u_j^{-*}\big) f_j \mid \sum_{l=1}^{n} \pi_A\big(u_l^{-*}\big) f_l\right)_{\mathcal{H}_A} \geqslant 0.\end{aligned}$$

On the other hand, by the above calculation we get

$$\begin{aligned}\big(K\big(s_l, s_j^{-*}\big)\xi_j \mid \xi_l\big)_{s_l, s_l^{-*}} &= \big(\pi_A\big(u_j^{-*}\big) f_j \mid \pi_A\big(u_l^{-*}\big) f_l\big)_{\mathcal{H}_A} = \overline{\big(\pi_A\big(u_l^{-*}\big) f_l \mid \pi_A\big(u_j^{-*}\big) f_j\big)}_{\mathcal{H}_A} \\ &= \overline{\big(K\big(s_j, s_l^{-*}\big)\xi_l \mid \xi_j\big)}_{s_j, s_j^{-*}} = \big(\xi_j \mid K\big(s_j, s_l^{-*}\big)\xi_l\big)_{s_j^{-*}, s_j}.\end{aligned}$$

Thus K is a reproducing $(-*)$-kernel. Again by the above calculation it follows that

$$(K_{\xi_j} \mid K_{\xi_l})_{\mathcal{H}^K} = \big(K\big(s_l, s_j^{-*}\big)\xi_j \mid \xi_l\big)_{s_l, s_l^{-*}} = \big(\pi_A\big(u_j^{-*}\big) f_j \mid \pi_A\big(u_l^{-*}\big) f_l\big)_{\mathcal{H}_A}. \tag{4.1}$$

Now, by Proposition 3.9, $\mathcal{H}^K \subset \mathcal{C}(G_A/G_B, D)$. Let F be a section in $\mathcal{H}^K$. By definition F is a limit, in the norm of $\mathcal{H}^K$, of a sequence of sections of the form $\sum_{j=1}^{n(m)} K_{\xi_j^m}$, where $\xi_j^m = [(v_j^m, f_j^m)] \in D$, $j = 1, \ldots, n(m)$, $m = 1, 2, \ldots$. By Eq. (4.1), $\sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m$ is a Cauchy sequence in $\mathcal{H}_A$, so that there exists $h := \lim_{m\to\infty} \sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m$ in $\mathcal{H}_A$. Now, by the proof of Proposition 3.9, convergence in $\mathcal{H}^K$ implies (locally uniform) convergence in $\mathcal{C}(G_A/G_B, D)$ whence, for every $s = uG_B$ in G_A/G_B, we get

$$\begin{aligned}F(s) = \lim_{m\to\infty} \sum_{j=1}^{n(m)} K_{\xi_j^m}(s) &= \lim_{m\to\infty} \sum_{j=1}^{n(m)} \big[\big(u, P\big(\pi_A(u)^{-1}\pi_A\big(v_j^m\big) f_j^m\big)\big)\big] \\ &= \lim_{m\to\infty} \left[\left(u, P\left(\pi_A(u)^{-1} \sum_{j=1}^{n(m)} \pi_A\big(v_j^m\big) f_j^m\right)\right)\right]\end{aligned}$$

in D_s. On the other hand, since the norm in D_s is the copy of the norm in $\mathcal{H}_A$, through the action of the basic mapping Φ associated with data (π_A, π_B, P) (see Example 2.5 and the bottom of Definition 2.10), we also have $\lim_{m\to\infty}[(u, P(\pi_A(u)^{-1} \sum_{j=1}^{n(m)} \pi_A(v_j^m) f_j^m))] = [(u, P(\pi_A(u)^{-1} h))]$. Thus we have shown that $F = F_h$. Also, for arbitrary $h \in \mathcal{H}_A$,

$$F_h = 0 \;\Leftrightarrow\; (\forall u \in G_A) \quad P\big(\pi_A\big(u^{-1}\big)h\big) = 0 \;\Leftrightarrow\; (\forall u \in G_A) \quad \pi_A\big(u^{-1}\big)h \perp \mathcal{H}_B.$$

Since π_A is a $*$-representation, it then follows that $F_h = 0$ if and only if $h \perp \operatorname{span}(\pi_A(G_A)\mathcal{H}_B)$. Hence $\mathcal{H}_A/([\operatorname{span}(\pi_A(G_A)\mathcal{H}_B)]^{\perp}) = \mathcal{H}^K = \{F_h \mid h \in \overline{\operatorname{span}}(\pi_A(G_A)\mathcal{H}_B)\}$.

Finally, note that $\mathcal{H}^K \subset \mathcal{C}^\infty(G_A/G_B, D)$ indeed, by definition of F_h ($h \in \mathcal{H}_A$). In the case where G_A and G_B are complex Banach–Lie groups then $\mathcal{H}^K \subset \mathcal{O}(G_A/G_B, D)$, by the definition of F_h as well.

Clearly, the mapping $F_h \mapsto h, \mathcal{H}^K \to \overline{\mathrm{span}}(\pi_A(G_A)\mathcal{H}_B)\} \subset \mathcal{H}_A$ is an isometry, which we denote by W, such that $W(K_\eta) = \pi_A(v)f$ for $\eta = [(v, f)] \in D$. In addition, if $\overline{\mathrm{span}}\,\pi_A(\mathrm{G}_A)\mathcal{H}_B = \mathcal{H}_A$ then the operator W is unitary. Recall the mapping $\widehat{K} : D \to \mathcal{H}^K$ given by $\widehat{K}(\xi) = K_\xi$ if $\xi \in D$, as in (3.2). Clearly $W \circ \widehat{K} = \Theta$, where Θ is the basic mapping for the data (π_A, π_B, P) (see Definition 2.10). □

The following result is an extension of [11, Theorem 5.4] and provides geometric realizations for $*$-representations of involutive Banach–Lie groups.

Theorem 4.2. *In the preceding setting, the following assertions hold*:

(a) *The linear operator*

$$\gamma : \mathcal{H}_A \to \mathcal{H}^K \subset \mathcal{C}^\infty(G_A/G_B, D), \quad \big(\gamma(h)\big)(uG_B) = \big[\big(u, P\big(\pi_A\big(u^{-1}\big)h\big)\big)\big],$$

satisfies $\mathrm{Ker}\,\gamma = (\mathrm{span}(\pi_A(G_B)\mathcal{H}_B))^\perp$ *and the operator* $\iota := \gamma \circ W$ *is the canonical inclusion* $\mathcal{H}^K \hookrightarrow \mathcal{C}^\infty(G_A/G_B, D)$. *Moreover,* $\gamma \circ \Theta = \widehat{K}$.

(b) *For every point* $t \in G_A/G_B$ *the evaluation map* $\mathrm{ev}_t^\iota = \iota(\cdot)(t) : \mathcal{H}^K \to D_t$ *is a continuous linear operator such that*

$$(\forall s, t \in G_A/G_B) \quad K(s, t^{-*}) = \mathrm{ev}_s^\iota \circ \big(\mathrm{ev}_t^\iota\big)^{-*}.$$

(c) *The mapping* γ *is a realization operator in the sense that it is an intertwiner between the* $*$*-representation* $\pi_A : G_A \to \mathcal{B}(\mathcal{H}_A)$ *and the natural representation of* G_A *on the space of cross sections* $\mathcal{C}^\infty(G_A/G_B, D)$.

Proof. (a) This part is just a reformulation of what has been shown prior to the statement of the theorem. The equality $\gamma \circ \Theta = \widehat{K}$ is obvious.

(b) Let $t \in G_A/G_B$ be arbitrary and then pick $u \in G_A$ such that $t = uG_B$. In particular, once the element u is chosen, we get a norm on the fiber D_t (see Definition 2.10) and then for every $F = F_h \in \mathcal{H}^K$, where $h \in \overline{\mathrm{span}}\,\pi_A(G_A)\mathcal{H}_B$, we have

$$\begin{aligned}\big\|\mathrm{ev}_t^\iota(F_h)\big\|_{D_t} &= \big\|\iota F_h(t)\big\|_{D_t} = \big\|\big[\big(u, P\big(\pi_A(u)^{-1}h\big)\big)\big]\big\|_{D_t} \\ &= \big\|\pi_A(u)P\big(\pi_A(u)^{-1}h\big)\big\|_{\mathcal{H}_A} \leqslant \big\|\pi_A(u)\big\| \cdot \big\|\pi_A\big(u^{-1}\big)\big\| \cdot \|h\|_{\mathcal{H}_A} = C_u\|F_h\|_{\mathcal{H}^K},\end{aligned}$$

so that the evaluation map $\mathrm{ev}_t^\iota : \mathcal{H}_A \to D_t$ is continuous.

Let us keep $s = uG_B$ fixed for the moment. We first prove that

$$(\widehat{K}|_{D_s})^{-*} = \mathrm{ev}_{s^{-*}}^\iota : \mathcal{H}^K \to D_{s^{-*}}. \tag{4.2}$$

To this end we check that condition (3.3) in Definition 3.4 is satisfied with $\theta_s = \mathrm{ev}_{s^{-*}}^\iota : \mathcal{H}^K \to D_{s^{-*}}$. In fact, let $\xi = [(u, f)] \in D_s$ arbitrary.

Then for all $h \in \overline{\mathrm{span}}(\pi_A(G_A)\mathcal{H}_B)$ we have

$$\begin{aligned}
(\xi \mid \theta_s F_h)_{s,s^{-*}} &= \big(\big[(u,f)\big] \mid \big[\big(u^{-*}, P\big(\pi_A\big(\big(u^{-*}\big)^{-1}\big)h\big)\big)\big]\big)_{s,s^{-*}} \\
&= \big(\pi_A(u)f \mid \pi_A\big(u^{-*}\big)P\big(\pi_A\big(\big(u^{-*}\big)^{-1}\big)h\big)\big)_{\mathcal{H}_A} \\
&= \big(f \mid P\big(\pi_A(u^*)h\big)\big)_{\mathcal{H}_A} = \big(\pi_A(u)f \mid h\big)_{\mathcal{H}_A} \\
&= \big(W(\gamma\circ\Theta)(\xi) \mid W\big(\gamma(h)\big)\big)_{\mathcal{H}_A} = \big((\gamma\Theta)(\xi) \mid F_h\big)_{\mathcal{H}^K} \\
&= \big(\widehat{K}(\xi) \mid F_h\big)_{\mathcal{H}^K} = (K_\xi \mid F_h)_{\mathcal{H}^K}.
\end{aligned}$$

Now let $s, t \in G_A/G_B$ be arbitrary and $u, v \in G_A$ such that $s = uG_B$ and $t = vG_B$. It follows by (4.2) that $(\mathrm{ev}_t^\iota)^{-*} = \widehat{K}_{D_{t^{-*}}}$, hence for every $\eta = [(v^{-*}, f)] \in D_{t^{-*}}$ we have

$$\mathrm{ev}_s^\iota \circ \big(\mathrm{ev}_t^\iota\big)^{-*}\eta = \mathrm{ev}_s^\iota\big(\widehat{K}(\eta)\big) = \big(\iota(K_\eta)\big)(s) = \big[\big(u, P\big(\pi_A\big(u^{-1}\big)\pi_A\big(v^{-*}\big)f\big)\big)\big] = K\big(s, t^{-*}\big)\eta.$$

(c) Let $h \in \mathcal{H}_A$ and $v \in G_A$ arbitrary. Then at every point $t = uG_B \in G_A/G_B$ we have

$$\begin{aligned}
\big(\gamma\big(\pi_A(v)h\big)\big)(t) &= \big[\big(u, P\big(\pi_A\big(u^{-1}\big)\pi_A(v)h\big)\big)\big] = \big[\big(u, P\big(\pi_A\big(\big(v^{-1}u\big)^{-1}\big)h\big)\big)\big] \\
&= v\cdot\big[\big(v^{-1}u, P\big(\pi_A\big(\big(v^{-1}u\big)^{-1}\big)h\big)\big)\big] = v\cdot\big(\gamma(h)\big)\big(v^{-1}t\big)
\end{aligned}$$

and the proof ends. □

Part (c) of the above theorem tells us that it is possible to realize representations like $\pi_A : G_A \to \mathcal{B}(\mathcal{H}_A)$ as natural actions on spaces of analytic sections. We next take advantage of this geometric model to point out some phenomena of holomorphic extension in bundle vectors and sections of them. Firstly, we record some auxiliary facts in the form of a lemma.

Lemma 4.3. *Let G_A be an involutive Banach–Lie group and G_B an involutive Banach–Lie subgroup of G_A, and denote by $\beta : (v, uG_B) \mapsto vuG_B$, $G_A \times G_A/G_B \to G_A/G_B$, the corresponding transitive action. Also denote $U_X = \{u \in G_X \mid u^{-*} = u\}$ for $X \in \{A, B\}$. Then the following assertions hold*:

(a) *There exists a correctly defined involutive diffeomorphism*

$$z \mapsto z^{-*}, \quad G_A/G_B \to G_A/G_B,$$

defined by $uG_B \mapsto u^{-*}G_B$. This diffeomorphism has the property $\beta(v^{-*}, z^{-*}) = \beta(v, z)^{-*}$ whenever $v \in G_A$ and $z \in G_A/G_B$.

(b) *The group U_X is a Banach–Lie subgroup of G_X for $X \in \{A, B\}$ and U_B is a Banach–Lie subgroup of U_A.*

(c) *If $G_B^+ = G_A^+ \cap G_B$, then the mapping*

$$\lambda : uU_B \mapsto uG_B, \quad U_A/U_B \to G_A/G_B,$$

is a diffeomorphism of U_A/U_B onto the fixed-point submanifold of the involutive diffeomorphism of G_A/G_B introduced above in assertion (a).

Proof. Assertion (a) follows since the mapping $u \mapsto u^{-*}$ is an automorphism of G_A (Remark 2.7). The proof of assertion (b) is straightforward.

As regards (c), what we really have to prove is the equality $\lambda(U_A/U_B) = \{z \in G_A/G_B \mid z^{-*} = z\}$. The inclusion $\subseteq$ is obvious. Conversely, let $z \in G_A/G_B$ with $z^{-*} = z$. Pick $u \in G_A$ arbitrary such that $z = uG_B$. Since $z^{-*} = z$, it follows that $u^{-1}u^{-*} \in G_B$. On the other hand, $u^{-1}u^{-*} \in G_A^+$, hence the hypothesis $G_B^+ = G_A^+ \cap G_B$ implies that $u^{-1}u^{-*} \in G_B^+$. That is, there exists $w \in G_B$ such that $u^{-1}u^{-*} = ww^*$. Hence $uw = u^{-*}(w^*)^{-1}$, so that $uw = (uw)^{-*}$. Consequently $uw \in U_A$, and in addition $z = uG_B = uwG_B = \lambda(uwU_B)$. □

The next theorem gives a holomorphic extension of the Hermitian vector bundles and kernels introduced in [11].

Theorem 4.4. *For $X \in \{A, B\}$, let G_X be a complex Banach–Lie group and G_B a Banach–Lie subgroup of G_A. As above, set $U_X = \{u \in G_X \mid u^{-*} = u\}$. Let $\pi_X : X \to \mathcal{B}(\mathcal{H}_X)$ be a holomorphic $*$-representation such that $\pi_B(u) = \pi_A(u)|_{\mathcal{H}_B}$ for all $u \in G_B$. Denote by $\Pi : D \to G_A/G_B$ the like-Hermitian vector bundle, K the reproducing $(-*)$-kernel, and $W : \mathcal{H}^K \to \mathcal{H}_A$ the isometry and $\gamma : \mathcal{H}_A \to \mathcal{C}^\infty(G_A/G_B, D)$ the realization operator associated with the data (π_A, π_B, P), where $P : \mathcal{H}_A \to \mathcal{H}_B$ is the orthogonal projection.*

Also denote by $\Pi^U : D^U \to U_A/U_B$ the like-Hermitian vector bundle, K^U the reproducing $(-*)$-kernel, and $W^U : \mathcal{H}^{K^U} \to \mathcal{H}_A$ the isometry and $\gamma^U : \mathcal{H}_A \to \mathcal{C}^\infty(U_A/U_B, D)$ the operators associated with the data $(\pi_A|_{U_A}, \pi_B|_{U_B}, P)$:

Assume in addition that $G_B^+ = G_A^+ \cap G_B$. Then the following assertions hold:

(a) *The inclusion $\iota := \gamma \circ W : \mathcal{H}^K \to \mathcal{O}(G_A/G_B, D)$ is the realization operator associated with the reproducing $(-*)$-kernel K. Moreover, γ intertwines the $*$-representation $\pi_A : G_A \to \mathcal{B}(\mathcal{H}_A)$ and the natural representation of G_A on the space of cross sections $\mathcal{O}(G_A/G_B, D)$.*
(b) *The like-Hermitian vector bundle $\Pi^U : D^U \to U_A/U_B$ is actually a Hermitian vector bundle. The mapping $\lambda : uU_B \mapsto uG_B$, $U_A/U_B \hookrightarrow G_A/G_B$, is a diffeomorphism of U_A/U_B onto a submanifold of G_A/G_B and we have*

$$\lambda(U_A/U_B) = \{z \in G_A/G_B \mid z^{-*} = z\}. \tag{4.3}$$

In addition, there exists an U_A-equivariant real analytic embedding $\Lambda : D^U \to D$ such that the diagrams

$$\begin{tikzcd} D^U \arrow[r, "\Lambda"] \arrow[d, "\Pi^U"'] & D \arrow[d, "\Pi"] \\ U_A/U_B \arrow[r, "\lambda"] & G_A/G_B \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} D^U \arrow[r, "\Lambda"] & D \\ U_A/U_B \arrow[r, "\lambda"] \arrow[u, "\gamma^U(h)"] & G_A/G_B \arrow[u, "\gamma(h)"'] \end{tikzcd}$$

for arbitrary $h \in \overline{\operatorname{span}}(\pi_A(G_A)\mathcal{H}_B)$ are commutative, the mapping Λ is a fiberwise isomorphism, and $\Lambda(D^U) = \Pi^{-1}(\lambda(U_A/U_B))$.
(c) *The inclusion $\iota^U := \gamma^U \circ W : \mathcal{H}^K \to \mathcal{C}^\omega(U_A/U_B, D^U)$ is the realization operator associated with the reproducing kernel K^U, where $\mathcal{C}^\omega(U_A/U_B, D^U)$ is the subspace of $\mathcal{C}^\infty(U_A/U_B, D^U)$ of real analytic sections. In addition, γ^U is an intertwiner between the unitary representation $\pi_A : U_A \to \mathcal{B}(\mathcal{H}_A)$ and the natural representation of U_A on the space of cross sections $\mathcal{C}^\omega(U_A/U_B, D^U)$.*

Proof. (a) This follows by Theorem 4.2 applied to the data (π_A, π_B, P). The fact that the range of the realization operator ι consists only of holomorphic sections follows either by Proposition 3.9(c) or directly by the definition of γ (see Theorem 4.2(a)).

(b) The fact that Π^U is a Hermitian vector bundle (Remark 2.4) follows by (2.2). Moreover, the asserted properties of λ follow by Lemma 4.3(c) as we are assuming that $G_B^+ = G_B \cap G_A^+$. As regards the U_A-equivariant embedding $\Lambda : D^U \to D$, it can be defined as the mapping that takes every equivalence class $[(u, f)] \in D^U$ into the equivalence class $[(u, f)] \in D$. Then Λ clearly has the wished-for properties.

(c) Use again Theorem 4.2 for the data $(\pi_A|_{U_A}, \pi_B|_{U_B}, P)$. It is clear from definitions (see also [11]) that K^U is a reproducing *kernel* indeed. The fact that the range of γ^U, or ι^U, consists of real analytic sections follows by the definition of γ^U (see Theorem 4.2(a) again). Alternatively, one can use assertions (a) and (b) above to see that for arbitrary $h \in \mathcal{H}_A$ the mapping $\Lambda \circ \gamma^U(h) \circ \lambda^{-1} : \lambda(U_A/U_B) \to D$ is real analytic since it is a section of Π over $\lambda(U_A/U_B)$ which extends to the *holomorphic* section $\gamma(h) : G_A/G_B \to D$. □

Remark 4.5. Theorem 4.4(b) says that the image of Λ is precisely the restriction of Π to the fixed-point set of the involution on the base G_A/G_B, and this restriction is a Hermitian vector bundle. This remark along with the alternative proof of assertion (c) show that there exists a close relationship between the setting of Theorem 4.4 and the circle of ideas related to complexifications of real analytic manifolds, and in particular complexifications of compact homogeneous spaces (see for instance [34,48,62] and the references therein for the case of finite-dimensional manifolds). Specifically, the manifold U_A/U_B can be identified with the fixed-point set of the antiholomorphic involution $z \mapsto z^{-*}$ of G_A/G_B. Thus we can view G_A/G_B as a *complexification* of U_A/U_B. By means of this identification, we can say that for arbitrary $h \in \mathcal{H}_A$ the real analytic section $\gamma^U(h) : U_A/U_B \to D^U$ can be holomorphically extended to the section $\gamma(h) : G_A/G_B \to D$.

In the next section we analyze the complex structure of G_A/G_B (with self-conjugate space U_A/U_B) in more detail, when G_X is the group of invertibles of a C^*-algebra $X = A$ or B and there exists a conditional expectation $E : A \to B$.

5. Complexifications in the C^*-algebra case

The complexification G_A/G_B of U_A/U_B can be suitably displayed in a C^*-algebra setting. Assume that $\mathbf{1} \in B \subseteq A$ are two C^*-algebras such that there exists a conditional expectation $E : A \to B$ from A onto B. Denote the groups of invertible elements in A and B by G_A and G_B, respectively, and consider the quotient map $q : a \mapsto a\mathrm{G}_B$, $\mathrm{G}_A \to \mathrm{G}_A/\mathrm{G}_B$.

Theorem 5.1. *Let* $\mathfrak{p} := (\operatorname{Ker} E) \cap \mathfrak{u}_A$, *which is a real Banach space acted on by* U_B *by means of the adjoint action* $(u, X) \mapsto uXu^{-1}$. *Consider the corresponding quotient map* $\kappa : (u, X) \mapsto [(u, X)]$, $\mathrm{U}_A \times \mathfrak{p} \to \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}$, *and define the mapping* $\Psi_0^E : (u, X) \mapsto u \exp(\mathrm{i}X)$, $\mathrm{U}_A \times \mathfrak{p} \to \mathrm{G}_A$. *Then there is a unique* U_A*-equivariant, real analytic diffeomorphism* $\Psi^E : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_B$ *such that the diagram*

$$\begin{tikzcd} \mathrm{U}_A \times \mathfrak{p} \arrow[r, "\Psi_0^E"] \arrow[d, "\kappa"'] & \mathrm{G}_A \arrow[d, "q"] \\ \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \arrow[r, "\Psi^E"] & \mathrm{G}_A/\mathrm{G}_B \end{tikzcd}$$

is commutative. Thus the complex homogeneous space $\mathrm{G}_A/\mathrm{G}_B$ *has the structure of a* U_A-*equivariant real vector bundle over its real form* $\mathrm{U}_A/\mathrm{U}_B$, *the corresponding projection being given by the composition* (*depending on the conditional expectation* E)

$$\mathrm{G}_A/\mathrm{G}_B \xrightarrow{(\Psi^E)^{-1}} \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \xrightarrow{\Xi} \mathrm{U}_A/\mathrm{U}_B,$$

where the typical fiber of the vector bundle $\Xi \circ (\Psi^E)^{-1}$ *is the real Banach space* $\mathfrak{p} = (\operatorname{Ker} E) \cap \mathfrak{u}_A$.

Proof. The uniqueness of Ψ^E follows since the mapping κ is surjective. For the existence of Ψ^E, note that for all $u \in \mathrm{U}_A$, $v \in \mathrm{U}_B$, and $X \in \mathfrak{p}$ we have

$$\begin{aligned} q\big(\Psi_0^E\big(uv, v^{-1}Xv\big)\big) &= q\big(uv \cdot \exp\big(\mathrm{i}v^{-1}Xv\big)\big) = q\big(uv \cdot v^{-1}\exp(\mathrm{i}X)v\big) = q\big(u\exp(\mathrm{i}X)v\big) \\ &= u\exp(\mathrm{i}X)vG_B = u\exp(\mathrm{i}X)G_B = q\big(\Psi_0^E(u, X)\big). \end{aligned}$$

This shows that the mapping

$$\Psi^E : \big[(u, X)\big] \mapsto u\exp(\mathrm{i}X)\mathrm{G}_B, \quad \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_B, \tag{5.1}$$

is well defined, and it is clearly U_A-equivariant. Moreover, since κ is a submersion and $\Psi^E \circ \kappa$ ($= q \circ \Psi_0^E$) is a real analytic mapping, it follows by Corollary 8.4(i) in [65] that Ψ^E is real analytic.

Now we prove that Ψ^E is bijective. To this end we need the following fact:

$$\text{for all } a \in \mathrm{G}_A \text{ there exists a unique } (u, X, b) \in \mathrm{U}_A \times \mathfrak{p} \times \mathrm{G}_B^+ \text{ such that } a = u \cdot \exp(\mathrm{i}X) \cdot b \tag{5.2}$$

(see [53, Theorem 8]). It follows by (5.1) and (5.2) that the mapping $\Psi^E : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_B$ is surjective. To see that it is also injective, assume that $u_1\exp(\mathrm{i}X_1)\mathrm{G}_B = u_2\exp(\mathrm{i}X_2)\mathrm{G}_B$, where $(u_j, X_j) \in \mathrm{U}_A \times \mathfrak{p}$ for $j = 1, 2$. Then there exists $b_1 \in \mathrm{G}_B$ such that $u_1\exp(\mathrm{i}X_1)b_1 = u_2\exp(\mathrm{i}X_2)$. Let $b_1 = vb$ be the polar decomposition of $b_1 \in \mathrm{G}_B$, where $v \in \mathrm{U}_B$ and $b \in \mathrm{G}_B^+$. Then

$$u_1\exp(\mathrm{i}X_1)b_1 = u_1\exp(\mathrm{i}X_1)vb = u_1v\exp\big(\mathrm{i}v^{-1}X_1v\big)b.$$

Note that $u_1v \in \mathrm{U}_A$ and $v^{-1}X_1v \in \mathfrak{p}$ since $E(v^{-1}X_1v) = v^{-1}E(X_1)v = 0$. Since $u_1\exp(\mathrm{i}X_1)b_1 = u_2\exp(\mathrm{i}X_2)$, it then follows by the uniqueness assertion in (5.2) that $u_2 = u_1v$ and $X_2 = v^{-1}X_1v$. Hence $[(u_1, X_1)] = [(u_2, X_2)]$, and thus the mapping $\Psi^E : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_B$ is injective as well.

Finally, we show that the inverse function

$$\left(\Psi^E\right)^{-1}: a\mathrm{G}_B = u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto \left[(u,X)\right], \quad \mathrm{G}_A/\mathrm{G}_B \to \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}$$

is also smooth. For this, note that u and X in (5.2) depend on a in a real analytic fashion (see [53]). Hence, the mapping $\sigma : a \mapsto [(u,X)]$, $\mathrm{G}_A \to \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}$ is smooth. Since $\sigma = (\Psi^E)^{-1} \circ q$ and q is a submersion, it follows again from Corollary 8.4(i) in [65] that $(\Psi^E)^{-1}$ is smooth. In conclusion, Ψ^E is a real analytic diffeomorphism (see [9, p. 268]), as we wanted to show. □

Remark 5.2. From the observation in the second part of the above statement, it follows that the mapping $\Xi \circ (\Psi^E)^{-1}$ can be thought of as an infinite-dimensional version of Mostow fibration; see [42,43] and Section 3 in [14] for more details on the finite-dimensional setting. See also Theorem 1 in [39, Section 3] for a related property of complexifications of compact symmetric spaces.

In fact the construction of the diffeomorphism Ψ^E in Theorem 5.1 relies on the representation (5.2), and so it depends on the decomposition of A obtained in terms of the expectation E, see [53]. It is interesting to see how Ψ^E depends explicitly on E at the level of tangent maps: we have

$$T_{(\mathbf{1},0)}\kappa : (Z,Y) \mapsto \big((\mathbf{1}-E)Z, Y\big), \quad \mathfrak{u}_A \times \mathfrak{p} \to T_{[(\mathbf{1},0)]}(\mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}) \simeq \mathfrak{p} \times \mathfrak{p},$$
$$T_{(\mathbf{1},0)}\big(\Psi_0^E\big) : (Z,Y) \mapsto Z + \mathrm{i}Y, \quad \mathfrak{u}_A \times \mathfrak{p} \to A,$$
$$T_{\mathbf{1}}q : Z \mapsto (\mathbf{1}-E)Z, \quad A \to \operatorname{Ker} E,$$

hence $T_{[(\mathbf{1},0)]}(\Psi^E)((\mathbf{1}-E)Z, Y) = (\mathbf{1}-E)(Z+\mathrm{i}Y) = (\mathbf{1}-E)Z + \mathrm{i}Y$ whenever $Z \in \mathfrak{u}_A$ and $Y \in \mathfrak{p}$. Thus

$$T_{[(\mathbf{1},0)]}\big(\Psi^E\big) : (Y_1, Y_2) \mapsto Y_1 + \mathrm{i}Y_2, \quad \mathfrak{p} \times \mathfrak{p} \to \operatorname{Ker} E,$$

which is an isomorphism of real Banach spaces since $\operatorname{Ker} E = \mathfrak{p} \dotplus \mathrm{i}\mathfrak{p}$.

Corollary 5.3. *Let A and B two C^*-algebras as in the preceding theorem. Then $\mathrm{G}_A/\mathrm{G}_B \simeq \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}$ is a complexification of $\mathrm{U}_A/\mathrm{U}_B$ with respect to the anti-holomorphic involutive diffeomorphism*

$$u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto u\exp(-\mathrm{i}X)\mathrm{G}_B, \quad \mathrm{G}_A/\mathrm{G}_B \to \mathrm{G}_A/\mathrm{G}_B$$

where $u \in \mathrm{U}_A$, $X \in \mathfrak{p}$ (alternatively, $[(u,X)] \mapsto [(u,-X)]$).

Proof. First, note that $\mathrm{G}_B^+ = \mathrm{G}_B \cap \mathrm{G}_A^+$. This is a direct consequence of the fact that the C^*-algebras are closed under taking square roots of positive elements. So Theorem 5.1 applies to get $\mathrm{U}_A/\mathrm{U}_B$ as the set of fixed points of the mapping $a\mathrm{G}_B \mapsto a^{-*}\mathrm{G}_B$ on G_A/G_B, where $a^{-*} := (a^{-1})^*$ for $a \in A$, and $*$ is the involution in A. By (5.2), every element $a\mathrm{G}_B$ in $\mathrm{G}_A/\mathrm{G}_B$ is of the form $a\mathrm{G}_B = u\exp(\mathrm{i}X)\mathrm{G}_B$ with $u \in \mathrm{U}_A$ and $X \in \mathfrak{p}$, and the correspondence $u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto [(u,X)]$ is a bijection. But then $(u\exp(\mathrm{i}X))^{-*} = u\exp(-\mathrm{i}X)$ since $X^* = -X$, and the proof ends. □

To put Theorem 5.1 and Corollary 5.3 in a proper perspective, we recall that for $X \in \{A, B\}$ the Banach–Lie group G_X is the universal complexification of U_X (see [46, Example VI.9], and also [32]). Besides this, we have seen in Theorem 4.4 that the homogeneous space $\mathrm{G}_A/\mathrm{G}_B$ is a complexification of $\mathrm{U}_A/\mathrm{U}_B$. Now Corollary 5.3 implements such a complexification in the explicit terms of a sort of polar decomposition (if $X \in \mathfrak{p}$ then $\exp(\mathrm{i}X)^* = \exp(-\mathrm{i}(-X)) = \exp(\mathrm{i}X)$ whence $\exp(\mathrm{i}X) = \exp(\mathrm{i}X/2)\ \exp(\mathrm{i}X/2)^*$ is positive). For the group case, see [32].

Remark 5.4. It is to be noticed that there is an alternative way to express the involution mapping considered in this section as multiplication by positive elements. This representation was suggested by Axiom 4 for involutions of homogeneous reductive spaces as studied in the paper [41].

Specifically, under the conditions assumed above the following condition is satisfied:

$$(\forall a \in \mathrm{G}_A)\big(\exists a_+ \in \mathrm{G}_A^+,\ b_+ \in \mathrm{G}_B^+\big) \quad a^{-*} = a_+ a b_+. \tag{5.3}$$

To see this first note that we can assume $\|a^*\| < \sqrt{2}$. Then, if $\mathcal{H}$ is a Hilbert space such that A is embedded in $\mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$,

$$\|a^*ax\|^2 < 2\|ax\|^2 \quad \Leftrightarrow \quad (x - a^*ax \mid x - a^*ax)_{\mathcal{H}} < (x \mid x)_{\mathcal{H}} \quad \Leftrightarrow \quad \big\|(\mathbf{1} - a^*a)x\big\|^2 < \|x\|^2.$$

Thus $\|\mathbf{1} - a^*a\| < 1$ and so $\|\mathbf{1} - E(a^*a)\| = \|E(\mathbf{1}) - E(a^*a)\| < 1$, whence $b_+ := E(a^*a) \in \mathrm{G}_B^+$. Now it is clear that (5.6) holds with $a_+ := (a^{-*})b_+^{-1}a^{-1} \in \mathrm{G}_A^+$.

As a consequence of (5.6), we have that $a^{-*}\mathrm{G}_B = a_+a\mathrm{G}_B$ for every $a \in \mathrm{G}_B$. Let us see the correspondence of such an identity with the decomposition of $a^{-*}\mathrm{G}_B$ given in Theorem 5.1. Since $a = ue^{iX}b$ in (5.5), we have $a^*a = (be^{iX}u^{-1})(ue^{iX}b) = be^{2iX}b$ and so $E(a^*a) = bE(e^{2iX})b$. It follows that $a^{-*} = a_+ab_+$ where $b_+ = bE(e^{2iX})b$ and $a_+ = ue^{-iX}b^{-2}E(e^{2iX})^{-1}b^{-2}e^{-iX}u^{-1}$.

There is a natural identification between the vector bundle $\Xi : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{U}_A/\mathrm{U}_B$ and the tangent bundle $T(\mathrm{U}_A/\mathrm{U}_B) \to \mathrm{U}_A/\mathrm{U}_B$. In view of Theorem 5.1, we get an interesting interpretation of the homogeneous space $\mathrm{G}_A/\mathrm{G}_B$ as the tangent bundle of $\mathrm{U}_A/\mathrm{U}_B$.

Corollary 5.5. *In the above notation, the vector bundle* $\Xi : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{U}_A/\mathrm{U}_B$ *is* U_A-*equivariantly isomorphic to the tangent bundle* $T(\mathrm{U}_A/\mathrm{U}_B) \to \mathrm{U}_A/\mathrm{U}_B$. *Hence, the composition*

$$\mathrm{G}_A/\mathrm{G}_B \xrightarrow{(\Psi^E)^{-1}} \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \xrightarrow{\simeq} T(\mathrm{U}_A/\mathrm{U}_B)$$

defines a U_A-*equivariant diffeomorphism between the complexification* $\mathrm{G}_A/\mathrm{G}_B$ *and the tangent bundle* $T(\mathrm{U}_A/\mathrm{U}_B)$ *of the homogeneous space* $\mathrm{U}_A/\mathrm{U}_B$.

Proof. Let $\alpha : (u, v\mathrm{U}_B) \mapsto uv\mathrm{U}_B$, $\mathrm{U}_A \times \mathrm{U}_A/U_B \to \mathrm{U}_A/\mathrm{U}_B$. Then let $p_0 = \mathbf{1}\mathrm{U}_B \in \mathrm{U}_A/\mathrm{U}_B$ and $\partial_2\alpha : \mathrm{U}_A \times T(\mathrm{U}_A/\mathrm{U}_B) \to T(\mathrm{U}_A/\mathrm{U}_B)$ the partial derivative of α with respect to the second variable. Since $T_{p_0}(\mathrm{U}_A/\mathrm{U}_B) \simeq \mathfrak{p}$, by restricting $\partial_2\alpha$ to $\mathrm{U}_A \times T_{p_0}(\mathrm{U}_A/U_B)$ we get a mapping $\alpha_0^E : \mathrm{U}_A \times \mathfrak{p} \to T(\mathrm{U}_A/\mathrm{U}_B)$. Then it is straightforward to show that there exists a unique U_A-equivariant diffeomorphism $\alpha^E : \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to T(\mathrm{U}_A/\mathrm{U}_B)$ such that $\alpha^E \circ \kappa = \alpha_0^E$.

Now it follows by Theorem 5.1 that the composition $\mathrm{G}_A/\mathrm{G}_B \xrightarrow{(\Psi^E)^{-1}} \mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \xrightarrow{\alpha^E} T(\mathrm{U}_A/\mathrm{U}_B)$ defines a U_A-equivariant diffeomorphism between the complexification $\mathrm{G}_A/\mathrm{G}_B$ and the tangent bundle $T(\mathrm{U}_A/\mathrm{U}_B)$ of the homogeneous space $\mathrm{U}_A/\mathrm{U}_B$. □

Remark 5.6. It is known that conditional expectations can be regarded as connection forms of principal bundles, see [2,21,30]. Thus Corollary 5.5 leads to numerous examples of real analytic Banach manifolds whose tangent bundles have complex structures associated with certain connections. See for instance [13,40,62] for the case of finite-dimensional manifolds.

6. Stinespring representations

In this section we are going to apply the preceding theory of reproducing $(-*)$-kernels, for homogeneous like-Hermitian bundles, to explore the differential geometric background of completely positive maps. Thus we shall find geometric realizations of the Stinespring representations which will entail an unexpected bearing on the Stinespring dilation theory. Specifically, it will follow that the classical constructions of extensions of representations and induced representations of C^*-algebras (see [25,54], respectively), which seemed to pass beyond the realm of geometric structures, actually have geometric interpretations in terms of reproducing kernels on vector bundles. See Remark 6.9 below for some more details.

Notation 6.1. For every linear map $\Phi : X \to Y$ between two vector spaces and every integer $n \geqslant 1$ we denote $\Phi_n = \Phi \otimes \mathrm{id}_{M_n(\mathbb{C})} : M_n(X) \to M_n(Y)$, that is, $\Phi_n((x_{ij})_{1\leqslant i,j\leqslant n}) = (\Phi(x_{ij}))_{1\leqslant i,j\leqslant n}$ for every matrix $(x_{ij})_{1\leqslant i,j\leqslant n} \in M_n(X)$.

Definition 6.2. Let A_1 and A_2 be two unital C^*-algebras and $\Phi : A_1 \to A_2$ a linear map. We say that Φ is *completely positive* if for every integer $n \geqslant 1$ the map $\Phi_n : M_n(A_1) \to M_n(A_2)$ is positive in the sense that it takes positive elements in the C^*-algebra $M_n(A_1)$ to positive ones in $M_n(A_2)$.

If moreover $\Phi(\mathbf{1}) = \mathbf{1}$ then we say that Φ is unital and in this case we have $\|\Phi_n\| = 1$ for every $n \geqslant 1$ by the Russo–Dye theorem (see e.g., [50, Corollary 2.9]).

Definition 6.3. Let A be a unital C^*-algebra, $\mathcal{H}_0$ a complex Hilbert space and $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ a unital completely positive map. Define a nonnegative sesquilinear form on $A \otimes \mathcal{H}_0$ by the formula

$$\left(\sum_{j=1}^{n} b_j \otimes \eta_j \mid \sum_{i=1}^{n} a_i \otimes \xi_i\right) = \sum_{i,j=1}^{n} \big(\Phi\big(a_i^* b_j\big)\eta_j \mid \xi_i\big)$$

for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}_0$ and $n \geqslant 1$. In particular

$$\left(\sum_{j=1}^{n} b_j \otimes \eta_j \mid \sum_{j=1}^{n} b_j \otimes \eta_j\right) = \big(\Phi_n\big(\big(b_i^* b_j\big)_{1\leqslant i,j\leqslant n}\big)\eta \mid \eta\big), \tag{6.1}$$

where $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in M_{n,1}(\mathbb{C}) \otimes \mathcal{H}_0$. Consider the linear space $N = \{x \in A \otimes \mathcal{H}_0 \mid (x \mid x) = 0\}$ and denote by $\mathcal{K}_0$ the Hilbert space obtained as the completion of $(A \otimes \mathcal{H}_0)/N$ with respect to the scalar product defined by $(\cdot \mid \cdot)$ on this quotient space.

On the other hand define a representation $\widetilde{\pi}$ of A by linear maps on $A \otimes \mathcal{H}_0$ by

$$(\forall a, b \in A)(\forall \eta \in \mathcal{H}_0) \quad \widetilde{\pi}(a)(b \otimes \eta) = ab \otimes \eta.$$

Then every linear map $\widetilde{\pi}(a): A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0$ induces a continuous map $(A \otimes \mathcal{H}_0)/N \to (A \otimes \mathcal{H}_0)/N$, whose extension by continuity will be denoted by $\pi_\Phi(a) \in \mathcal{B}(\mathcal{K}_0)$. We thus obtain a unital $*$-representation $\pi_\Phi : A \to \mathcal{B}(\mathcal{K}_0)$ which is called the *Stinespring representation associated with* Φ.

Additionally, denote by $V : \mathcal{H}_0 \to \mathcal{K}_0$ the bounded linear map obtained as the composition

$$V : \mathcal{H}_0 \to A \otimes \mathcal{H}_0 \to (A \otimes \mathcal{H}_0)/N \hookrightarrow \mathcal{K}_0,$$

where the first map is defined by $A \ni h \mapsto \mathbf{1} \otimes h \in A \otimes \mathcal{H}_0$ and the second map is the natural quotient map. Then $V : \mathcal{H}_0 \to \mathcal{K}_0$ is an isometry satisfying $\Phi(a) = V^*\pi(a)V$ for all $a \in A$.

Remark 6.4. The construction sketched in Definition 6.3 essentially coincides with the proof of the Stinespring theorem on dilations of completely positive maps [60]; see for instance [27, Theorem 5.2.1] or [50, Theorem 4.1]. Minimal Stinespring representations are uniquely determined up to a unitary equivalence; see [50, Proposition 4.2].

We now start the preparations necessary for obtaining the realization theorem for Stinespring representations (Theorem 6.10).

Lemma 6.5. *Let $\Phi : A \to B$ be a unital completely positive map between two C^*-algebras. Then for every $n \geqslant 1$ and every $a \in M_n(A)$ we have $\Phi_n(a)^*\Phi_n(a) \leqslant \Phi_n(a^*a)$.*

Proof. Note that $\Phi_n : M_n(A) \to M_n(B)$ is in turn a unital completely positive map, hence after replacing A by $M_n(A)$, B by $M_n(B)$, and Φ by Φ_n, we may assume that $n = 1$. In this case we may assume $B \subseteq \mathcal{B}(\mathcal{H}_0)$ for some complex Hilbert space $\mathcal{H}_0$ and then, using the notation in Definition 6.3 we have

$$\Phi(a^*a) = V^*\pi_\Phi(a^*a)V = V^*\pi_\Phi(a)^*\mathrm{id}_{\mathcal{K}_0}\pi_\Phi(a)V \geqslant V^*\pi_\Phi(a)^*VV^*\pi_\Phi(a)V = \Phi(a)^*\Phi(a),$$

where the second equality follows since the Stinespring representation $\pi_\Phi : A \to \mathcal{B}(\mathcal{K}_0)$ is in particular a $*$-homomorphism. See for instance [27, Corollary 5.2.2] for more details. □

For later use we now recall the theorem of Tomiyama on conditional expectations.

Remark 6.6. Let $\mathbf{1} \in B \subseteq A$ be two C^*-algebras and such that there exists a *conditional expectation* $E : A \to B$, that is, E is a linear map satisfying $E^2 = E$, $\|E\| = 1$ and $\operatorname{Ran} E = B$. Then for every $a \in A$ and $b_1, b_2 \in B$ we have $E(a^*) = E(a)^*$, $0 \leqslant E(a)^*E(a) \leqslant E(a^*a)$, and $E(b_1 a b_2) = b_1 E(a) b_2$. (See for instance [64] or [56].) Additionally, E is completely positive and $E(\mathbf{1}) = \mathbf{1}$, and this explains why E has the Schwarz property stated in the previous Lemma 6.5.

Lemma 6.7. *Assume that $\mathbf{1} \in B \subseteq A$ are C^*-algebras with a conditional expectation $E : A \to B$ and a unital completely positive map $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ satisfying $\Phi \circ E = \Phi$, where $\mathcal{H}_0$ is a complex Hilbert space. Denote by $\pi_A : A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$ the Stinespring representations associated with the unital completely positive maps Φ and $\Phi|_B$, respectively. Then $\mathcal{H}_B \subseteq \mathcal{H}_A$, and for every $h_0 \in \mathcal{H}_0$ and $b \in B$ we have the commutative diagrams*

$$\begin{tikzcd}
A \arrow[r, "\iota_{h_0}"] \arrow[d, "E"'] & \mathcal{H}_A \arrow[r, "\pi_A(b)"] \arrow[d, "P"'] & \mathcal{H}_A \arrow[d, "P"'] \\
B \arrow[r, "\iota_{h_0}"] & \mathcal{H}_B \arrow[r, "\pi_B(b)"] & \mathcal{H}_B
\end{tikzcd}$$

where $P : \mathcal{H}_A \to \mathcal{H}_B$ is the orthogonal projection, and $\iota_{h_0} : A \to \mathcal{H}_A$ is the map induced by $a \mapsto a \otimes h_0$.

Proof. We first check that the right-hand square is a commutative diagram. In fact, it is clear from the construction in Definition 6.3 that $\mathcal{H}_B \subseteq \mathcal{H}_A$ and for every $b \in B$ we have $\pi_A(b^*)|_{\mathcal{H}_B} = \pi_B(b^*)$. In other words, if we denote by $I : \mathcal{H}_B \hookrightarrow \mathcal{H}_A$ the inclusion map, then $\pi_A(b^*) \circ I = I \circ \pi_B(b^*)$. Now note that $I^* = P$ and take the adjoints in the previous equation to get $P \circ \pi_A(b) = \pi_B(b) \circ P$.

To check that the left-hand square is commutative, first note that $E \otimes \mathrm{id}_{\mathcal{H}_0} : A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0$ is an idempotent mapping. To investigate the continuity of this map, let $x = \sum_{i=1}^n a_i \otimes \xi_i \in A \otimes \mathcal{H}_0$ and note that $((E \otimes \mathrm{id}_{\mathcal{H}_0})x \mid (E \otimes \mathrm{id}_{\mathcal{H}_0})x) = (\Phi_n((E(a_i^*)E(a_j))_{1 \leqslant i,j \leqslant n})\xi \mid \xi)$, where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in M_{n,1}(\mathbb{C}) \otimes \mathcal{H}_0$. On the other hand, $(E(a_i^*)E(a_j))_{1 \leqslant i,j \leqslant n} = E_n(a^*)E_n(a) \leqslant E_n(a^*a) = E_n((a_i^*a_j)_{1 \leqslant i,j \leqslant n})$, where

$$a = \begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in M_n(A)$$

and the above inequality follows by Lemma 6.5. Now, since $\Phi_n : M_n(A) \to M_n(B)$ is a positive map, we get $\Phi_n((E(a_i^*)E(a_j))_{1 \leqslant i,j \leqslant n}) \leqslant \Phi_n(E_n((a_i^*a_j)_{1 \leqslant i,j \leqslant n}))$. Furthermore we have $\Phi_n \circ E_n = (\Phi \circ E)_n = \Phi_n$ by hypothesis, hence $((E \otimes \mathrm{id}_{\mathcal{H}_0})x \mid (E \otimes \mathrm{id}_{\mathcal{H}_0})x) \leqslant (\Phi_n((a_i^*a_j)_{1 \leqslant i,j \leqslant n})\xi \mid \xi) = (x \mid x)$. Thus the linear map $E \otimes \mathrm{id}_{\mathcal{H}_0} : A \otimes \mathcal{H}_0 \to A \otimes \mathcal{H}_0$ is continuous (actually contractive) with respect to the semi-scalar product $(\cdot \mid \cdot)$ and then it induces a bounded linear operator $\widetilde{E} : \mathcal{H}_A \to \mathcal{H}_A$. Moreover, since $E^2 = E$ and $E(A) = B$, it follows that $\widetilde{E}^2 = \widetilde{E}$ and $\widetilde{E}(\mathcal{H}_A) = \mathcal{H}_B$. On the other hand, it is obvious that for every $h_0 \in \mathcal{H}$ we have $\widetilde{E} \circ \iota_{h_0} = \iota_{h_0} \circ E$. Hence it will be enough to prove that $\widetilde{E} = P$.

To this end let $x = \sum_{i=1}^n a_i \otimes \xi_i \in A \otimes \mathcal{H}_0$ and $y = \sum_{j=1}^n b_j \otimes \eta_j \in B \otimes \mathcal{H}_0$ arbitrary. We have

$$\big((E \otimes \mathrm{id}_{\mathcal{H}_0})x \mid y\big) = \left(\sum_{i=1}^n E(a_i) \otimes \xi_i \mid \sum_{j=1}^n b_j \otimes \eta_j\right)$$

$$= \sum_{i,j=1}^{n} \bigl(\Phi\bigl(b_j^* E(a_i)\bigr) \mid \eta_j\bigr) = \sum_{i,j=1}^{n} \bigl(\Phi\bigl(E\bigl(b_j^* a_i\bigr)\bigr) \mid \eta_j\bigr)$$

$$= \sum_{i,j=1}^{n} \bigl(\Phi\bigl(b_j^* a_i\bigr) \mid \eta_j\bigr) = (x \mid y),$$

where the third equality follows since $E(ba) = bE(a)$ for all $a \in A$ and $b \in B$, while the next-to-last equality follows by the hypothesis $\Phi \circ E = \Phi$. Since $y \in B \otimes \mathcal{H}_0$ is arbitrary, the above equality shows that $(E \otimes \mathrm{id}_{\mathcal{H}_0})x - x \perp B \otimes \mathcal{H}_0$. This implies that $\widetilde{E}(\tilde{x}) - \tilde{x} \perp \mathcal{H}_B$, whence $\widetilde{E}(\tilde{x}) = P(\tilde{x})$ for all $x \in A \otimes \mathcal{H}_0$, where $x \mapsto \tilde{x}$, $A \otimes \mathcal{H}_0 \to \mathcal{H}_A$, is the canonical map obtained as the composition $A \otimes \mathcal{H}_0 \to (A \otimes \mathcal{H}_0)/N \hookrightarrow \mathcal{H}_A$. (See Definition 6.3.) Since $\{\tilde{x} \mid x \in A \otimes \mathcal{H}_0\}$ is a dense linear subspace of $\mathcal{H}_A$, it follows that $\widetilde{E} = P$ throughout $\mathcal{H}_A$, and we are done. □

Remark 6.8. Under the assumptions of the previous lemma, we also obtain that $\mathcal{H}_A = \overline{\operatorname{span}}\, \pi_A(\mathrm{U}_A)\mathcal{H}_B$: by standard arguments in C^*-algebras, we have that $A = \operatorname{span} \mathrm{U}_A$ or, equivalently, $A = \operatorname{span} \mathrm{U}_A B$ since we have $\mathbf{1} \in B$. So $A \otimes \mathcal{H}_0 = \operatorname{span} \mathrm{U}_A(B \otimes \mathcal{H}_0)$ whence by quotienting and then by passing to the completion we get $\mathcal{H}_A = \overline{\operatorname{span}}\, \pi_A(\mathrm{U}_A)\mathcal{H}_B$.

Hence the mapping γ is an isometry from $\mathcal{H}_A$ onto $\mathcal{H}^K$ and the inverse mapping γ^{-1} coincides with W, see the remark prior to Theorem 4.2.

Remark 6.9. In the setting of Lemma 6.7, if the restriction of Φ to B happens to be a nondegenerate $*$-representation of B on $\mathcal{H}_0$, then $\mathcal{H}_B = \mathcal{H}_0$ and $\pi_B = \Phi|_B$ by the uniqueness property of the minimal Stinespring dilation (see Remark 6.4). In this special case our Lemma 6.7 is related to the constructions of extensions of representations (see [25, Proposition 2.10.2]) and induced representations of C^*-algebras (see [54, Lemma 1.7, Theorem 1.8, and Definition 1.9]).

In the following theorem we are using notation of Section 4.

Theorem 6.10. *Assume that $B \subseteq A$ are two unital C^*-algebras such that there exists a conditional expectation $E : A \to B$ from A onto B, and let $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ be a unital completely positive map satisfying $\Phi \circ E = \Phi$, where $\mathcal{H}_0$ is a complex Hilbert space. Let $(\pi_A|_{\mathrm{G}_A}, \pi_B|_{\mathrm{G}_B}, P)$ be the Stinespring data associated with E and Φ. Set $\lambda : u\mathrm{U}_B \mapsto u\mathrm{G}_B$, $\mathrm{U}_A/\mathrm{U}_B \hookrightarrow \mathrm{G}_A/\mathrm{G}_B$.*

Then the following assertions hold:

(a) *There exists a real analytic diffeomorphism $a \mapsto (u(a), X(a), b(a))$, $\mathrm{G}_A \to \mathrm{U}_A \times \mathfrak{p} \times \mathrm{G}_B^+$ so that $a = u(a)\exp(\mathrm{i}X(a))b(a)$ for all $a \in A$, which induces the polar decomposition in $\mathrm{G}_A/\mathrm{G}_B$,*

$$a\mathrm{G}_B = u(a)\exp\bigl(\mathrm{i}X(a)\bigr)\mathrm{G}_B, \quad a \in G_A.$$

(b) *The mapping $-* : u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto u\exp(-\mathrm{i}X)\mathrm{G}_B$, $\mathrm{G}_A/\mathrm{G}_B \to \mathrm{G}_A/\mathrm{G}_B$ is an anti-holomorphic involutive diffeomorphism of $\mathrm{G}_A/\mathrm{G}_B$ such that*

$$\lambda(\mathrm{U}_A/\mathrm{U}_B) = \{s \in \mathrm{G}_A/\mathrm{G}_B \mid s = s^{-*}\}.$$

(c) *The projection*

$$u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto u\mathrm{U}_B, \quad \mathrm{G}_A/\mathrm{G}_B \to \mathrm{U}_A/\mathrm{U}_B$$

has the structure of a vector bundle isomorphic to the tangent bundle $\mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p} \to \mathrm{U}_A/\mathrm{U}_B$ *of the manifold* $\mathrm{U}_A/\mathrm{U}_B$. *The corresponding isomorphism is given by* $u\exp(\mathrm{i}X)\mathrm{G}_B \mapsto [(u,X)]$ *for all* $u \in \mathrm{U}_A$, $X \in \mathfrak{p}$.

(d) *Set* $\mathcal{H}(E,\Phi) := \{\gamma(h) \mid h \in \mathcal{H}_A\} \subset \mathcal{O}(G_A/G_B, D)$ *where* $\gamma : \mathcal{H}_A \to \mathcal{O}(G_A/G_B, D)$ *is the realization operator defined by* $\gamma(h)(a\mathrm{G}_B) = [(a, P(\pi_A(a)^{-1}h))]$ *for* $a \in \mathrm{G}_A$ *and* $h \in \mathcal{H}_A$. *Put* $\gamma^{\mathrm{U}} := \gamma(\cdot)|_{\mathrm{U}_A/\mathrm{U}_B} : \mathcal{H}_A \to \mathcal{C}^{\omega}(\mathrm{U}_A/\mathrm{U}_B, D^{\mathrm{U}})$ *and* $\mathcal{H}^{\mathrm{U}}(E,\Phi) := \{\gamma^{\mathrm{U}}(h) \mid h \in \mathcal{H}_A\}$. *Denote by* $\mu(a)$ *the operator on the spaces* $\mathcal{C}^{\omega}(\mathrm{U}_A/\mathrm{U}_B, D^{\mathrm{U}})$ *and* $\mathcal{O}(G_A/G_B, D)$ *defined by natural multiplication by* $a \in \mathrm{G}_A$. *Then* $\mathcal{H}(E,\Phi)$ *and* $\mathcal{H}^{\mathrm{U}}(E,\Phi)$ *are Hilbert spaces isometric with* $\mathcal{H}_A$. *Moreover, for every* $a \in \mathrm{G}_A$ *the following diagram*

$$\begin{tikzcd}
\mathcal{H}_A \arrow[r, "\gamma^{\mathrm{U}}"] \arrow[d, "\pi(a)"'] & \mathcal{H}^{\mathrm{U}}(E,\Phi) \arrow[r, "\simeq"] \arrow[d, "\mu(a)"] & \mathcal{H}(E,\Phi) \arrow[d, "\mu(a)"] \\
\mathcal{H}_A \arrow[r, "\gamma^{\mathrm{U}}"] & \mathcal{H}^{\mathrm{U}}(E,\Phi) \arrow[r, "\simeq"] & \mathcal{H}(E,\Phi),
\end{tikzcd}$$

is commutative, that is, $\gamma \circ \pi(a) = \mu(a) \circ \gamma$.

(e) *There exists an isometry* $V_{E,\Phi} : \mathcal{H}_0 \to \mathcal{H}(E,\Phi)$ *such that*

$$\Phi(a) = V_{E,\Phi}^*(T_{\mathbf{1}}\mu)(a)V_{E,\Phi}, \quad a \in A,$$

where $T_{\mathbf{1}}\mu$ *is the tangent map of* $\mu(\cdot)|_{\mathcal{H}(E,\Phi)}$ *at* $\mathbf{1} \in \mathrm{G}_A$. *In fact,* $T_{\mathbf{1}}\mu$ *is a Banach algebra representation of* A *which extends* μ.

Proof. (a) Let $(\pi_A|_{\mathrm{G}_A}, \pi_B|_{\mathrm{G}_B}, P)$ be the Stinespring data introduced in Lemma 6.7, so that $\mathcal{H}_A = \overline{\operatorname{span}}\,\pi_A(\mathrm{G}_A)\mathcal{H}_B$ according to Remark 6.8. We have that $\mathrm{G}_B^+ = \mathrm{G}_B \cap \mathrm{G}_A^+$ as a direct consequence of the fact that the C^*-algebras are closed under taking square roots of positive elements. Then parts (a)–(d) of the theorem follow immediately by application of Theorem 5.1, Corollaries 5.3, 5.5 and Theorem 4.4.

As regards (e) note that for every $a \in A$ and $h \in \mathcal{H}_A$,

$$\begin{aligned}
T_{\mathbf{1}}\mu(a)\gamma(h) &= (d/dt)|_{t=0}\mu\big(e^{ta}\big)\gamma(h) \\
&= (d/dt)|_{t=0}e^{ta}\big[\big(e^{-ta}(\cdot), P\big(\pi_A(\cdot)^{-1}\pi_A\big(e^{ta}\big)h\big)\big)\big] = \gamma\big(\pi_A(a)(h)\big).
\end{aligned}$$

Since γ is bijective (and isometric) we have that $T_{\mathbf{1}}\mu(a) = \gamma^{-1}\pi_A(a)\gamma$ for all $a \in A$, whence it is clear that $T_{\mathbf{1}}\mu$ becomes a Banach algebra representation (and not only a Banach–Lie algebra representation).

Now take $V_{E,\Phi} := \gamma \circ V$ where V is the isometry $V : \mathcal{H}_0 \to \mathcal{H}_A$ given in Definition 6.3. It is clear that $\gamma^* = \gamma^{-1}$ and then that $V_{E,\Phi}$ is the isometry we wanted to find. □

Remark 6.11. Theorem 6.10 extends to the holomorphic setting, and for Stinespring representations, the geometric realization framework given in of [11, Theorem 5.4] for GNS representations. As part of such an extension we have found that the real analytic sections obtained in

[11] are always restrictions of holomorphic sections of suitable (like-Hermitian) vector bundles on fairly natural complexifications.

Part (e) of the theorem provides us with a strong geometric view of the completely positive mappings on C^*-algebras A: such a map is the compression of the "natural action of A" (in the sense that it is obtained by differentiating the non-ambiguous natural action of G_A) on a Hilbert space formed by holomorphic sections of a vector bundle of the formerly referred to type.

7. Further applications, examples and links

7.1. Banach algebraic amenability

Example 7.1. Let $\mathfrak{A}$ be a Banach algebra. A *virtual diagonal* of $\mathfrak{A}$ is by definition an element M in the bidual $\mathfrak{A}$-bimodule $(\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ such that

$$\mathrm{y}\cdot M = M\cdot\mathrm{y} \quad \text{and} \quad \mathrm{m}(M)\cdot\mathrm{y} = \mathrm{y} \quad (\mathrm{y}\in\mathfrak{A})$$

where m is the extension to $(\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ of the multiplication map in $\mathfrak{A}$, $\mathrm{y}\otimes\mathrm{y}'\mapsto\mathrm{y}\mathrm{y}'$. The algebra $\mathfrak{A}$ is called *amenable* when it possesses a virtual diagonal as above. When $\mathfrak{A}$ is a C^*-algebra, then $\mathfrak{A}$ is amenable if and only it is nuclear. Analogously, a *dual* Banach algebra $\mathfrak{M}$ is called *Connes-amenable* if $\mathfrak{A}$ has a virtual diagonal which in addition is *normal*. Then a von Neumann algebra $\mathfrak{A}$ is Connes-amenable if and only it is injective. For all these concepts and results, see [55].

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{B}$ be a von Neumann algebra. By $\mathrm{Rep}(\mathfrak{A},\mathfrak{B})$ we denote the set of bounded representations $\rho:\mathfrak{A}\to\mathfrak{B}$ such that $\overline{\rho(\mathfrak{A})\mathfrak{B}_*}=\mathfrak{B}_*$ where $\mathfrak{B}_*$ is the (unique) predual of $\mathfrak{B}$ (recall that $\mathfrak{B}_*$ is a left Banach $\mathfrak{B}$-module). In the case $\mathfrak{B}=\mathcal{B}(\mathcal{H})$, for a complex Hilbert space $\mathcal{H}$, the property that $\overline{\rho(\mathfrak{A})\mathfrak{B}_*}=\mathfrak{B}_*$ is equivalent to have $\overline{\rho(\mathfrak{A})\mathcal{H}}=\mathcal{H}$, that is, ρ is nondegenerate. Let $\mathrm{Rep}_*(\mathfrak{A},\mathfrak{B})$ denote the subset of $*$-representations in $\mathrm{Rep}(\mathfrak{A},\mathfrak{B})$. For a von Neumann algebra $\mathfrak{M}$, we denote by $\mathrm{Rep}^{\omega}(\mathfrak{M},\mathfrak{B})$ the subset of homomorphisms in $\mathrm{Rep}(\mathfrak{M},\mathfrak{B})$ which are ultraweakly continuous, or *normal* for short. As above, the set of $*$-representations of $\mathrm{Rep}^{\omega}(\mathfrak{M},\mathfrak{B})$ is denoted by $\mathrm{Rep}^{\omega}_*(\mathfrak{M},\mathfrak{B})$.

From now on, $\mathfrak{A}$, $\mathfrak{M}$ will denote a nuclear C^*-algebra and an injective von Neumann algebra, respectively. Fix $\rho\in\mathrm{Rep}(\mathfrak{A},\mathfrak{B})$. The existence of a virtual diagonal M for $\mathfrak{A}$ allows us to define an operator $E_\rho:\mathfrak{B}\to\mathfrak{B}$ by

$$\bigl(E_\rho(T)x\mid x'\bigr) := M\bigl(\mathrm{y}\otimes\mathrm{y}'\mapsto\bigl(\bigl(\rho(\mathrm{y})T\bigr)\rho(\mathrm{y}')x\mid x'\bigr)\bigr) \equiv \int\limits_{\mathfrak{A}\otimes\mathfrak{A}} \bigl(\bigl(\rho(\mathrm{y})T\bigr)\rho(\mathrm{y}')x\mid x'\bigr)\,dM(\mathrm{y},\mathrm{y}')$$

where x, x' belong to a Hilbert space $\mathcal{H}$ such that $\mathfrak{B}\hookrightarrow\mathcal{B}(\mathcal{H})$, and $T\in\mathfrak{B}$. In the formula, the "integral" corresponds to the Effros notation, see [20]. The operator E_ρ is a bounded projection such that

$$E_\rho(\mathfrak{B}) = \rho(\mathfrak{A})' := \bigl\{T\in\mathfrak{B}\mid T\rho(\mathrm{y})=\rho(\mathrm{y})T,\ \mathrm{y}\in\mathfrak{A}\bigr\}.$$

In fact, it is readily seen that $\|E_\rho\|\leqslant\|M\|\|\rho\|^2$, so that E_ρ becomes a conditional expectation provided that $\|M\|=\|\rho\|=1$. For instance, if ρ is a $*$-homomorphism then its norm is one, see [50, p. 7]. The existence of (normal) virtual diagonals of norm one in (dual) Banach algebras is not a clear fact in general, but it is true, and not simple, that such (normal) virtual diagonals exist for (injective von Neumann) nuclear C^*-algebras, see [55, p. 188].

For $\rho \in \mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$ and $T \in \mathfrak{B}$, let $T\rho T^{-1} \in \mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$ defined as $(T\rho T^{-1})(\mathrm{y}) := T\rho(\mathrm{y})T^{-1}$ ($\mathrm{y} \in \mathfrak{A}$). Put

$$\mathfrak{S}(\rho) := \{T\rho T^{-1} \mid T \in \mathrm{G}_{\mathfrak{B}}\} \quad \text{and} \quad \mathfrak{U}(\rho) := \{T\rho T^{-1} \mid T \in \mathrm{U}_{\mathfrak{B}}\}.$$

The set $\mathfrak{S}(\rho)$ is called the *similarity orbit* of ρ, and $\mathfrak{U}(\tau)$ is called the *unitary orbit* of $\tau \in \mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B})$. It is known that $\mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$, endowed with the norm topology, is the discrete union of orbits $\mathfrak{S}(\rho)$. Moreover, each orbit $\mathfrak{S}(\rho)$ is a homogeneous Banach manifold with a reductive structure induced by the connection form E_ρ. In the same way, $\mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B})$ is the disjoint union of orbits $\mathfrak{U}(\tau)$, and the restriction of E_ρ on $\mathfrak{u}_{\mathfrak{B}}$ is a connection form which induces a homogeneous reductive structure on $\mathfrak{U}(\tau)$—see [2,20,30]. We next compile some more information, about the similarity and unitary orbits, which is obtained on the basis of results of the preceding sections.

Let $\rho \in \mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$. As it was proved in [18] and independently in [19] (see also [33]), there exists $\tau \in \mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B}) \cap \mathfrak{S}(\rho)$, whence $\mathfrak{S}(\rho) = \mathfrak{S}(\tau)$. Hence, without loss of generality, ρ can be assumed to be a $*$-representation, so that $\|\rho\| = 1$. Moreover, since we are assuming that $\mathfrak{A}$ is nuclear, we can choose a virtual diagonal M of $\mathfrak{A}$ of norm one. Thus the operator E_ρ is a conditional expectation. Set $A := \mathfrak{B}$, $B := \rho(\mathfrak{A})'$. With this notation, $\mathfrak{S}(\rho) = \mathrm{G}_A/\mathrm{G}_B$ and $\mathfrak{U}(\rho) = \mathrm{U}_A/\mathrm{U}_B$ diffeomorphically.

For $X \in \mathfrak{p}_\rho := \mathrm{Ker}\, E_\rho \cap \mathfrak{u}_A$, let $[X]$ denote the equivalence class of X under the adjoint action of U_B on $\mathfrak{p}_\rho$ considered in Theorem 5.1. Also, set $e^{\mathrm{i}X} := \exp(\mathrm{i}X)$.

Corollary 7.2. *Let $\mathfrak{A}$ be a nuclear C^*-algebra and let $\mathfrak{B}$ be a von Neumann algebra. The following assertions hold*:

(a) *Each connected component of* $\mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$ *is a similarity orbit $\mathfrak{S}(\rho)$, for some $\rho \in$* $\mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B})$. *Moreover, each orbit $\mathfrak{S}(\rho)$ is the disjoint union*

$$\mathfrak{S}(\rho) = \bigcup_{[X] \in \mathfrak{p}_\rho/\mathrm{U}_B} \mathfrak{U}(e^{\mathrm{i}X}\rho e^{-\mathrm{i}X})$$

where $\mathfrak{U}(e^{\mathrm{i}X}\rho e^{-\mathrm{i}X})$ is connected, for all $[X] \in \mathfrak{p}_\rho/\mathrm{U}_B$.

(b) *The similarity orbit $\mathfrak{S}(\rho)$ is a complexification of the unitary orbit $\mathfrak{U}(\rho)$ with respect to the involutive diffeomorphism $ue^{\mathrm{i}X}\rho e^{-\mathrm{i}X}u^{-1} \mapsto ue^{-\mathrm{i}X}\rho e^{\mathrm{i}X}u^{-1}$ ($u \in \mathrm{U}_{\mathfrak{B}}$).*

(c) *The mapping $ue^{\mathrm{i}X}\rho e^{-\mathrm{i}X}u^{-1} \mapsto u\rho u^{-1}$, $\mathfrak{S}(\rho) \to \mathfrak{U}(\rho)$ is a continuous retraction which defines a vector bundle diffeomorphic to the tangent bundle $\mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}_\rho \to \mathfrak{U}(\rho)$ of $\mathfrak{U}(\rho)$.*

(d) *Let $\mathcal{H}_0$ be a Hilbert space such that $\mathfrak{B} \hookrightarrow \mathcal{B}(\mathcal{H}_0)$. For every $\rho \in \mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$ there exists a Hilbert space $\mathcal{H}_0(\rho)$ isometric with $\mathcal{H}_0$, which is formed by holomorphic sections of a like-Hermitian vector bundle with base $\mathfrak{S}(\rho)$. Moreover, $\mathfrak{B}$ acts continuously by natural multiplication on $\mathcal{H}_0(\rho)$, and the representation R obtained by transferring ρ on $\mathcal{H}_0(\rho)$ coincides with multiplication by ρ; that is, $R(\mathrm{y})F = \rho(\mathrm{y}) \cdot F$ for all $\mathrm{y} \in \mathfrak{A}$ and section $F \in \mathcal{H}_0(\rho)$.*

Proof. (a) As said before, every similarity orbit of $\mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$ is of the form $\mathfrak{S}(\rho)$ for some $\rho \in \mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B})$. Since $A = \mathfrak{B}$ is a von Neumann algebra, the set of unitaries $\mathrm{U}_A = \mathrm{U}_{\mathfrak{B}}$ is connected whence it follows that the orbits $\mathfrak{S}(\rho)$ and $\mathfrak{U}(e^{\mathrm{i}X}\rho e^{-\mathrm{i}X})$ are connected for all $X \in \mathfrak{p}_\rho$ (see for instance [10] for more details). For $X, Y \in \mathfrak{p}_\rho$, we have $\mathfrak{U}(e^{\mathrm{i}X}\rho e^{-\mathrm{i}X}) = \mathfrak{U}(e^{\mathrm{i}Y}\rho e^{-\mathrm{i}Y})$

if and only if there exists $u \in \mathrm{U}_A$ such that $e^{\mathrm{i}Y}\rho e^{-\mathrm{i}Y} = ue^{\mathrm{i}X}\rho e^{-\mathrm{i}X}$, which means that $u \in \mathrm{U}_B$ and $Y = uXu^{-1}$ (see Theorem 5.1). Hence $[X] = [Y]$. Finally, by Theorem 5.1 again we have $\mathfrak{S}(\rho) = \bigcup_{[X]\in\mathfrak{p}_\rho/\mathrm{U}_B} \mathfrak{U}(e^{\mathrm{i}X}\rho e^{-\mathrm{i}X})$.

(b) This is Theorem 6.10(b).

(c) This follows by Theorem 6.10(c).

(d) Given ρ in $\mathrm{Rep}(\mathfrak{A}, \mathfrak{B})$, there is $\tau = \tau(\rho)$ in $\mathrm{Rep}_*(\mathfrak{A}, \mathfrak{B})$ such that $\mathfrak{S}(\rho) = \mathfrak{S}(\tau)$. Now we fix a virtual diagonal of $\mathfrak{A}$ of norm one and then define the conditional expectation $E_\rho \equiv E_{\tau(\rho)}$ as prior to this corollary. So $E_\rho : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_0)$ is a completely positive mapping and one can apply Theorem 6.10(d). As a result, one gets a Hilbert space $\mathcal{H}(\rho) := \mathcal{H}(E_\rho, E_\rho)$ of holomorphic sections of a like-Hermitian bundle on $\mathfrak{S}(\rho) = \mathrm{G}_{\mathfrak{B}}/\mathrm{G}_B$ (where $B = \rho(\mathfrak{A})'$), and an isometry $V_\rho := V_{E_\rho} : \mathcal{H}_0 \to \mathcal{H}(\rho)$, satisfying $E_\rho(\rho(\mathrm{y})) = V_\rho^* T_{\mathbf{1}}\mu(\rho(\mathrm{y}))V_\rho$ for all $\mathrm{y} \in \mathfrak{A}$, in the notations of Theorem 6.10. Note that $V_\rho^* V_\rho = \mathbf{1}$ and therefore the correspondence $\mathrm{y} \mapsto V_\rho \rho(\mathrm{y}) V_\rho^*$ defines a (bounded) representation of $\mathcal{H}(\rho)$. Now take $\mathcal{H}_0(\rho) := V(\mathcal{H}_0)$ and define $R(\mathrm{y})$ as the restriction of $V_\rho\rho(\mathrm{y})V_\rho^*$ on $\mathcal{H}_0(\rho)$ for every $\mathrm{y} \in \mathfrak{A}$. Clearly, R is the transferred representation of ρ from $\mathcal{H}_0$ to $\mathcal{H}_0(\rho)$.

Also, for every $F \in \mathcal{H}_0(\rho)$ there exists $h_0 \in \mathcal{H}_0$ such that $F = V_\rho(\mathbf{1} \otimes h_0)$, that is, $F(aG_B) = [(a, P(a^{-1} \otimes h_0)]$ for all $a \in \mathrm{G}_{\mathfrak{B}}$, where P is as in Lemma 6.7. Then, for $\mathrm{y} \in \mathfrak{A}$,

$$\begin{aligned} R(\mathrm{y})F &= R(\mathrm{y})V_\rho(\mathbf{1} \otimes h_0) = V_\rho\ \rho(\mathrm{y})V_\rho^* V_\rho(\mathbf{1} \otimes h_0) = V_\rho\ \rho(\mathrm{y})(\mathbf{1} \otimes h_0) \\ &= V_\rho\big(\rho(\mathrm{y}) \otimes h_0\big) = T_{\mathbf{1}}\mu\big(\rho(\mathrm{y})\big)V_\rho(\mathbf{1} \otimes h_0) = \rho(\mathrm{y}) \cdot F, \end{aligned}$$

as we wanted to show. □

Remark 7.3. (i) The first part of Corollary 7.2(a) was already well known (see for example [2]). In the decomposition of the second part, the orbit $\mathfrak{U}(e^{iX}\rho e^{-iX})$ for $X = 0$ corresponds to the unitary orbit of ρ. So the disjoint union supplies a sort of configuration of the similarity orbit $\mathfrak{S}(\rho)$ by relation with the unitary orbit $\mathfrak{U}(\rho)$.

(ii) Parts (a)–(c) of Corollary 7.2 are consequences of the Porta–Recht decomposition given in [53], see (5.2). Such a decomposition has been considered previously in relation with similarity orbits of nuclear C^*-algebras, though in a different perspective, see [2, Theorem 5.7], for example.

(iii) Corollary 7.2 admits a version entirely analogous for injective von Neumann algebras $\mathfrak{M}$ (replacing the nuclear C^* algebra $\mathfrak{A}$ of the statement) and representations in $\mathrm{Rep}^\omega(\mathfrak{M}, \mathfrak{B})$ and $\mathrm{Rep}_*^\omega(\mathfrak{M}, \mathfrak{B})$. Proofs are similar to the nuclear, C^*, case. For the analog of (d) one needs to take a normal virtual diagonal of $\mathfrak{M}$ of norm one.

(iv) Corollary 7.2 applies in particular to locally compact groups G for which the group C^*-algebra $C^*(G)$ is amenable, see [2,21]. When the group is compact the method to define the expectation E_ρ works for every representation ρ taking values in any Banach algebra A. We shall see a particular example of this below, involving Cuntz algebras.

7.2. *Completely positive mappings*

Let A be a complex unital C^*-algebra, with unit $\mathbf{1}$, included in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. Assume that $\Phi : A \to \mathcal{B}(\mathcal{H})$ is a unital, completely bounded mapping. (In the sequel we shall assume freely that $\mathcal{H}$ is separable, when necessary.) The following lemma is standard. We give a proof for the sake of completeness.

Lemma 7.4. *Given Φ as above and $u \in \mathrm{G}_A$, let Φ_u denote the mapping $\Phi_u := u\Phi(u^{-1} \cdot u)u^{-1}$. Then*

(i) *For every $u \in \mathrm{G}_A$, Φ_u is completely bounded and $\|\Phi_u\|_{cb} \leqslant \|\Phi\|_{cb}\|u^*\|\|u\|\|u^{-1}\|^2$.*
(ii) *If Φ is completely positive then Φ_u is completely positive for every $u \in \mathrm{U}_A$.*

Proof. (i) Let n be a natural number. Take $f = (f_1, \dots, f_n), h = (h_1, \dots, h_n) \in \mathcal{H}^n$ and $(a_{ij})_{ij} \in M_n(A)$ all of them of respective norms less than or equal to 1. In the following we shall think of f and g in their column version. Then, for $u \in \mathrm{G}_A$, we have

$$\begin{aligned}\left|\left(\Phi_u^{(n)}(a_{ij})_{ij} f \mid h\right)_{\mathcal{H}^n}\right| &= \left|\sum_{i,j}\left(\Phi\left(u^{-1}a_{ij}u\right)\left(u^{-1}f_j\right) \mid u^*h_i\right)_{\mathcal{H}}\right| \\ &\leqslant \left\|\Phi^{(n)}\left(u^{-1}a_{ij}u\right)_{ij}\right\|_{\mathcal{B}(\mathcal{H}^n)}\left\|\left(u^{-1}f_j\right)_j\right\|_{\mathcal{H}^n}\left\|\left(u^*h_i\right)_i\right\|_{\mathcal{H}^n} \\ &\leqslant \|\Phi\|_{cb}\left\|\left(u^{-1}I\right)(a_{ij})_{ij}(uI)\right\|_{\mathcal{B}(\mathcal{H}^n)}\left\|u^{-1}\right\|\|u^*\| \\ &\leqslant \|\Phi\|_{cb}\left\|u^{-1}\right\|^2\|u\|\|u^*\|.\end{aligned}$$

(ii) Assume now that Φ is completely positive. For natural n, take $(a_{ij})_{ij} \geqslant 0$ in $M_n(A)$ and $h = (h_1, \dots, h_n) \in \mathcal{H}^n$. Then

$$\begin{aligned}\left(\Phi_u^{(n)}(a_{ij})_{ij}h \mid h\right)_{\mathcal{H}^n} &= \left(\left(\sum_{j=1}^n \Phi_u(a_{1j})h_j, \dots, \sum_{j=1}^n \Phi_u(a_{nj})h_j\right) \mid h\right)_{\mathcal{H}^n} \\ &= \sum_{i,j=1}^n \left(\Phi_u(a_{ij})h_j \mid h_i\right)_{\mathcal{H}} = \sum_{i,j=1}^n \left(\Phi^{(n)}(b_{ij}f \mid f)\right)_{\mathcal{H}^n}\end{aligned}$$

where $f = u^{-1}h$, $b_{ij} = u^{-1}a_{ij}u$, $u \in \mathrm{U}_A$. So $(b_{ij})_{ij} \geqslant 0$ in $M_n(A)$ and, since Φ is completely positive, we conclude that $\Phi^{(n)}(b_{ij}) \geqslant 0$ as we wanted to show. □

Now let $\Phi : A \to \mathcal{B}(\mathcal{H})$ be a fixed, unital *completely positive* mapping. By [50, Proposition 3.5], Φ is completely bounded. According to the preceding Lemma 7.4, if $\mathcal{U}(\Phi)$ and $\mathcal{S}(\Phi)$ denote, respectively, the *unitary* orbit $\mathcal{U}(\Phi) := \{\Phi_u \mid u \in \mathrm{U}_A\}$ and the *similarity* orbit $\mathcal{S}(\Phi) := \{\Phi_u \mid u \in \mathrm{G}_A\}$ of Φ, there are natural actions of G_A on $\mathcal{S}(\Phi)$ and of U_A on $\mathcal{U}(\Phi)$, under usual conjugation. Note that the elements of the orbit $\mathcal{S}(\Phi)$ are completely bounded maps but they need not be completely positive.

Put $\mathrm{G}(\Phi) := \{u \in \mathrm{G}_A \mid \Phi_u = \Phi\}$ and $\mathrm{U}(\Phi) := \mathrm{G}(\Phi) \cap \mathrm{U}_A$.

Corollary 7.5. *In the above notation, $\mathcal{S}(\Phi) = \mathrm{G}_A/\mathrm{G}(\Phi)$ and $\mathcal{U}(\Phi) = \mathrm{U}_A/\mathrm{U}(\Phi)$.*

Proof. It is enough to observe that $\mathrm{G}(\Phi)$ and $\mathrm{U}(\Phi)$ are the isotropy subgroups of the actions of G_A on $\mathcal{S}(\Phi)$ and of U_A on $\mathcal{U}(\Phi)$, respectively. □

Note that $\mathrm{G}(\Phi)$ is defined by the family of polynomial equations

$$\varphi\left(\Phi\left(axa^{-1}\right) - a\Phi(x)a^{-1}\right) = 0, \quad x \in A,\ \varphi \in \mathcal{B}(\mathcal{H})_*,\ a \in \mathrm{G}_A$$

on $\mathrm{G}_A \times \mathrm{G}_A$, so $\mathrm{G}(\Phi)$ is algebraic and a Banach–Lie group with respect to the relative topology of A (see for instance the Harris–Kaup theorem in [65]). To see when the isotropy groups $\mathrm{G}(\Phi)$ and $\mathrm{U}(\Phi)$ are Banach–Lie subgroups of G_A, we need to compute their Lie algebras $\mathfrak{g}(\Phi)$ and $\mathfrak{u}(\Phi)$, respectively, and to see whether they are complemented subspaces of A.

Lemma 7.6. *In the above notation we have* $\mathfrak{g}(\Phi) = \{X \in A \mid (\forall a \in A)\ \Phi([a, X]) = [\Phi(a), X]\}$, *and therefore* $\mathfrak{u}(\Phi) = \{X \in \mathfrak{u}_A \mid (\forall a \in A)\ \Phi([a, X]) = [\Phi(a), X]\}$.

Proof. To prove the inclusion "$\subseteq$" just note that if $X \in A$ and $e^{tX} := \exp(tX) \in \mathrm{G}(\Phi)$, then for every $a \in A$ we get $\Phi(e^{tX} a e^{-tX}) = e^{tX}\Phi(a)e^{-tX}$ for all $t \in \mathbb{R}$. Hence by differentiating in t and taking values at $t = 0$ we obtain $\Phi(aX - Xa) = \Phi(a)X - X\Phi(a)$; that is, $\Phi([a, X]) = [\Phi(a), X]$.

Now let X in the right-hand side of the first equality from the statement. Then A is an invariant subspace for the mapping $\mathrm{ad}\, X = [X, \cdot] : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, since $X \in A$. In addition, $\Phi \circ (\mathrm{ad}\, X)|_A = (\mathrm{ad}\, X) \circ \Phi$. Hence for every $t \in \mathbb{R}$ and $n \geqslant 0$ we get $\Phi \circ (t\, \mathrm{ad}\, X)^n|_A = (t\, \mathrm{ad}\, X)^n \circ \Phi$, whence $\Phi \circ \exp(t(\mathrm{ad}\ X)|_A) = \exp(t\ \mathrm{ad}\ X) \circ \Phi$. Since $\exp(t\ \mathrm{ad}\ X)b = \mathrm{e}^{tX} b\, \mathrm{e}^{-tX}$ for all $b \in \mathcal{B}(\mathcal{H})$, it then follows that $\mathrm{e}^{tX} \in \mathrm{G}(\Phi)$ for all $t \in \mathbb{R}$, whence $X \in \mathfrak{g}(\Phi)$. The remainder of the proof is now clear. □

As regards the description of the isotropy Lie algebra $\mathfrak{g}(\Phi)$ in Lemma 7.6, let us note the following fact:

Proposition 7.7. *The isotropy Lie algebra* $\mathfrak{g}(\Phi)$ *is a closed involutive Lie subalgebra of* A. *If the range of* Φ *is contained in the commutant of* A *then* $\mathfrak{g}(\Phi)$ *is actually a unital* C^**-subalgebra of* A, *given by* $\mathfrak{g}(\Phi) = \{X \in A \mid \Phi(aX) = \Phi(Xa) \text{ for all } a \in A\}$. *In this case,* $\mathrm{G}_{\mathfrak{g}(\Phi)} = \mathrm{G}(\Phi)$.

Proof. It is clear from Lemma 7.6 that $\mathfrak{g}(\Phi)$ is a closed linear subspace of A which contains the unit $\mathbf{1}$. Moreover, since $\Phi(a^*) = \Phi(a)^*$ for all $a \in A$ (this is automatic by the Stinespring's dilation theorem, for instance), then for $X \in \mathfrak{g}(\Phi)$ and $a \in A$ we have $\Phi([X^*, a]) = \Phi([a^*, X]^*) = \Phi([a^*, X])^* = [\Phi(a^*), X]^* = [X^*, \Phi(a)]$ whence $\mathfrak{g}(\Phi)$ is stable under involution as well. The fact that $\mathfrak{g}(\Phi)$ is a Lie subalgebra of A follows by [9, Theorem 4.13] (see the proof there).

If the range of Φ is contained in the commutant of A, then $\mathfrak{g}(\Phi) = \{X \in A \mid \Phi(aX) = \Phi(Xa) \text{ for all } a \in A\}$, and so $\mathfrak{g}(\Phi)$ is a C^*-subalgebra of A. Finally, note that $u \in \mathrm{G}_{\mathfrak{g}(\Phi)}$ if and only if $u \in \mathrm{G}_A$ and $\Phi(uau^{-1}) = \Phi(a) = u\Phi(a)u^{-1}$ (since $\Phi(A) \subseteq A'$), if and only if $u \in \mathrm{G}(\Phi)$. □

The condition in the above statement for Φ to be contained in the commutant of A holds if for instance, Φ is a state of A. Next, we give another example suggested by Example 7.1. For a C^*-algebra $\mathfrak{A}$ and von Neumann algebra A with predual A_*, let $\rho : \mathfrak{A} \to A$ be a bounded $*$-homomorphism such that $\overline{\rho(\mathfrak{A})A_*} = A_*$. Denote w^* the (generic) weak operator topology in a von Neumann algebra.

Corollary 7.8. *Assume that* $\mathfrak{A}$ *is a nuclear* C^**-algebra or an injective von Neumann algebra* (*in the second case we assume in addition that* ρ *is normal*), *and that* $A = \overline{\rho(\mathfrak{A})}^{w^*}$. *Let* $\Phi = E_\rho : A \to A$ *be a conditional expectation associated with* ρ *as in Example* 7.1. *Then* $B := \Phi(A) \subseteq A'$ *and therefore* $\mathfrak{g}(\Phi)$ *is a von Neumann subalgebra of* A. *Also,* B *is commutative and so it is isomorphic to an algebra of* L^∞ *type.*

Proof. From $\Phi(A) = \rho(\mathfrak{A})'$ and $A = \overline{\rho(\mathfrak{A})}^{w^*}$ it is readily seen (recall that A_* is an A-bimodule for the natural module operations) that $\Phi(a)$ commutes with every element of A for all $a \in A$. The remainder of the corollary is clear. □

It is not difficult to find representations as those of the preceding corollary. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a representation as in Example 7.1, then it is enough to take $A := \overline{\pi(\mathfrak{A})}^{w^*}$ in $\mathfrak{B}$, and $\rho : \mathfrak{A} \to A$ defined by $\rho(y) := \pi(y)$ $(y \in \mathfrak{A})$, to obtain a representation satisfying the hypotheses of Corollary 7.8. It is straightforward to check that A is a C^*-algebra and, moreover, that A is a dual Banach space. In effect, if $\mathfrak{B}_*$ is the predual of $\mathfrak{B}$ and ${}^\perp A$ is the pre-annihilator subspace of A in $\mathfrak{B}_*$, then the quotient $\mathfrak{B}_*/{}^\perp A$ is a predual of A, and an A-submodule of A^*, such that $\overline{\rho(\mathfrak{A})(\mathfrak{B}_*/{}^\perp A)} = \mathfrak{B}_*/{}^\perp A$. Note that in the case when $\mathfrak{B} = \mathcal{B}(\mathcal{H})$ the von Neumann's bicommutant theorems say that $A = \pi(\mathfrak{A})''$.

7.3. Conditional expectations

We are going to see that the isotropy group $\mathrm{G}(\Phi)$ of a completely positive map $\Phi : A \to \mathcal{B}(\mathcal{H})$ is actually a Banach–Lie subgroup of G_A in the important special case when Φ is a faithful normal conditional expectation. This will provide us with a wide class of completely positive mappings whose similarity orbits illustrate the main results of the present paper.

Thus, let assume in this subsection that $\Phi = E$ is a faithful, normal, *conditional expectation* $E : A \to B$, where A and B are von Neumann algebras with $B \subseteq A \subseteq \mathcal{B}(\mathcal{H})$. In this case all of the elements in the unitary orbit $\mathcal{U}(E)$ are conditional expectations, whereas all we can say about the elements in the similarity orbit $\mathcal{S}(E)$ is that they are completely bounded quasi-expectations. We would like to present $\mathcal{S}(E)$ and $\mathcal{U}(E)$ as examples of the theory given in the previous Theorems 4.4 and/or 5.1, or even Section 5, of this paper.

Denote $A_E := \{x \in B' \cap A \mid E(ax) = E(xa), a \in A\}$ and fix a faithful, normal state φ on B. (Such a faithful state exists if the Hilbert space $\mathcal{H}$ is separable.) The set A_E is a von Neumann subalgebra of A and, using the modular group of A induced by the gauge state $\psi := \varphi \circ E$, it can be proven that there exists a faithful, normal, conditional expectation $F : A \to A_E$ such that $E \circ F = F \circ E$ and $\psi \circ F = \psi$ (see [4, Proposition 4.5]). Set $\Delta = E + F - EF$. Then Δ is a bounded projection from A onto

$$\Delta(A) = A_E + B = (A_E \cap \ker E) \oplus B.$$

By considering the *connected* $\mathbf{1}$*-component* $\mathrm{G}(E)^0 = \mathrm{G}_{A_E} \cdot \mathrm{G}_B$ of the isotropy group $\mathrm{G}(E)$ (see [4, Proposition 3.3]), the existence of Δ implies that $\mathrm{G}(E)$ is in fact a Banach–Lie subgroup of G_A, the orbits $\mathcal{S}(E)$ and $\mathcal{U}(E)$ are homogeneous Banach manifolds, and the quotient map $\mathrm{G}_A \to \mathcal{S}(E) \simeq \mathrm{G}_A/\mathrm{G}(E)$ is an analytic submersion, see [4, Corollary 4.7 and Theorem 4.8]. Also, the following assertions hold:

Proposition 7.9. *In the notations from above and from Section* 7.1, $\Delta(A) = \mathfrak{g}(E)$. *In particular, A splits through $\mathfrak{g}(E)$ and $\mathfrak{g}(E)$ is a w^*-closed Lie subalgebra of A.*

Proof. By [4, Theorem 4.8] the quotient mapping $\mathrm{G}_A \to \mathcal{S}(E) = \mathrm{G}_A/\mathrm{G}(E)$ is an analytic submersion. In fact the kernel of its differential is $\mathfrak{g}(E)$ (see [65, Theorem 8.19]). Also, $\mathfrak{g}(E) := T_{\mathbf{1}}(\mathrm{G}(E)) = \Delta(A)$ by [4, Proposition 4.6]. □

Now let $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ be any unital completely positive map such that $\Phi \circ E = \Phi$ and apply Stinespring's dilation procedure to the mapping Φ and the conditional expectation $E : A \to B$. Thus, for $J = A, B$ there are the Hilbert spaces $\mathcal{H}_J(\Phi)$ and (Stinespring) representations $\pi_J : J \to \mathcal{B}(\mathcal{H}_J(\Phi))$ such that $\mathcal{H}_B(\Phi) \subseteq \mathcal{H}_A(\Phi)$ and $\pi_B(u) = \pi_A(u)|_{\mathcal{H}_B(\Phi)}$ for each $u \in B$, as given in Lemma 6.7. Denote by $P : \mathcal{H}_A(\Phi) \to \mathcal{H}_B(\Phi)$ the corresponding orthogonal projection.

We are going to construct representations of the intermediate groups in the sequence

$$\mathrm{G}_B \subseteq \mathrm{G}(E)^0 \subseteq \mathrm{G}(E) \subseteq \mathrm{G}_A.$$

For this purpose set $\mathcal{H}_E(\Phi) := \overline{\operatorname{span}}(\pi_A(\mathrm{G}(E))\mathcal{H}_B(\Phi))$ and P_E the orthogonal projection from $\mathcal{H}_A(\Phi)$ onto $\mathcal{H}_E(\Phi)$. We have, $\overline{\operatorname{span}}(\pi_A(\mathrm{G}_A)\mathcal{H}_E(\Phi)) = \mathcal{H}_A(\Phi)$, since $\overline{\operatorname{span}}(\pi_A(\mathrm{G}_A)\mathcal{H}_B(\Phi)) = \mathcal{H}_A(\Phi)$ by Remark 6.8. For every $u \in \mathrm{G}(E)$, put $\pi_E(u) := \pi_A(u)|_{\mathcal{H}_E(\Phi)}$. Then $\pi_E(u)(\mathcal{H}_E(\Phi)) \subseteq \mathcal{H}_E(\Phi)$ and so (π_A, π_E, P_E) is a *data* in the sense of Definition 2.10 (with holomorphic π_A and π_E). Similarly, set $\mathcal{H}_E^0(\Phi) := \overline{\operatorname{span}}(\pi_A(\mathrm{G}(E)^0)\mathcal{H}_B(\Phi))$ and P_E^0 the orthogonal projection from $\mathcal{H}_A(\Phi)$ onto $\mathcal{H}_E^0(\Phi)$, and then for every $u \in \mathrm{G}(E)^0$, define $\pi_E^0(u) := \pi_A(u)|_{\mathcal{H}_E^0(\Phi)}$.

Next set $D_B := \mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B(\Phi)$, $D_E^0 := \mathrm{G}_A \times_{\mathrm{G}(E)^0} \mathcal{H}_E(\Phi)$, and $D_E := \mathrm{G}_A \times_{\mathrm{G}(E)} \mathcal{H}_E(\Phi)$. Let $\mathcal{H}_B(P, \Phi)$, $\mathcal{H}_E^0(P_E^0, \Phi)$ and $\mathcal{H}_E(P_E, \Phi)$ denote the (reproducing kernel) Hilbert spaces of holomorphic sections in these bundles, respectively, given by Theorems 4.2 and 6.10(d).

Corollary 7.10. *Let $B \subseteq A$ be unital von Neumann algebras, $E : A \to B$ be a faithful, normal, conditional expectation, and use the above notations. Then the inclusion maps $\mathcal{H}_B(\Phi) \hookrightarrow \mathcal{H}_E^0(\Phi) \hookrightarrow \mathcal{H}_E(\Phi)$ and $\mathrm{G}_B \hookrightarrow \mathrm{G}(E)^0 \hookrightarrow \mathrm{G}(E)$ induce bundle homomorphisms*

$$\begin{array}{ccccc} D_B & \longrightarrow & D_E^0 & \longrightarrow & D_E \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{G}_A/\mathrm{G}_B & \longrightarrow & G_A/\mathrm{G}(E)^0 & \longrightarrow & \mathcal{S}(E), \end{array}$$

which leads to G_A-equivariant isometric isomorphisms $\mathcal{H}_B(P, \Phi) \to \mathcal{H}_E^0(P_E^0, \Phi) \to \mathcal{H}_E(P_E, \Phi)$. In particular, the Stinespring representation $\pi_A|_{\mathrm{G}_A} : \mathrm{G}_A \to \mathcal{B}(\mathcal{H}_A(\Phi))$ can be realized as the natural representation $\mu : \mathrm{G}_A \to \mathcal{B}(\mathcal{H}_E(P_E, \Phi))$ on the vector bundle D_E over the similarity orbit $\mathcal{S}(E)$.

Proof. Recall that $\mathcal{S}(E) \simeq \mathrm{G}_A/\mathrm{G}(E)$ and the elements or sections of the spaces $\mathcal{H}_B(P, \Phi)$, $\mathcal{H}_E^0(P_E^0, \Phi)$ and $\mathcal{H}_E(P_E, \Phi)$ are of the form

$$u\mathrm{G}_B \mapsto \big[\big(u, P\big(\pi_A\big(u^{-1}\big)h\big)\big)\big]; \qquad u\mathrm{G}(E)^0 \mapsto \big[\big(u, P_E^0\big(\pi_A\big(u^{-1}\big)h\big)\big)\big];$$
$$E_u \cong u\mathrm{G}(E) \mapsto \big[\big(u, P_E\big(\pi_A\big(u^{-1}\big)h\big)\big)\big],$$

respectively, for h running over $\mathcal{H}_A(\Phi)$. This gives us the quoted isometries. The fact that $\pi_A|_{\mathrm{G}_A}$ is realized as μ acting on $\mathcal{H}_E(P_E, \Phi)$ is a consequence of Theorem 4.4. □

Corollary 7.10 admits a version in the unitary setting, that is, for the unitary groups U_A, U_B, $\mathrm{U}(E)^0$, $\mathrm{U}(E)$ and unitary orbit $\mathcal{U}(E)$ playing the role of the corresponding invertible groups and orbit. The following result answers in the affirmative the natural question of whether the similarity orbit $\mathcal{S}(E) \simeq \mathrm{G}_A/\mathrm{G}(E)$ endowed with the involutive diffeomorphism $a\mathrm{G}(E) \mapsto a^{-*}\mathrm{G}(E)$ is the complexification of the unitary orbit $\mathcal{U}(E)$ of the conditional expectation E.

Corollary 7.11. *In the above situation, the similarity orbit $\mathcal{S}(E)$ of the conditional expectation E is a complexification of its unitary orbit $\mathcal{U}(E)$, and it is also U_A-equivariantly diffeomorphic to the tangent bundle of $\mathcal{U}(E)$.*

Proof. Since the tangent bundles of $\mathcal{U}(E)$ and $\mathrm{U}_A/\mathrm{U}(E)$ coincide the assertion that the tangent bundle of $\mathcal{U}(E)$ is U_A-equivariantly diffeomorphic to $\mathrm{G}_A/\mathrm{G}(E)$ is a consequence of Corollary 5.5. On the other hand, as recalled above, to prove the fact that $\mathrm{G}_A/\mathrm{G}(E)$ is the complexification of $\mathrm{U}_A/\mathrm{U}(E)$ it will be enough to check that $\mathrm{G}(E)^+ = \mathrm{G}_A^+ \cap \mathrm{G}(E)$ (and then to apply Lemma 4.3). The inclusion $\subseteq$ is obvious. Now let $c \in \mathrm{G}_A^+ \cap \mathrm{G}(E)$. By Definition 2.6, there exists $g \in \mathrm{G}_A$ such that $c = g^*g \in \mathrm{G}(E)$. Then the reasoning from the proof of [4, Theorem 3.5] shows that $g^*g = ab$ with $a \in \mathrm{G}_{A_E}^+$ and $b \in \mathrm{G}_B^+$, whence $c = ab \in \mathrm{G}_{A_E}^+ \cdot \mathrm{G}_B^+ \subseteq \mathrm{G}(E)^+$. □

Remark 7.12. In connection with the commutative diagram of Corollary 7.10, note that since $\mathrm{G}(E)^0$ is the connected $\mathbf{1}$-component of $\mathrm{G}(E)$, it follows that the arrow $\mathrm{G}_A/\mathrm{G}(E)^0 \to \mathrm{G}_A/\mathrm{G}(E) = \mathcal{S}(E)$ is actually a covering map whose fiber is the *Weyl group* $\mathrm{G}(E)/\mathrm{G}(E)^0$ of the conditional expectation E (cf. [4] and the references therein).

Remark 7.13. It is interesting to observe how Corollary 7.10 looks in the case when $\mathfrak{g}(E)$ is an associative algebra, as in the second part of Proposition 7.7.

Thus let assume that for a conditional expectation $E : A \to A$ as in former situations we have that $B := E(A) \subseteq A'$. Then B is commutative and $A \subseteq B'$ (note that B' need not be commutative; in other words, B is not maximal abelian). Hence, by Proposition 7.7,

$$\mathfrak{g}(E) = \{X \in A \mid E(aX) = E(Xa),\ a \in A\} = \{X \in B' \cap A \mid E(aX) = E(Xa),\ a \in A\} := A_E.$$

By Proposition 7.9, $A_E + B = \Delta(A) = \mathfrak{g}(E) = A_E$ whence $B \subseteq A_E$. Also, as regards to groups, Proposition 7.9 applies to give $\mathrm{G}_{\mathfrak{g}(E)} = \mathrm{G}(E)$ whence we have

$$\mathrm{G}(E) = \mathrm{G}_{\mathfrak{g}(E)} = \mathrm{G}_{A_E} \subseteq \mathrm{G}_{A_E} \cdot \mathrm{G}_B = \mathrm{G}(E)^0 \subseteq \mathrm{G}(E),$$

and we obtain that $\mathrm{G}(E)^0 = \mathrm{G}(E)$. This implies that the bundles $D_E^0 \to \mathrm{G}_A/\mathrm{G}(E)^0$ and $D_E \to \mathcal{S}(E)$ of Corollary 7.10 coincide.

Moreover, from the fact that $B \subseteq A_E = \mathfrak{g}(E)$ it follows that $F \circ E = E$ where F is the conditional expectation given prior to Proposition 7.9. In fact, for $a \in A$, $E(a) \in A_E = F(A)$ so there is some $a' \in A$ such that $E(a) = F(a')$. Then $(FE)(a) = F(F(a')) = (FF)(a') = F(a') = E(a)$ as required. Since $FE = EF$ we have eventually that $FE = EF = E$.

Suppose now that $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ is a completely positive mapping such that $\Phi \circ E = \Phi$. Then $\Phi \circ F = (\Phi \circ E) \circ F = \Phi \circ (EF) = \Phi \circ E = \Phi$, and one can use again the argument preceding Corollary 7.10 to find vector bundles with corresponding Hilbert spaces (fibers) and representations $\pi_A : A \to \mathcal{B}(\mathcal{H}_A(\Phi))$, $\pi_{A_E} : A_E \to \mathcal{B}(\mathcal{H}_{A_E}(\Phi))$ and $\pi_B : B \to \mathcal{B}(\mathcal{H}_B(\Phi))$, and so on. In particular, from $E|_{A_E} : A_E \to B$ one gets $\mathcal{H}_{A_E}(\Phi) = \overline{\mathrm{span}}\ \pi_{A_E}(\mathrm{G}(E))\mathcal{H}_B(\Phi) = \overline{\mathrm{span}}\, \pi_A(\mathrm{G}(E))\mathcal{H}_B(\Phi) = \mathcal{H}_E(\Phi)$. Hence, in this case, the bundle $D_E \to \mathcal{S}(E)$ is a Stinespring bundle with respect to data $\pi_{A_E}|_{\mathrm{G}(E)}$, $\pi_B|_{\mathrm{G}_B}$ (and the corresponding projection) to which Theorem 6.10 can be applied.

More precisely, part (c) of that theorem implies that $\mathcal{S}(E)$ is diffeomorphic to the tangent bundle $\mathrm{U}_A \times_{\mathrm{U}(E)} \mathfrak{p}^F$ of $\mathcal{U}(E)$, where $\mathfrak{p}^F = \ker F \cap \mathfrak{u}_A$, in the same way as $\mathrm{G}_A/\mathrm{G}_B$ is diffeomorphic to $\mathrm{U}_A \times_{\mathrm{U}_B} \mathfrak{p}^E$, $\mathfrak{p}^E = \ker E \cap \mathfrak{u}_A$.

7.4. Representations of Cuntz algebras

We wish to illustrate the theorem on geometric realizations of Stinespring representations by an application to representations of Cuntz algebras. For the sake of simplicity we shall be working in the classical setting [22], although a part of what we are going to do can be extended to more general versions of these C^*-algebras (see [23,51], and also [26]) or to more general C^*-dynamical systems.

Example 7.14. Let $N \in \{2, 3, \dots\} \cup \{\infty\}$ and denote by $\mathcal{O}_N$ the C^*-algebra generated by a family of isometries $\{v_j\}_{0 \leqslant j < N}$ that act on the same Hilbert space and satisfy the condition that their ranges are mutually orthogonal, and in addition $v_0 v_0^* + \dots + v_{N-1} v_{N-1}^* = \mathbf{1}$ in the case $N \neq \infty$. The Cuntz algebra $\mathcal{O}_N$ has a canonical uniqueness property with respect to the choice of the generators $\{v_j\}_{1 \leqslant j < N}$ subject to the above conditions (see [22]). In particular, this implies that there exists a pointwise continuous *gauge $*$-automorphism group* parameterized by the unit circle, $\lambda \mapsto \tau(\lambda) = \tau_\lambda$, $\mathbb{T} \to \mathrm{Aut}(\mathcal{O}_N)$, such that $\tau_\lambda(v_j) = \lambda v_j$ if $0 \leqslant j < N$ and $\lambda \in \mathbb{T}$. For every $m \in \mathbb{Z}$ we denote

$$\mathcal{O}_N^{(m)} = \bigl\{x \in \mathcal{O}_N \mid (\forall \lambda \in \mathbb{T})\ \tau_\lambda(x) = \lambda^m x\bigr\} \tag{7.1}$$

the spectral subspace associated with m, and then $\mathcal{F}_N := \mathcal{O}_N^{(0)}$ (the fixed-point algebra of the gauge group). For every $m \in \mathbb{Z}$ we have a contractive surjective linear idempotent mapping $E^{(m)} : \mathcal{O}_N \to \mathcal{O}_N^{(m)}$ defined by

$$(\forall x \in \mathcal{O}_N) \quad E^{(m)}(x) = \int_{\mathbb{T}} \lambda^{-m} \tau_\lambda(x)\, \mathrm{d}\lambda, \tag{7.2}$$

which is a faithful conditional expectation in the case $m = 0$ (see for instance [24, Theorem V.4.3]). We shall denote $E^{(0)} = E$ for the sake of simplicity.

The following statement is inspired by some remarks from Section 2 in [37]. It shows that, under specific hypothesis on the C^*-algebras from Theorem 6.10, the corresponding reproducing kernel Hilbert space has a circular symmetry that resembles the one of the classical spaces of holomorphic functions on the unit disk. Thus we get series expansions and the natural setting of harmonic analysis in the spaces of bundle sections associated with completely positive maps.

Corollary 7.15. *Let $N \in \{2, 3, \dots\} \cup \{\infty\}$, a completely positive unital map $\Phi : \mathcal{O}_N \to \mathcal{B}(\mathcal{H}_0)$, and the corresponding Stinespring representation $\pi_\Phi : \mathcal{O}_N \to \mathcal{B}(\mathcal{K}_0)$, and isometry $V : \mathcal{H}_0 \to \mathcal{K}_0$ such that $\Phi = V^* \pi V$. Put $\widetilde{\mathcal{H}}_0 := V(\mathcal{H}_0)$. Then the condition $\Phi \circ E = \Phi$ is satisfied if and only if Φ is gauge invariant, in the sense that for each $\lambda \in \mathbb{T}$ we have $\Phi \circ \tau_\lambda = \Phi$. In addition, if this is the case, then the following assertions hold*:

(a) *Consider the geometric realization* $\gamma:\mathcal{K}_0\to\mathcal{H}(E,\Phi)$ *of the Stinespring representation* π_Φ *and let* $\Pi:D\to \mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$ *be the corresponding homogeneous vector bundle. Then the gauge automorphism group of* $\mathcal{O}_N$ *induces smooth actions* $\widetilde{\tau}$ *and* $\bar{\tau}$ *of the circle group* $\mathbb{T}$ *on the total space* D *and the base* $\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$, *respectively, such that the diagram*

$$\begin{tikzcd}
\mathbb{T}\times D \arrow[r, "\widetilde{\tau}"] \arrow[d, "\mathrm{id}_{\mathbb{T}}\times\Pi"'] & D \arrow[d, "\Pi"] \\
\mathbb{T}\times(\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}) \arrow[r, "\bar{\tau}"] & \mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}
\end{tikzcd}$$

is commutative. The action on the base of the vector bundle also commutes with the natural involutive diffeomorphism thereof.

(b) *If for all* $m\in\mathbb{Z}$ *we denote by* $\mathcal{H}(E,\Phi)^{(m)}$ *the closed linear subspace generated by* $\gamma(\pi_\Phi(\mathcal{O}_N^{(m)})\widetilde{\mathcal{H}}_0)$, *then we have the orthogonal direct sum decomposition* $\mathcal{H}(E,\Phi)=\bigoplus_{m\in\mathbb{Z}}\mathcal{H}(E,\Phi)^{(m)}$ *and each term of this decomposition is* $\mathrm{G}_{\mathcal{F}_N}$*-invariant.*

(c) *For each* $m\in\mathbb{Z}$, *the orthogonal projection* $P^{(m)}_{E,\Phi}:\mathcal{H}(E,\Phi)\to\mathcal{H}(E,\Phi)^{(m)}$ *is given by the formula*

$$\big(P^{(m)}_{E,\Phi}\Delta\big)(z)=\int_{\mathbb{T}}\lambda^{-m}\big(\widetilde{\tau}_\lambda\circ\Delta\circ\bar{\tau}_\lambda^{-1}\big)(z)\,\mathrm{d}\lambda$$

whenever $z\in\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$ *and* $\Delta\in\mathcal{H}(E,\Phi)$.

Proof. Firstly note that (7.2) implies that $E\circ\tau_\lambda=\tau_\lambda\circ E=E$ for all $\lambda\in\mathbb{T}$, because of the invariance property of the Haar measure $\mathrm{d}\lambda$ on the unit circle $\mathbb{T}$. Consequently, if we assume that Φ is gauge invariant, then for all $x\in\mathcal{O}_N$ we have $\Phi(E(x))=\Phi(\int_{\mathbb{T}}\tau_\lambda(x)\,\mathrm{d}\lambda)=\int_{\mathbb{T}}\Phi(\tau_\lambda(x))\,\mathrm{d}\lambda=\int_{\mathbb{T}}\Phi(x)\,\mathrm{d}\lambda=\Phi(x)$. Conversely, if $\Phi\circ E=\Phi$, then for every $\lambda\in\mathbb{T}$ we have $\Phi\circ\tau_\lambda=\Phi\circ E\circ\tau_\lambda=\Phi\circ E=\Phi$.

(a) To define the action of $\mathbb{T}$ upon the base $\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$ we use the fact that each gauge automorphism τ_λ leaves $\mathcal{F}_N$ pointwise invariant and therefore induces a mapping of $\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$ onto itself. It is straightforward to show that in this way we get an action $(\lambda, a\mathrm{G}_{\mathcal{F}_N})\mapsto\tau_\lambda(a)\mathrm{G}_{\mathcal{F}_N}$ of $\mathbb{T}$ as claimed. The action of the circle group upon the total space D can be defined by the formula $[(a,f)]\mapsto[(\tau_\lambda(a),f)]$ for all $[(a,f)]\in D$ and $\lambda\in\mathbb{T}$.

(b) The realization operator $\gamma:\mathcal{K}_0\to\mathcal{H}(E,\Phi)$ is unitary, hence it will be enough to prove that $\mathcal{K}_0=\bigoplus_{m\in\mathbb{Z}}\mathcal{K}_0^{(m)}$ and that each term of this decomposition is invariant under all of the operators in the C^*-algebra $\pi_\Phi(\mathcal{F}_N)$, where $\mathcal{K}_0^{(m)}$ is the closed linear subspace of $\mathcal{K}_0$ spanned by $\pi_\Phi(\mathcal{O}_N^{(m)})\widetilde{\mathcal{H}}_0$ for all $m\in\mathbb{Z}$. (Note that $\mathcal{K}_0=\overline{\operatorname{span}}\,\pi_\Phi(\mathcal{F}_N)\widetilde{\mathcal{H}}_0$ by construction.)

The proof of this assertion follows the lines of [37, Section 1] and relies on the fact that, as an easy consequence of (7.1), we have $\mathcal{O}_N^{(m)}\mathcal{O}_N^{(n)}\subseteq\mathcal{O}_N^{(m+n)}$ and $(\mathcal{O}_N^{(m)})^*\subseteq\mathcal{O}_N^{(-m)}$ for all $m,n\in\mathbb{Z}$ (where $(\cdot)^*$ stands for the image under the involution of $\mathcal{O}_N$). Note that $VV^*:\mathcal{K}_0\to\widetilde{\mathcal{H}}_0$ is the orthogonal projection from $\mathcal{K}_0$ onto $\widetilde{\mathcal{H}}_0$. It follows that for all $m,n\in\mathbb{Z}$ with $m\neq n$, and $x\in\mathcal{O}_N^{(m)}$, $y\in\mathcal{O}_N^{(n)}$, $\xi,\eta\in\mathcal{H}_0$ we have

$$\big(\pi_\Phi(x)V\xi \mid \pi_\Phi(y)V\eta\big) = \big(\pi_\Phi(y^*x)V\xi \mid V\eta\big) = \big(VV^*(\pi_\Phi(y^*x)V\xi) \mid V\eta\big)$$
$$= \big(\Phi\big(y^*x\big)\xi \mid \eta\big)_{\mathcal{H}_0} = \big(\Phi\big(E\big(y^*x\big)\big)\xi \mid \eta\big)_{\mathcal{H}_0} = 0,$$

where the latter equality follows since $y^*x \in \mathcal{O}_N^{(m-n)}$ with $m-n \neq 0$, so that $E(y^*x) = 0$ as an easy consequence of (7.2). The above computation shows that $\mathcal{K}_0^{(m)} \perp \mathcal{K}_0^{(n)}$ whenever $m \neq n$.

To see that $\bigcup_{m\in\mathbb{Z}} \mathcal{K}_0^{(m)}$ spans the whole $\mathcal{K}_0$, just recall from [22] that the set $\bigcup_{m\in\mathbb{Z}} \mathcal{O}_N^{(m)}$ spans a dense linear subspace of $\mathcal{O}_N$, and use the image of this set under the unital $*$-homomorphism $\pi_\Phi : \mathcal{O}_N \to \mathcal{B}(\mathcal{K}_0)$. Consequently the asserted orthogonal direct sum decomposition of $\mathcal{K}_0$ is proved. On the other hand, since $\mathcal{F}_N = \mathcal{O}_N^{(0)}$, it follows that for all $m \in \mathbb{Z}$ we have $\mathcal{F}_N\mathcal{O}_N^{(m)} = \mathcal{O}_N^{(0)}\mathcal{O}_N^{(m)} \subseteq \mathcal{O}_N^{(m)}$, so $\pi_\Phi(\mathcal{F}_N)\mathcal{K}_0^{(m)} \subseteq \mathcal{K}_0^{(m)}$ according to the construction of $\mathcal{K}_0^{(m)}$.

(c) For every $\lambda \in \mathbb{T}$ denote by $a \otimes f \mapsto \overline{a \otimes f}$, $\mathcal{O}_N \otimes \mathcal{H}_0 \to \mathcal{K}_0$, the canonical map. Since $\Phi \circ \tau_\lambda = \Phi$, it follows that the mapping $a \otimes f \mapsto \tau_\lambda(a) \otimes f$, $\mathcal{O}_N \otimes \mathcal{H}_0 \to \mathcal{O}_N \otimes \mathcal{H}_0$, induces a unitary operator $V_\lambda : \overline{a \otimes f} \mapsto \overline{\tau_\lambda(a) \otimes f}$, $\mathcal{K}_0 \to \mathcal{K}_0$. As the closure of the image of $\mathcal{O}_N^{(m)} \otimes \mathcal{H}_0$ in $\mathcal{K}_0$ is equal to $\mathcal{K}_0^{(m)}$, it follows that for all $m \in \mathbb{Z}$ we have

$$\mathcal{K}_0^{(m)} = \big\{h \in \mathcal{K}_0 \mid (\forall \lambda \in \mathbb{T})\ V_\lambda h = \lambda^m h\big\}. \tag{7.3}$$

On the other hand, for all $c, a \in \mathcal{O}_N$ and $f \in \mathcal{H}_0$ we have

$$\pi_\Phi\big(\tau_\lambda(c)\big)\overline{a \otimes f} = \overline{\tau_\lambda(c)a \otimes f} = \overline{\tau_\lambda\big(c\tau_{\lambda^{-1}}(a)\big) \otimes f} = V_\lambda\overline{c\tau_{\lambda^{-1}}(a) \otimes f} = V_\lambda\pi_\Phi(c)\overline{\tau_{\lambda^{-1}}(a) \otimes f}$$
$$= V_\lambda\pi_\Phi(c)V_{\lambda^{-1}}\overline{a \otimes f},$$

whence

$$(\forall \lambda \in \mathbb{T},\ c \in \mathcal{O}_N) \quad \pi_\Phi\big(\tau_\lambda(c)\big) = V_\lambda\pi_\Phi(c)V_{\lambda^{-1}}. \tag{7.4}$$

It follows by (7.3) and (7.4) that

$$\big(\forall \lambda \in \mathbb{T},\ n \in \mathbb{Z},\ c \in \mathcal{O}_N,\ h \in \mathcal{K}_0^{(n)}\big) \quad \pi_\Phi\big(\tau_\lambda(c)\big)h = \lambda^{-n}V_\lambda\pi_\Phi(c)h. \tag{7.5}$$

Now for $\lambda \in \mathbb{T}$, $\eta \in \mathcal{K}_0$, and $c \in \mathrm{G}_{\mathcal{O}_N}$ we get

$$\big(\widetilde{\tau}_\lambda \circ \gamma(\eta) \circ \bar{\tau}_\lambda^{-1}\big)(c\mathrm{G}_{\mathcal{F}_N}) = \widetilde{\tau}_\lambda\big(\gamma(\eta)\big(\tau_{\lambda^{-1}}(c)\mathrm{G}_{\mathcal{F}_N}\big)\big) = \widetilde{\tau}\big[\big(\tau_{\lambda^{-1}}(c), P\big(\pi_\Phi\big(\tau_{\lambda^{-1}}(c^{-1})\eta\big)\big)\big)\big]$$
$$= \big[\big(c, P\big(\pi_\Phi\big(\tau_{\lambda^{-1}}\big(c^{-1}\big)\eta\big)\big)\big)\big],$$

whence by (7.5) it follows that $(\widetilde{\tau}_\lambda \circ \gamma(h) \circ \bar{\tau}_\lambda^{-1})(c\,\mathrm{G}_{\mathcal{F}_N}) = \lambda^n[(c, P(V_{\lambda^{-1}}\pi_\Phi(c^{-1})h))]$, for $h \in \mathcal{K}_0^{(n)}$ where for $m \in \mathbb{Z}$, $P^{(m)} : \mathcal{K}_0 \to \mathcal{K}_0^{(m)}$ is the orthogonal projection and $P = P^{(0)}$.

On the other hand, it follows by (7.3) that $\int_{\mathbb{T}} \lambda^{n-m}(V_\lambda h)\,\mathrm{d}\lambda = P^{(m-n)}h$ for all $h \in \mathcal{K}_0$, and thus by the above equality we get for $h \in \mathcal{K}_0^{(n)}$,

$$\int_{\mathbb{T}} \lambda^{-m}\big(\widetilde{\tau}_\lambda \circ \gamma(h) \circ \widetilde{\tau}_\lambda^{-1}\big)(c\mathrm{G}_{\mathcal{F}_N})\,\mathrm{d}\lambda = \int_{\mathbb{T}} \lambda^{n-m}\big[\big(c, P\big(V_{\lambda^{-1}}\pi_\Phi\big(c^{-1}\big)h\big)\big)\big]\,\mathrm{d}\lambda$$
$$= \big[\big(c, P^{(0)}P^{(n-m)}\big(\pi_\Phi\big(c^{-1}\big)h\big)\big)\big]$$
$$= \begin{cases} \gamma(h)(c\mathrm{G}_{\mathcal{F}_N}) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Thus we have proved the asserted formula for $\Delta = \gamma(h)$ with $h \in \bigcup_{n\in\mathbb{Z}} \mathcal{K}_0^{(n)}$. On the other hand, the reproducing $(-*)$-kernel associated with the Stinespring representation π_Φ is continuous, hence the realization operator $\gamma : \mathcal{K}_0 \to \mathcal{O}(\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}, D)$ is continuous with respect to the uniform convergence on the compact subsets of the base $\mathrm{G}_{\mathcal{O}_N}/\mathrm{G}_{\mathcal{F}_N}$. Since $\mathcal{K}_0 = \bigoplus_{n\in\mathbb{Z}} \mathcal{K}_0^{(n)}$ and the right-hand side of the asserted formula is linear and continuous with respect to the latter topology, it follows that the corresponding equality extends by linearity and continuity to the whole space $\mathcal{H}(E, \Phi) = \gamma(\mathcal{K}_0)$. □

Remark 7.16. It is noteworthy that orthogonal decompositions similar to the one of Corollary 7.15(b) also show up in connection with representations of Cuntz algebras that do not necessarily occur as Stinespring dilations of gauge invariant maps; see for instance the representations studied in [17].

There is a close relationship between $*$-endomorphisms of algebras $\mathcal{B}(\mathcal{H})$ and representations of Cuntz algebras, see for instance [36,37]. In the remainder of this subsection, we point out that some of the notions underlying this relationship provide us with more examples of the theory proposed in the present paper. Thus let $\mathcal{H}$ be a separable Hilbert space, let $A := \mathcal{B}(\mathcal{H})$ and let $\alpha : A \to A$ a unital $*$-endomorphism; then α is normal, as noted for instance in [36]. By a celebrated result of W. Arveson there exist $N \in \{1, \dots, +\infty\}$ and a $*$-representation $\rho : \mathcal{O}_N \to A$, where $\mathcal{O}_N$ is the Cuntz algebra generated by N isometries v_j, on $\mathcal{H}$, such that

$$\alpha(T) = \sum_{j=1}^{N} \rho(v_j) T \rho\big(v_j^*\big), \quad T \in A. \tag{7.6}$$

(See [5, Proposition 2.1].) Since α is unital we have that $\sum_{j=1}^{N} \rho(v_j)\rho(v_j^*) = \mathbf{1}$ even for $N = \infty$ (in the strong operator topology, in the latter case). In the sequel we assume that $N \geqslant 2$.

We do observe that, in order to make a link between the geometry of ρ and the one of α, the endomorphism α can be regarded either as a $*$-representation of the injective von Neumann algebra A on $\mathcal{H}$ or as a completely positive mapping from A into $\mathcal{B}(\mathcal{H})$. In fact, it seems natural at first glance to take into consideration the second option, since (7.6) induces the correspondence $u\rho u^{-1} \mapsto u\alpha(u^{-1} \cdot u)u^{-1}$ $(u \in \mathrm{G}_A)$; that is, the canonical map $\mathfrak{S}(\rho) \to \mathcal{S}(\alpha)$. Nevertheless, in such a case we cannot be sure that the algebraic isomorphism $\mathcal{S}(\alpha) \simeq \mathrm{G}_A/\mathrm{G}(\alpha)$ entails a structure of smooth homogeneous manifold. Namely, the Lie algebra of $\mathrm{G}(\alpha)$ is $\mathfrak{g}(\alpha) = \{X \in A \mid X - \alpha(X) \in \alpha(A)'\}$, see Lemma 7.6, and it is not clear that $\mathfrak{g}(\alpha)$ is topologically complemented in A.

On the other hand, by looking at α as a $*$-representation and by using a bit of the structure of von Neumann algebras, it is possible to relate the orbits $\mathfrak{S}(\rho)$ and $\mathfrak{S}(\alpha)$ to each other, as we are going to see.

Since A is Connes-amenable (that is, A is injective, see [55]) an appropriate virtual diagonal M for A can be fixed, so that the mapping

$$E_\alpha(T) := \int_{A\otimes A} \alpha(a)T\alpha(b)\,\mathrm{d}M(a,b) \quad (T \in A),$$

is a conditional expectation from A onto $B_\alpha := E_\alpha(A)$. This endows $\mathfrak{S}(\alpha)$ with the corresponding smooth homogeneous manifold structure.

Similarly, $\mathcal{O}_N$ is a nuclear (amenable) C^*-algebra, see [22], and therefore we can fix a contractive virtual diagonal M_N for $\mathcal{O}_N$ so that the mapping

$$E_\rho(T) := \int_{\mathcal{O}_N\otimes\mathcal{O}_N} \rho(s)T\rho(t)\,\mathrm{d}M_N(s,t) \quad (T \in A)$$

is a conditional expectation from A onto $B_\rho := E_\rho(A)$. In this way, we regard $\alpha : A \to \mathcal{B}(\mathcal{H})$ and $\rho : \mathcal{O}_N \to A$ as special cases of Example 7.1.

There are two C^*-subalgebras of A which are important in the study of the endomorphism α. These are $\{a \in A \mid \alpha(a) = a\}$ and $\bigcap_{n>0} \alpha^n(A)$. Recall that $B_\alpha = \{a \in A \mid E_\alpha(a) = a\} = \alpha(A)'$ and $B_\rho = \{a \in A \mid E_\rho(a) = a\} = \rho(\mathcal{O}_N)'$, see [20]. (In this notation, $\mathfrak{g}(\alpha) = (I - \alpha)^{-1}(E_\alpha(A))$.) By [36, Proposition 3.1(i)] we have $B_\rho = \rho(\mathcal{O}_N)' = \{a \in \mathcal{B}(\mathcal{H}) \mid \alpha(a) = a\}$ (which means in particular that $B_\rho = \mathbb{C}\mathbf{1}$ if the representation ρ is irreducible or, equivalently, if α is an *ergodic* endomorphism of $A = \mathcal{B}(\mathcal{H})$). Hence, for every $b \in B_\rho$ we get $b = \alpha(b) \in E_\alpha(A)'$. Shortly,

$$B_\rho \subseteq \alpha(A)'' = \alpha(A). \tag{7.7}$$

Note that $\alpha(A)'$ is w^*-closed in A and so it is a von Neumann subalgebra of A. Moreover $\alpha(A)'$ is the range of the conditional expectation E_α and then it turns out to be injective. Let Δ be the anti-unitary operator (Δ is an antilinear isometry with $\Delta^2 = \mathrm{id}$) which appears in the standard form of $\alpha(A)'$, see [63, vol. II]. Then the mapping $E_{\Delta\alpha\Delta} : A \to A$ given by

$$E_{\Delta\alpha\Delta}(T) := \Delta E_\alpha(\Delta T \Delta)\Delta \quad (T \in A)$$

is a conditional expectation such that $\alpha(A)'' = E_{\Delta\alpha\Delta}(A)$. It follows that $\alpha(A)''$ is also injective, see [63, vol. III]. The above notation is not by chance since $E_{\Delta\alpha\Delta}$ corresponds exactly to the expectation defined by the virtual diagonal M and representation $\Delta\alpha\Delta$. Thus we have

$$B_\rho \subseteq B_{\Delta\alpha\Delta}. \tag{7.8}$$

Passing to quotients, (7.8) implies that we obtain a canonical surjection $\mathfrak{S}(\rho) \to \mathfrak{S}(\Delta\alpha\Delta)$. In fact, it is a submersion mapping: the above map is G_A-equivariant and its tangent map at ρ is implemented by the bounded projection

$$\mathrm{id} - E_{\Delta\alpha\Delta} : (\mathrm{id} - E_\rho)(a) \mapsto (\mathrm{id} - E_{\Delta\alpha\Delta})(a)$$

from A/B_ρ onto $A/B_{\Delta\alpha\Delta}$.

On the other hand, the involutive transformation $\beta \mapsto \Delta\beta(\cdot)\Delta$ carries diffeomorphically similarity (unitary) orbits onto similarity (unitary) orbits of the space of representations $\mathrm{Rep}^{\omega}(A)$. In particular $\mathfrak{S}(\alpha)$ and $\mathfrak{S}(\Delta\alpha\Delta)$ are diffeomorphic through the map

$$a\alpha a^{-1} \mapsto (\Delta a\Delta)(\Delta\alpha\Delta)(\Delta a^{-1}\Delta), \quad \mathfrak{S}(\alpha) \to \mathfrak{S}(\Delta\alpha\Delta)$$

(note that $u\alpha = \alpha u$ if and only if $(\Delta u\Delta)(\Delta\alpha\Delta) = (\Delta\alpha\Delta)(\Delta u\Delta)$). Putting all the above facts together we obtain the analytic submersion $\mathfrak{S}(\rho) \to \mathfrak{S}(\alpha)$ given by

$$a\rho a^{-1} \mapsto (\Delta a\Delta)\alpha\big(\Delta a^{-1}\Delta\big) \quad (a \in \mathrm{G}_A).$$

At this point one can return to (7.7), which also tells us that

$$B_\alpha = \alpha(A)' \subseteq B'_\rho. \tag{7.9}$$

Thus we can proceed as formerly, just replacing α with ρ. In particular, the von Neumann algebra $B_\rho = \rho(\mathcal{O}_N)'$ is injective and then its commutant subalgebra B'_ρ in A is also injective. In effect, it is the image $B'_\rho = E_{R\rho R}(A)$ of the conditional expectation $E_{R\rho R}$ defined by the virtual diagonal M_N and representation $R\rho R$, where R is the anti-unitary operator involved in the standard form of $\rho(\mathcal{O}_N)'$. Hence, there exists an analytic submersion $\mathfrak{S}(\alpha) \to \mathfrak{S}(\rho)$ given by

$$u\rho u^{-1} \mapsto (RuR)\rho\big(Ru^{-1}R\big) \quad (u \in \mathrm{G}_A).$$

In summary, we have proved the following result.

Theorem 7.17. *Let Δ, R be the anti-unitary operators associated with the standard forms of the injective von Neumann algebras $\alpha(A)'$, $\rho(\mathcal{O}_N)'$, respectively. Then $B_\rho \subseteq B_{\Delta\alpha\Delta}$ and $B_\alpha \subseteq B_{R\rho R}$, and the mappings*

$$q_\rho : a\rho a^{-1} \mapsto (\Delta a\Delta)\alpha\big(\Delta a^{-1}\Delta\big), \quad \mathfrak{S}(\rho) \to \mathfrak{S}(\alpha),$$
$$q_\alpha : u\alpha u^{-1} \mapsto (RuR)\rho\big(Ru^{-1}R\big), \quad \mathfrak{S}(\alpha) \to \mathfrak{S}(\rho)$$

are analytic submersions.

Remark 7.18. The joint action of suitably chosen mappings in the proposition yields new submersions

$$a\rho a^{-1} \mapsto \big(VaV^{-1}\big)\rho\big(Va^{-1}V^{-1}\big), \quad \mathfrak{S}(\rho) \to \mathfrak{S}(\rho)$$

and

$$u\alpha u^{-1} \mapsto \big(V^{-1}uV\big)\alpha\big(V^{-1}u^{-1}V\big), \quad \mathfrak{S}(\alpha) \to \mathfrak{S}(\alpha),$$

where V is the unitary operator $V = R\Delta$. Such submersions need not be diffeomorphisms.

Because of the inclusion $B_\rho \subseteq B_{\Delta\alpha\Delta}$ we have $E_{\Delta\alpha\Delta} \circ E_\rho = E_\rho$. Set $F_\rho := E_\rho E_{\Delta\alpha\Delta}$. Then F_ρ is also a conditional expectation and F_ρ and E_ρ are equivalent: $F_\rho E_\rho = E_\rho$ and $E_\rho F_\rho = F_\rho$, so that $F_\rho(A) = B_\rho$. In addition, F_ρ and $E_{\Delta\alpha\Delta}$ commute: $E_{\Delta\alpha\Delta} F_\rho = F_\rho = F_\rho E_{\Delta\alpha\Delta}$.

Let $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ be a completely positive map, for some Hilbert space $\mathcal{H}_0$. Put $\Phi_\rho := \Phi \circ F_\rho$. Then $\Phi_\rho F_\rho = \Phi_\rho$ and $\Phi_\rho E_{\Delta\alpha\Delta} = \Phi_\rho$. Applying Stinespring's dilation theorem we find Hilbert spaces $\mathcal{H}_J(\Phi_\rho)$, for $J = A, B_{\Delta\alpha\Delta}, B_\rho$ with $\mathcal{H}_{B_\rho}(\Phi_\rho) \subseteq \mathcal{H}_{B_{\Delta\alpha\Delta}}(\Phi_\rho) \subseteq \mathcal{H}_A(\Phi_\rho)$, and $*$-representations $\pi_J : J \to \mathcal{B}(\mathcal{H}_J(\Phi_\rho))$ satisfying $\pi_{B_{\Delta\alpha\Delta}}(u) = \pi_A(u)|_{\mathcal{H}_{B_{\Delta\alpha\Delta}}(\Phi_\rho)}$ for each $u \in B_{\Delta\alpha\Delta}$, and $\pi_{B_\rho}(u) = \pi_{B_{\Delta\alpha\Delta}}(u)|_{\mathcal{H}_{B_\rho}(\Phi_\rho)}$ for each $u \in B_\rho$.

Corollary 7.19. *In the above setting, there exists the commutative diagram*

$$
\begin{tikzcd}
\mathrm{G}_A \times_{\mathrm{G}_{B_\rho}} \mathcal{H}_{B_\rho}(\Phi_\rho) \arrow[r] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_{B_\alpha}} \mathcal{H}_B(\Phi_\rho) \arrow[d] \\
\mathfrak{S}(\rho) \arrow[r, "q_\rho"] & \mathfrak{S}(\alpha)
\end{tikzcd}
$$

whose arrows are G_A*-equivariant and compatible with the involutive diffeomorphisms* $-*$ *on both* $\mathfrak{S}(\rho)$ *and* $\mathfrak{S}(\alpha)$*. Moreover, the representation* π_A *of* G_A *on* $\mathcal{H}_A(\Phi_\rho)$ *can be extended to* A *and realized as multiplication on a reproducing kernel Hilbert space formed by holomorphic sections of the left-side vector bundle in the diagram.*

Proof. Firstly, a diagram similar to that one of the statement, and concerning the algebras $B_\rho \subseteq B_{\Delta\alpha\Delta}$, is immediately obtained by mimicking the proof of Corollary 7.10. Then using the diffeomorphism $\Delta(\cdot)\Delta$ one gets the wanted result or diagram, where the action of B_α on $\mathcal{H}_B(\Phi_\rho)$ is given just by transferring the action of $B_{\Delta\alpha\Delta}$ through $\Delta(\cdot)\Delta$ (note that $B_{\Delta\alpha\Delta} = \Delta B_\alpha \Delta$). □

A diagram entirely analog to the previous one of the corollary can be obtained by interchanging roles of representations ρ and α.

Remark 7.20. Using the representation ρ, we can make a link between Corollary 7.15 and the preceding setting. Let τ be the gauge automorphism group of $\mathcal{O}_N$, and let E_τ be the expectation defined by (7.2) for $m = 0$. Corollary 7.15 applies to completely positive mappings $\Phi : \mathcal{O}_N \to \mathcal{B}(\mathcal{H}_0)$ such that $\Phi \circ E_\tau = \Phi$. Assume that ρ is a $*$-representation of $\mathcal{O}_N$ on a von Neumann algebra A, and $E_\rho : A \to A$ the conditional expectation associated with some, fixed, virtual diagonal M of norm one for A. Let $\Phi : E_\rho(A) \to \mathcal{B}(\mathcal{H}_0)$ be completely positive, and let consider $\Phi_{\rho,\tau} := \Phi \circ E_\rho \circ \rho \circ E_\tau$. Then $\Phi_{\rho,\tau} E_\tau = \Phi_{\rho,\tau}$ and we obtain Hilbert spaces and their decompositions like those of Corollary 7.15, associated with the representation ρ and algebra A.

Finally, the subalgebra $\rho(\mathcal{F}_N)' = \bigcap_{n>0} \alpha^n(A)$ (see [36, Proposition 3.1(ii)]) suggested us to form sequences of vector bundles in the following manner. Let $\alpha : A \to A$ be a normal, $*$-representation where $A = \mathcal{B}(\mathcal{H})$ as above. Then $\alpha^*(A_*) \subseteq A_*$ where A_* denotes the predual of A formed by the trace-class operators on $\mathcal{H}$, and α^* is the transpose mapping of α. For $n \in \mathbb{N}$ we are going to consider the iterative mappings $\beta_n := \alpha^n \circ \rho$ and corresponding expectations denoted by $E_n := E_{\beta_n}$ and put $E_0 = E_\rho$. Then, for $\xi \in A_*$ and $T \in A$,

$$
\begin{aligned}
(E_n \circ \alpha)(T)(\xi) &= \int\limits_{\mathcal{O}_N \otimes \mathcal{O}_N} \big(\alpha^n \rho\big)(s)\alpha(T)\big(\alpha^n \rho\big)(t)(\xi)\,\mathrm{d}M_N(s,t) \\
&= \int\limits_{\mathcal{O}_N \otimes \mathcal{O}_N} \big(\alpha^{n-1} \rho\big)(s)T\big(\alpha^{n-1} \rho\big)(t)\big(\alpha^*\xi\big)\,\mathrm{d}M_N(s,t) \\
&= E_{n-1}(T)\big(\alpha^*\xi\big) = (\alpha \circ E_{n-1})(T)(\xi),
\end{aligned}
$$

see [20]. More specifically, $(E_\rho \alpha)(T) = \alpha(\int_{\mathcal{O}_N \otimes \mathcal{O}_N} \rho(s)T\rho(t)\,\mathrm{d}M_N(s,t)) = \varphi(T) \in \mathbb{C}$, where φ is the state given by $\varphi(T) := \int_{\mathcal{O}_N \otimes \mathcal{O}_N} \rho(s)T\rho(t)\,\mathrm{d}M_N(s,t) \in \mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbf{1}$, $T \in A$. Hence

$$E_n \circ \alpha = \alpha \circ E_{n-1}, \quad n \in \mathbb{N}, \tag{7.10}$$

whence, by a reiterative process and since $\alpha E_\rho = E_\rho$, we get

$$E_n \circ \alpha^n = E_\rho, \quad n \in \mathbb{N}. \tag{7.11}$$

Hence we get $E_n E_\rho = E_\rho$ and therefore $B_\rho \subseteq B_n$, where $B_n := E_n(A)$, for all n. Further, we have $\alpha E_n(A) = E_{n+1}\alpha(A) \subseteq E_{n+1}(A)$ by (7.10), that is, $\alpha(B_n) \subseteq B_{n+1}$, $n \in \mathbb{N}$.

Now consider a countable family $(\Phi_n)_{n \geqslant 0}$ of completely positive mappings $\Phi_n : A \to \mathcal{B}(\mathcal{H}_0)$, for some Hilbert space $\mathcal{H}_0$, such that

$$\Phi_{n+1} \circ \alpha = \Phi_n, \qquad \Phi_n \circ E_n = \Phi_n, \quad n \in \mathbb{N}. \tag{7.12}$$

Such a family exists. Take for instance $\phi_n := E_\rho E_1 \ldots E_n$, and a completely positive map $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$. Then the family $\Phi_n := \Phi \circ \phi_n$, $n \geqslant 0$, satisfies (7.12). In these conditions the diagram

$$
\begin{array}{ccccccccccc}
A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\Phi_{n+1}} & \mathcal{B}(\mathcal{H}_0) \\
\downarrow{\scriptstyle E_0} & & \downarrow{\scriptstyle E_1} & & \downarrow & & \downarrow{\scriptstyle E_n} & & \downarrow{\scriptstyle E_{n+1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
B_0 & \xrightarrow{\alpha} & B_1 & \xrightarrow{\alpha} & \cdots & \xrightarrow{\alpha} & B_n & \xrightarrow{\alpha} & B_{n+1} & \xrightarrow{\Phi_{n+1}} & \mathcal{B}(\mathcal{H}_0)
\end{array}
$$

is commutative in each of its subdiagrams.

For every $n \geqslant 0$, by applying Theorem 6.10 to the conditional expectation $E_n : A \to B_n$ and mapping Φ_n one finds the corresponding Hilbert space $\mathcal{H}_{B_n}(\Phi_n)$ for the representation which is the Stinespring dilation of Φ_n. Take a finite set of elements b_j in B_n. As $\Phi_n(b_i^* b_j) = \Phi_{n+1}(\alpha(b_i)^*\alpha(b_j))$ it follows that

$$\Big\| \sum_j b_j \otimes f_j \Big\|_{\Phi_n} = \Big\| \sum_j \alpha(b_j) \otimes f_j \Big\|_{\Phi_{n+1}}$$

for all $\sum_j b_j \otimes f_j \in B_n \otimes \mathcal{H}_0$, see Section 5. Hence, $\alpha(\mathcal{H}_{B_n}(\Phi_n)) \subseteq \mathcal{H}_{B_{n+1}}(\Phi_{n+1})$. This implies that we have found the (countable) system of vector bundle homomorphisms

$$\begin{tikzcd}
\mathrm{G}_A \times_{\mathrm{G}_{B_\rho}} \mathcal{H}_{B_\rho}(\Phi_0) \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_{B_1}} \mathcal{H}_{B_1}(\Phi_1) \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d] & \cdots \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_{B_n}} \mathcal{H}_{B_n}(\Phi_n) \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d] & \cdots \\
\mathfrak{S}(\rho) \arrow[r, "\tilde{\alpha}_0"] & \mathfrak{S}(\alpha^1\rho) \arrow[r, "\tilde{\alpha}_1"] & \cdots \arrow[r, "\tilde{\alpha}_{n-1}"] & \mathfrak{S}(\alpha^n\rho) \arrow[r, "\tilde{\alpha}_n"] & \cdots
\end{tikzcd}$$

where $\tilde{\alpha}_n$ is the canonical submersion induced by $\alpha|_{B_n} : B_n \to B_{n+1}$, $n \geqslant 0$.

Of course the above sequence of diagrams gives rise to the corresponding statements about complexifications, and realizations of representations on spaces of holomorphic sections.

7.5. Non-commutative stochastic analysis

We have just shown a sample of how to find sequences of homogeneous vector bundles of the type dealt with in this paper. As a matter of fact, continuous families of such bundles are also available, which could hopefully be of interest in other fields. More precisely, the geometric models developed in the present paper might prove useful in order to get a better understanding of the phenomena described by the various theories of non-commutative probabilities. By way of illustrating this remark, we shall briefly discuss from our geometric perspective a few basic ideas related to the stochastic calculus on full Fock spaces as developed in [7,8]. (See also [28,66] for a complementary perspective that highlights the role of the Cuntz algebras in connection with full Fock spaces.)

In the paper [7], a family of conditional expectations $\{E_t\}_{t>0}$ is built on the von Neumann algebra A of bounded operators on the full Fock space, generated by the annihilation, creation, and gauge operators. Set $A_t := E_t(A)$ for $t > 0$. It is readily seen that $A_t \subseteq A_s$ and that $E_t E_s = E_t$ whenever $0 < s \leqslant t$ (check first for the so-called in [7] *basic* elements). Applying the Stinespring dilation procedure to the conditional expectation E_s and completely positive mapping E_t one gets Hilbert spaces $\mathcal{H}_{A_s}(E_t) \subseteq \mathcal{H}_A(E_t)$ and the consequent Stinespring representations $\pi_{A_j} : A_j \to \mathcal{B}(\mathcal{H}_{A_j}(E_t))$, where $j = 0, t$, and $A_0 = A$. This entails the commutative diagram

$$\begin{tikzcd}
\mathrm{G}_A \times_{\mathrm{G}_t} \mathcal{H}_{A_t}(E_t) \arrow[r] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_s} \mathcal{H}_{A_s}(E_t) \arrow[r] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_r} \mathcal{H}_{A_r}(E_t) \arrow[d] \\
\mathrm{G}_A/\mathrm{G}_t \arrow[r] & \mathrm{G}_A/\mathrm{G}_s \arrow[r] & \mathrm{G}_A/\mathrm{G}_r,
\end{tikzcd}$$

for $r < s < t$, where $\mathrm{G}_j = \mathrm{G}_{A_j}$ for $j = r, s, t$. Moreover, as usual, the geometrical framework of the present paper works to produce a Hilbert space $\mathcal{H}_A(E_s, E_t)$, formed by holomorphic sections on $\mathrm{G}_A/\mathrm{G}_s$, which is isometric to $\mathcal{H}_A(E_t)$ and enables us to realize π_A as natural multiplication.

On the other hand, from the point of view of the quantum stochastic analysis (see for instance [12,49]), it is worth considering unital completely positive mappings $\Phi : A \to \mathcal{B}(\mathcal{H}_0)$ with the following filtration property: There exists a family $\{\Phi_t : A \to \mathcal{B}(\mathcal{H}_t)\}_{t \geqslant 0}$ of completely positive mappings which approximate Φ in some sense and satisfy $\Phi_t \circ E_t = \Phi_t$ for all $t > 0$. Then we get commutative diagrams

$$\begin{tikzcd} \mathrm{G}_A \times_{\mathrm{G}_t} \mathcal{H}_{A_t}(\Phi_t) \arrow[r] \arrow[d] & \mathrm{G}_A \times_{\mathrm{G}_s} \mathcal{H}_{A_s}(\Phi_s) \arrow[d] \\ \mathrm{G}_A/\mathrm{G}_t \arrow[r] & \mathrm{G}_A/\mathrm{G}_s \end{tikzcd}$$

whenever $s < t$. By means of the realizations of the full Fock space as reproducing kernel Hilbert spaces of sections in appropriate holomorphic vector bundles we find geometric interpretations for most concepts usually related to the Fock spaces (for instance, annihilation, creation, and gauge operators). We thus arrive at the challenging perspective of a relationship between the noncommutative stochastic analysis and the infinite-dimensional complex geometry, which certainly deserves to be understood in more detail. For one thing, this might provide useful geometric insights in areas like the theory of quantum Markov processes.

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