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PHYSICS LETTERS B
Physics Letters B 571 (2003) 235–244
www.elsevier.com/locate/npe

Gravitational localization of all local fields on the brane

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Received 13 July 2003; accepted 28 July 2003

Editor: G.F. Giudice

Abstract

We present a new $(p-1)$-brane solution to Einstein's equations in a general spacetime dimension. This solution is a natural generalization of the stringlike defect solution with codimension 2 in 6 spacetime dimensions, which has been recently discovered by Gogberashvili and Singleton, to a general $(p-1)$-brane solution with codimension n in general $D = p + n$ spacetime dimensions. It is shown that all the local fields are localized on the brane only through the gravitational interaction although this solution does not have a warp factor and takes a finite value in the radial infinity. Thus, this solution is a solution in an arbitrary spacetime dimension realizing the idea of "gravitational trapping" of the whole bulk fields on the brane within the framework of a local field theory. Some problems associated with this solution and localization are pointed out.

The idea that our four-dimensional world is a three-brane embedded in a higher-dimensional spacetime with non-factorizable warped geometry has been much investigated since the appearance of papers [1–3]. (See also [4–6] for the pioneering works and [7] for many brane model.) In this idea, the key observation is that the graviton, which is allowed to be free to propagate in the bulk, is confined to the brane because of the warped geometry, thereby implying that the gravitational law on the brane obeys the usual four-dimensional Newton's law as desired.

On the other hand, the other local fields except the gravitational field are not always localized on the brane even in the warped geometry. Indeed, in the Randall–Sundrum model in five dimensions [2], the following facts are well known: spin 0 field is localized on a brane with positive tension which also localizes the graviton [8]. Spin 1 field is not localized neither on a brane with positive tension nor on a brane with negative tension [9]. (In six spacetime dimensions, the spin 1 gauge field is also localized on the brane [10].) Moreover, spin $1/2$ and $3/2$ fields are localized not on a brane with positive tension but on a brane with negative tension [8]. Thus, in order to fulfill the localization of Standard Model particles on a brane with positive tension, it seems that some additional interactions except the gravitational interaction must be also introduced in the bulk. (See the review [11] and the papers [12] for the localization of the bulk fields in various brane world models.)

The introduction of such additional interactions, however, is not only unnatural from the physical viewpoint but also can be applied to only specific situations. (For instance, for the localization of fermionic fields one must introduce a mass term with a 'kink' profile [13].) Thus, it is very welcoming if we could find a model in the brane world where all the local bulk fields are localized on the 3-brane only by the universal interaction, i.e., the gravity.

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0370-2693
doi:10.1016/j.physletb.2003.07.076

It is of interest that Gogberashvili and Singleton [14,15] have recently found such a solution to Einstein's equations in six spacetime dimensions and pointed out that all the local fields ranging from the spin 0 scalar field to the spin 2 gravitational field are localized on the 3-brane in this background geometry.

The aim of the present Letter is twofold. One aim is to extend their 3-brane model in six spacetime dimensions to the case of a general $(p-1)$-brane model in a general spacetime dimension. We explicitly show that even in this general model whole local fields, those are, spin 0 scalar field, spin $1/2$ spinor field, spin 1 gauge field, spin $3/2$ gravitino field and spin 2 gravitational field as well as totally antisymmetric tensor fields, are confined on the $(p-1)$-brane without appealing to the additional bulk interactions except the gravity. The other aim is to mention some problems associated with the solution and the localization. In particular, we will stress that the 'mild' localization of the wave function of the zero-modes might give rise to a conflict with experiments and arguments about the stability of the brane is completely ignored.

The action which we consider in this Letter is that of gravity in general $D = p + n$ dimensions, with the conventional Einstein–Hilbert action plus the bulk cosmological constant and some matter action [16]:

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g}\,(R - 2\Lambda) + \int d^D x \sqrt{-g}\, L_m, \tag{1}$$

where κ_D denotes the D-dimensional gravitational constant with the relation $\kappa_D^2 = 8\pi G_N = \frac{8\pi}{M_*^{D-2}}$ with G_N and M_* being the D-dimensional Newton constant and the D-dimensional Planck mass scale, respectively. Throughout this Letter we follow the standard conventions and notation of the textbook of Misner, Thorne and Wheeler [17].

The variation of the action (1) with respect to the D-dimensional metric tensor g_{MN} leads to Einstein's equations:

$$R_{MN} - \frac{1}{2} g_{MN} R = -\Lambda g_{MN} + \kappa_D^2 T_{MN}, \tag{2}$$

where the energy–momentum tensor is defined as

$$T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{MN}} \int d^D x \sqrt{-g}\, L_m. \tag{3}$$

We adopt the following metric ansatz:

$$\begin{aligned} ds^2 &= g_{MN}\, dx^M\, dx^N \\ &= g_{\mu\nu}\, dx^\mu\, dx^\nu + \tilde{g}_{ab}\, dx^a\, dx^b = \phi^2(r) \hat{g}_{\mu\nu}\big(x^\lambda\big)\, dx^\mu\, dx^\nu + g(r)\big(dr^2 + r^2\, d\Omega_{n-1}^2\big), \end{aligned} \tag{4}$$

where $M, N, \ldots$ denote D-dimensional spacetime indices, $\mu, \nu, \ldots$ p-dimensional brane ones, and $a, b, \ldots$ n-dimensional extra spatial ones, so the equality $D = p + n$ holds. (We assume $p \geqslant 4$.) And $d\Omega_{n-1}^2$ stands for the metric on a unit $(n-1)$-sphere, which is concretely expressed in terms of the angular variables θ_i as

$$d\Omega_{n-1}^2 = d\theta_2^2 + \sin^2\theta_2\, d\theta_3^2 + \sin^2\theta_2 \sin^2\theta_3\, d\theta_4^2 + \cdots + \prod_{i=2}^{n-1} \sin^2\theta_i\, d\theta_n^2, \tag{5}$$

with the volume element $\int d\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

Moreover, following Gogberashvili and Singleton [14] we take the ansatz for the energy–momentum tensor respecting the spherical symmetry (see [10] for the more general ansatz):

$$T_{\mu\nu} = g_{\mu\nu} F(r), \qquad T_{ab} = g_{ab} K(r), \tag{6}$$

where F and K are functions of only the radial coordinate r.

Under these ansatze, after a straightforward calculation, Einstein's equations reduce to

$$\frac{1}{g}\left[p(p-1)\left(\frac{\phi'}{\phi}\right)^2 + p(n-1)\frac{\phi'}{\phi}\frac{(r^2g)'}{r^2g} + \frac{(n-1)(n-2)}{4}\left(\frac{(r^2g)'}{r^2g}\right)^2 - (n-1)(n-2)\frac{1}{r^2}\right] - \frac{1}{\phi^2}\hat{R} + 2\Lambda = 2\kappa_D^2 K, \tag{7}$$

$$\frac{1}{g}\left[(p-1)\left(2\frac{\phi''}{\phi} - \frac{g'}{g}\frac{\phi'}{\phi}\right) + (p-1)(p-2)\left(\frac{\phi'}{\phi}\right)^2 + (p-1)(n-1)\frac{\phi'}{\phi}\frac{(r^2g)'}{r^2g} + (n-1)\left\{\frac{(r^2g)''}{r^2g} + \frac{n-4}{4}\left(\frac{(r^2g)'}{r^2g}\right)^2 - \frac{1}{2}\frac{g'}{g}\frac{(r^2g)'}{r^2g} - (n-2)\frac{1}{r^2}\right\}\right] + \frac{2-p}{p}\frac{1}{\phi^2}\hat{R} + 2\Lambda = 2\kappa_D^2 F, \tag{8}$$

$$\frac{1}{g}\left[p\left(2\frac{\phi''}{\phi} - \frac{g'}{g}\frac{\phi'}{\phi}\right) + p(p-1)\left(\frac{\phi'}{\phi}\right)^2 + p(n-2)\frac{\phi'}{\phi}\frac{(r^2g)'}{r^2g} + (n-2)\left\{\frac{(r^2g)''}{r^2g} + \frac{n-5}{4}\left(\frac{(r^2g)'}{r^2g}\right)^2 - \frac{1}{2}\frac{g'}{g}\frac{(r^2g)'}{r^2g} - (n-3)\frac{1}{r^2}\right\}\right] - \frac{1}{\phi^2}\hat{R} + 2\Lambda = 2\kappa_D^2 K, \tag{9}$$

where the prime denotes the differentiation with respect to r, and $\hat{R}$ is the scalar curvature associated with the brane metric $\hat{g}_{\mu\nu}$. Here we define the cosmological constant Λ_p on the $(p-1)$-brane by the equation

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} = -\Lambda_p \hat{g}_{\mu\nu}. \tag{10}$$

In deriving Eq. (8), we have used $\hat{R}_{\mu\nu} = \frac{1}{p}\hat{g}_{\mu\nu}\hat{R}$, which is obtained by taking the trace of Eq. (10). Note that since $T_{\mu\nu}$ is proportional to $\hat{g}_{\mu\nu}$, $\hat{R}$ is a constant [16]. In addition, the conservation law for the energy–momentum tensor, $\nabla^M T_{MN} = 0$, takes the form

$$K' + p\frac{\phi'}{\phi}(K - F) = 0. \tag{11}$$

One of our purposes in this Letter is to find a new $(p-1)$-brane solution to Einstein's equations and the conservation law in the above. To do that, the first step is to subtract Eq. (7) from Eq. (9). The result is given by

$$2p\left[\frac{\phi''}{\phi} - \frac{g'}{g}\frac{\phi'}{\phi} - \frac{\phi'}{r\phi}\right] + (n-2)\left[\frac{g''}{g} - \frac{3}{2}\left(\frac{g'}{g}\right)^2 - \frac{1}{r}\frac{g'}{g}\right] = 0. \tag{12}$$

Next we require the terms in each square bracket to vanish separately, that is,

$$\frac{\phi''}{\phi} - \frac{g'}{g}\frac{\phi'}{\phi} - \frac{\phi'}{r\phi} = 0, \qquad \frac{g''}{g} - \frac{3}{2}\left(\frac{g'}{g}\right)^2 - \frac{1}{r}\frac{g'}{g} = 0. \tag{13}$$

Here we notice that $n = 2$ is special in that the latter equation does not arise from Eq. (12) owing to the factor $n - 2$. In this sense, the stringlike defect solution with codimension 2, which has been found by Gogberashvili and Singleton [14] is distinct from the other defect solutions. Nevertheless, we will see that there is a similar solution even in $n \neq 2$, whose solution precisely corresponds to $b = 2$ case in [14].

It turns out that the solution to the former equation in Eq. (13) is given by [14]

$$g(r) = \rho^2\frac{\phi'(r)}{r}, \tag{14}$$

where ρ is an integration constant. The latter equation in Eq. (13) is then solved and the explicit forms of ϕ and g are given by

$$\phi(r) = a\frac{r^2 - c^2}{r^2 + c^2}, \qquad g(r) = 4ac^2\rho^2 \frac{1}{(r^2 + c^2)^2}, \tag{15}$$

where a, c and ρ are integration constants. (See below about boundary conditions which we take.) Furthermore, we can show that the remaining Einstein's equations and the conservation law of the energy–momentum tensor are satisfied if we choose the following form of the source functions:

$$\begin{aligned} K(r) &= \frac{1}{2\kappa_D^2}\left[\frac{4c^2}{a\rho^2}p(p-1)\frac{r^2}{(r^2-c^2)^2} - \frac{1}{a\rho^2}(n-1)(n+2p-2) - \frac{1}{a^2}\left(\frac{r^2+c^2}{r^2-c^2}\right)^2 \hat{R} + 2\Lambda\right], \\ F(r) &= \frac{1}{2\kappa_D^2}\left[\frac{4c^2}{a\rho^2}p(p-1)\frac{r^2}{(r^2-c^2)^2} - \frac{1}{a\rho^2}(n-1)(n+2p-2) - \frac{2}{a\rho^2}(p-1)\left(\frac{r^2+c^2}{r^2-c^2}\right)^2 \right. \\ &\quad \left. + \frac{1}{a^2}\frac{2-p}{p}\left(\frac{r^2+c^2}{r^2-c^2}\right)^2 \hat{R} + 2\Lambda\right]. \end{aligned} \tag{16}$$

Note that these source functions approach a definite value at the infinity $r \to \infty$. Although we could take $K(\infty) = F(\infty) = 0$ by selecting both $\hat{R}$ and ρ (or a) appropriately, we do not so since ϕ in the solution (15) does not take the vanishing value at $r \to \infty$, either.

Here we should mention one subtle point, that is, what boundary conditions on the brane (and/or at the infinity) we should impose. For instance, in the previous work of the stringlike defect model with codimension 2 [10], we have required that the extra two dimensions are conical around the brane with a deficit angle in order to describe the "local cosmic string" sitting at the origin $r = 0$. In the case at hand, we take the different boundary conditions which require us only to avoid singularities on the brane [14]. Then, the suitable boundary conditions which we take in this Letter are

$$\phi(\varepsilon) = 1, \qquad \phi(\infty) = a, \tag{17}$$

where ε denotes the "brane width", which now takes a fixed value. The former boundary condition allows us to express the constant c in terms of the "brane width" as $c = \sqrt{\frac{a-1}{a+1}}\varepsilon$, which implies $a > 1$ under the assumption of a being positive. Let us count the number of independent integration constants in the solution (15). Originally we have two second-order differential equations with respect to ϕ and g. Since we have set up two boundary conditions (17), the number of the remaining independent constants should be two, which are nothing but a and ρ. Furthermore, it is worthwhile to mention that in this Letter the brane is assumed to have the nonvanishing "brane width" since the "brane width" ε appears in the later arguments of localization of the bulk fields and plays a role as the short-distance cutoff. In this context, let us note that the core physics inside $r = \varepsilon$ is in essence controlled by the short-distance and high-energy physics, so the complete understanding of the core physics calls for quantum gravity. Because we have at present no satisfying theory of quantum gravity, it is physically reasonable to introduce such a cutoff via the boundary conditions into our theory where the cutoff has the physical meaning as the brane width.

Closely relating to the problem of boundary conditions, it is worthwhile to see how our solution (15) can be described in the coordinate system where the defects with a deficit angle are usually described. The result is given by

$$\begin{aligned} ds^2 &= \varphi^2(R)\hat{g}_{\mu\nu}\,dx^\mu\,dx^\nu + dR^2 + h(R)\,d\Omega_{n-1}^2 \\ &= a^2\cos^2\left(\frac{1}{\rho\sqrt{a}}R\right)\hat{g}_{\mu\nu}\,dx^\mu\,dx^\nu + dR^2 + 4a\rho^2\sin^2\left(\frac{1}{2\rho\sqrt{a}}R\right)d\Omega_{n-1}^2. \end{aligned} \tag{18}$$

In this coordinate system, the line element has especially a simple form in that the scale factor is expressed by (the square of) the cosine whereas the angular factor is the sine. Then, a deficit angle can be calculated to $\delta = 2\pi \frac{\varepsilon^2}{8\rho^3 a}$, which means that we have no deficit angle around the brane when $\varepsilon \approx 0$ as expected [10].

Now we turn our attention to the problem of the localization of the bulk fields on the brane in the background geometry (15). Of course, in due analysis, we will neglect the back-reaction on the geometry induced by the existence of the bulk fields. We proceed our study of the localization in order according to the size of spin of local fields and finally investigate totally antisymmetric tensor fields.

Let us start with a massless, spin 0, real scalar coupled to gravity:

$$S_0 = -\frac{1}{2} \int d^D x \sqrt{-g}\, g^{MN} \partial_M \Phi \partial_N \Phi, \tag{19}$$

from which the equation of motion can be derived:

$$\frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g}\, g^{MN} \partial_N \Phi \right) = 0. \tag{20}$$

From now on, without loss of generality, we shall take a flat metric on the brane, that is, $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. It turns out that $\Phi(x^M) = \phi(x^\mu) u_0$ which satisfies the Klein–Gordon equation on the brane $\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) = 0$ is a solution to the equation of motion (20) in the background metric (15). Substituting this solution into the starting action (19), the action can be cast to

$$S_0 = -\frac{1}{2} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} u_0^2 \int_\varepsilon^\infty dr\, \phi^{p-2} g^{\frac{n}{2}} r^{n-1} \int d^p x\, \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \cdots. \tag{21}$$

Now we wish to show that this zero-mode is localized on the brane sitting around the origin $r = 0$. The condition for having localized p-dimensional scalar field is that the solution is normalizable. It is of importance to notice that normalizability of the ground state wave function is equivalent to the condition that the "coupling" constant is non-vanishing. In other words, in order to show that the bulk zero-modes which satisfy the equation of motion in the bulk is in fact confined to a brane, the zero-modes must give us the kinetic terms on the brane, from which we can understand that the bulk zero-modes are truely dynamical and propagate on the brane. Thus, provided that we define

$$I_0 = \int_\varepsilon^\infty dr\, \phi^{p-2} g^{\frac{n}{2}} r^{n-1} = (2c\rho)^n a^{p+\frac{n}{2}-2} \int_\varepsilon^\infty dr\, \frac{(r^2 - c^2)^{p-2}}{(r^2 + c^2)^{n+p-2}} r^{n-1}, \tag{22}$$

the condition of having localized p-dimensional scalar field on the brane requires that I_0 should be finite. The integral in I_0 scales as $\frac{1}{r^{n+1}}$ at the radial infinity and is a smooth function between $r = \varepsilon$ and $r = \infty$, so I_0 is finite even if the analytic expression is not available. (In the case of $\varepsilon = c$, I_0 can be expressed in terms of the hypergeometric function, which is of course finite.) Hence, the p-dimensional scalar field ϕ is localized on the brane by the gravitational interaction.

Next, let us consider spin $1/2$ spinor field. Our starting action in this case is the Dirac action given by

$$S_{\frac{1}{2}} = \int d^D x \sqrt{-g}\, \bar{\Psi} i \Gamma^M D_M \Psi, \tag{23}$$

from which the equation of motion is given by

$$0 = \Gamma^M D_M \Psi = \left(\Gamma^\mu D_\mu + \Gamma^r D_r + \Gamma^{\theta_i} D_{\theta_i} \right) \Psi. \tag{24}$$

We introduce the vielbein $e_M^{\bar{M}}$ (and its inverse $e_{\bar{M}}^M$) through the usual definition $g_{MN} = e_M^{\bar{M}} e_N^{\bar{N}} \eta_{\bar{M}\bar{N}}$ where $\bar{M}, \bar{N}, \ldots$ denote the local Lorentz indices. Γ^M in a curved spacetime is related to $\gamma^{\bar{M}}$ in a flat spacetime by

$\Gamma^M = e^M_{\bar{M}} \gamma^{\bar{M}}$. In addition, the spin connection $\omega_M^{\bar{M}\bar{N}}$ in the covariant derivative $D_M \Psi = (\partial_M + \frac{1}{4}\omega_M^{\bar{M}\bar{N}} \gamma_{\bar{M}\bar{N}})\Psi \equiv (\partial_M + \omega_M)\Psi$ is defined as

$$\omega_M^{\bar{M}\bar{N}} = \frac{1}{2} e^{N\bar{M}} \left(\partial_M e_N^{\bar{N}} - \partial_N e_M^{\bar{N}}\right) - \frac{1}{2} e^{N\bar{N}} \left(\partial_M e_N^{\bar{M}} - \partial_N e_M^{\bar{M}}\right) - \frac{1}{2} e^{P\bar{M}} e^{Q\bar{N}} (\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}}) e_M^{\bar{R}}, \tag{25}$$

so the covariant derivative can be calculated to

$$\begin{aligned} D_\mu \Psi &= \left(\partial_\mu + \frac{1}{2}\frac{\phi'}{g\phi}\Gamma_\mu \Gamma_r\right)\Psi, \\ D_r \Psi &= \partial_r \Psi, \\ D_{\theta_i}\Psi &= \left[\partial_{\theta_i} - \frac{1}{2}\frac{1}{g^{\frac{3}{2}} r}\partial_r\left(g^{\frac{1}{2}} r\right)\Gamma_r \Gamma_{\theta_i} + \tilde{\omega}_{\theta_i}(\theta)\right]\Psi, \end{aligned} \tag{26}$$

where $\tilde{\omega}_{\theta_i}(\theta)$ is a contribution from S^{n-1}, whose explicit form is now irrelevant so is omitted to write down.

Let us look for a solution with the form of $\Psi(x^M) = \psi(x^\mu) u(r) \chi(\theta)$, where $\psi(x^\mu)$ satisfies the massless p-dimensional Dirac equation $\gamma^\mu \partial_\mu \psi = 0$ and the chiral condition $\gamma^r \psi(x^\mu) = \psi(x^\mu)$, and χ satisfies the equation $\gamma^{\theta_i}(\partial_{\theta_i} + \tilde{\omega}_{\theta_i})\chi = 0$. With this ansatz, the Dirac equation (24) is reduced to

$$\left[\partial_r + \frac{p}{2}\frac{\phi'}{\phi} + \frac{n-1}{2}\frac{\partial_r(g^{\frac{1}{2}} r)}{g^{\frac{1}{2}} r}\right] u(r) = 0. \tag{27}$$

The solution to this equation then reads:

$$u(r) = c_{\frac{1}{2}} \phi^{-\frac{p}{2}} \left(g^{\frac{1}{2}} r\right)^{-\frac{n-1}{2}}, \tag{28}$$

with $c_{\frac{1}{2}}$ being an integration constant.

Now we are willing to show that the solution (28) is normalizable so that the spin $1/2$ spinor field is localized on the brane. Inserting the above solution to the action gives rise to

$$S_{\frac{1}{2}} = \int_\varepsilon^\infty dr\, \phi^{p-1} g^{\frac{n}{2}} r^{n-1} u^2(r) \int d\Omega_{n-1}\, \chi^2(\theta) \int d^p x\, \bar{\psi} i \gamma^\mu \partial_\mu \psi + \cdots. \tag{29}$$

In order to localize the spin $1/2$ fermion, the integral $I_{\frac{1}{2}}$, which is defined as

$$I_{\frac{1}{2}} = \int_\varepsilon^\infty dr\, \phi^{p-1} g^{\frac{n}{2}} r^{n-1} u^2(r) \tag{30}$$

should be finite. (Here note that the integral over S^{n-1} is finite.) Indeed, this integral can be easily calculated as

$$I_{\frac{1}{2}} = \frac{c_{\frac{1}{2}}^2 \rho}{\sqrt{a}} \log\left|\frac{\varepsilon + c}{\varepsilon - c}\right|, \tag{31}$$

which is obviously finite as long as the brane width ε is non-vanishing. The important point here is that the divergence at $r = \infty$, which usually makes the zero-mode solutions unnormalizable, does not occur in the model at hand.

Let us turn to spin 1 gauge field. We consider the action of $U(1)$ vector field:

$$S_1 = -\frac{1}{4}\int d^D x\, \sqrt{-g}\, g^{MN} g^{RS} F_{MR} F_{NS}, \tag{32}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ as usual. From this action the equation of motion is given by

$$\frac{1}{\sqrt{-g}}\partial_M\big(\sqrt{-g}\, g^{MN} g^{RS} F_{NS}\big) = 0. \tag{33}$$

It is easily checked that $A_\mu(x^M) = a_\mu(x^\lambda)u_0$, $A_r(x^M) = \text{const}$, and $A_{\theta_i}(x^M) = \text{const}$ is a solution to this equation of motion if $\partial^\mu f_{\mu\nu} = 0$ where $f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$.

Once we have found the solution, let us ask ourselves whether this solution is a normalizable one or not by substituting it into the action (32). It turns out that the action is reduced to

$$S_1 = -\frac{1}{4}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}u_0^2 \int_\varepsilon^\infty dr\, \phi^{p-4} g^{\frac{n}{2}} r^{n-1} \int d^p x\, \eta^{\mu\nu}\eta^{\lambda\sigma} f_{\mu\lambda} f_{\nu\sigma} + \cdots. \tag{34}$$

The integral defined by

$$I_1 = \int_\varepsilon^\infty dr\, \phi^{p-4} g^{\frac{n}{2}} r^{n-1} = (2c\rho)^n a^{p+\frac{n}{2}-4} \int_\varepsilon^\infty dr\, \frac{(r^2-c^2)^{p-4} r^{n-1}}{(r^2+c^2)^{n+p-4}} \tag{35}$$

is finite as in the scalar field. Thus, the vector field can be also localized on the brane only by the gravitational interaction.

Next we are ready to consider spin 3/2 fermionic field, in other words, the gravitino. Let us begin with the action of the Rarita–Schwinger gravitino field:

$$S_{\frac{3}{2}} = \int d^D x \sqrt{-g}\, \bar{\Psi}_M i \Gamma^{[M}\Gamma^N\Gamma^{R]} D_N \Psi_R, \tag{36}$$

from which the equation of motion is given by

$$\Gamma^{[M}\Gamma^N\Gamma^{R]} D_N \Psi_R = 0. \tag{37}$$

Here the square bracket denotes the total antisymmetrization and the covariant derivative is defined with the affine connection $\Gamma^R_{MN} = e^R_{\bar{M}}(\partial_M e^{\bar{M}}_N + \omega^{\bar{M}\bar{N}}_M e_{N\bar{N}})$ by $D_M\Psi_N = \partial_M\Psi_N - \Gamma^R_{MN}\Psi_R + \frac{1}{4}\omega^{\bar{M}\bar{N}}_M \gamma_{\bar{M}\bar{N}}\Psi_N$. We look for a solution with the form of $\Psi_\mu(x^M) = \psi_\mu(x^\lambda)u(r)\chi(\theta)$ and $\Psi_r = 0 = \Psi_{\theta_i}$ where ψ and χ are assumed to satisfy the equations $\gamma^\mu\psi_\mu = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\partial_\nu\psi_\rho = 0$, $\gamma^r\psi_\mu = \psi_\mu$ and $\gamma^{\theta_i}(\partial_{\theta_i} + \tilde{\omega}_{\theta_i})\chi = 0$. Then, the equation of motion (37) takes the form

$$\left[\partial_r + \frac{p-1}{2}\frac{\phi'}{\phi} + \frac{n-1}{2}\frac{\partial_r(g^{\frac{1}{2}}r)}{g^{\frac{1}{2}}r}\right]u(r) = 0. \tag{38}$$

The solution to this equation reads:

$$u(r) = c_{\frac{3}{2}}\phi^{-\frac{p-1}{2}}\big(g^{\frac{1}{2}}r\big)^{-\frac{n-1}{2}}, \tag{39}$$

with $c_{\frac{3}{2}}$ being an integration constant.

We shall show that as in the case of spin 1/2 field, this solution is normalizable so the gravitino field is also localized on the brane. To do so, let us substitute the solution into the action, whose result is of form

$$S_{\frac{3}{2}} = \int_\varepsilon^\infty dr\, \phi^{p-3} g^{\frac{n}{2}} r^{n-1} u^2(r) \int d\Omega_{n-1}\, \chi^2(\theta) \int d^p x\, \bar{\psi}_\mu i \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\partial_\nu\psi_\rho + \cdots. \tag{40}$$

It is certain that the integral $I_{\frac{3}{2}}$, which is defined as

$$I_{\frac{3}{2}} = \int_{\varepsilon}^{\infty} dr\, \phi^{p-3} g^{\frac{n}{2}} r^{n-1} u^2(r) = \frac{c_{\frac{3}{2}}^2 c\rho}{a\sqrt{a}} \frac{2\varepsilon}{\varepsilon^2 - c^2} \tag{41}$$

is finite as long as the brane width ε is non-zero. (Here note that the integral over S^{n-1} is also finite.)

Next let us consider spin 2 gravitational field. As in the cases treated so far, we can search for a solution to the equation of motion in the background (15), insert the solution in the Einstein–Hilbert action and then examine the finiteness of the radial integral. However, in this case, it is well known that the localization property of the graviton is the same as in the scalar field [10], so we can conclude that the bulk graviton is trapped on the brane as in case of the scalar field.

Finally, we take account of totally antisymmetric tensor fields. In general, the action of k-rank totally antisymmetric tensor field A_k is of the form in the form notation

$$S_k = -\frac{1}{2} \int F_{k+1} \wedge *F_{k+1}, \tag{42}$$

where $F_{k+1} = dA_k$. The equation of motion is simply given by

$$d \wedge *F_{k+1} = 0. \tag{43}$$

We can show that $A_{\mu_1\mu_2\cdots\mu_k} = a_{\mu_1\mu_2\cdots\mu_k}(x^\lambda)u_0$ is a solution to this equation of motion if $d \wedge *f = 0$ where $f = da$. Substituting this solution in the action (42) leads to the following expression:

$$S_k = I_k \int f_{k+1} \wedge *f_{k+1} + \cdots, \tag{44}$$

where I_k is defined as

$$I_k \propto \int_{\varepsilon}^{\infty} dr\, \phi^{p-2-2k} g^{\frac{n}{2}} r^{n-1} \propto \int_{\varepsilon}^{\infty} dr\, \frac{(r^2 - c^2)^{p-2-2k} r^{n-1}}{(r^2 + c^2)^{n+p-2-2k}}. \tag{45}$$

It is obvious that I_k is finite so the totally antisymmetric tensor fields are also localized on the brane by the gravitational interaction.

In conclusion, in this Letter, we have presented a new $(p-1)$-brane solution in an arbitrary spacetime dimension. This solution is a natural generalization of the 3-brane solution in six dimensions recently discovered by Gogberashvili and Singleton [14] to general D spacetime dimensions. We have also clarified that the stringlike defect model with codimension 2 is specific due to the terms proportional to the factor $n-2$ in Einstein's equations. Moreover, we have presented a complete analysis of localization of all bulk fields on a brane and showed that all the bulk fields are trapped on the brane only via the gravitational interaction. It is well known that in the warped geometry spin 1/2 and 3/2 fermionic fields are not trapped by the gravitational interaction so it is necessary to introduce a non-trivial Higgs coupling, thereby generating a bulk mass term with a 'kink' profile and leads to the localization of these fermionic fields on the brane [13]. It is remarkable that in the present model, we do not have to include such an additional interaction for the localization of the fermionic fields.

At this stage, it is useful to ask why our solution gives rise to the localization of all the bulk fields on the brane. The technical reason is very much simple. Namely, the scale factor $\phi(r)$ has a property such that it approaches a definite value at the infinity and a smooth function without singularities from the edge of the brane to the radial infinity. So the normalizability of the ground state wave function, which is equivalent to a finite integral over the radial coordinate r, is assured for all the bulk fields. On the other hand, in the warped geometry, the integral over r associated with the fermionic fields includes e^{+cr} $(c > 0)$ factors coming from $g^{\mu\nu}$ (and $e^{\mu}_{\bar{\mu}}$), for which the integral over r diverges at $r \to \infty$ so it leads to the non-localization of these fermionic fields.

As problems in this model, we should first recall one important point about the localization. As stressed before [12], the normalizable condition, in other words, the convergence of the integral over r, is usually thought to be a condition for the localization, but the story is not so simple. As in the locally localized gravity models (see the third paper in [12]) the present model provides us with an example such that the zero-mode solutions of the bulk fields are normalizable, but their wave functions spread rather widely in the bulk owing to the lack of the warp factor. Thus, in order not to contradict with the strict experiments such as the charge conservation law, some parameters in our model must be chosen in a proper way. At present, we have no idea whether there is such a suitable choice of the parameters.

As the second problem, we wish to point out a problem associated with the source functions. In our model, the presence of a solution to Einstein's equations heavily depends on the form of the source functions. Therefore, there would be a possibility that we might have more solutions by changing the form of the source functions. The real problem is then how to construct such source functions from fundamental matter fields so that the brane is a stable localized object. For instance, a set of n scalar functions with the Higgs potential, thereby breaking the *global* $SO(n)$ symmetry to $SO(n-1)$ symmetry, are used to make the topologically *stable* brane since a topological argument guarantees the stability because of a mathematical formula $\Pi_{n-1}(SO(n-1)) = Z$ [12].

Let us close by mentioning some interesting future works related to the present study. For instance, one interesting work would be to construct a supergravity model corresponding to the situation at hand and investigate the SUSY-breaking and the cosmological constant problem, etc. The other problem is to make the source functions concretely from some local field such as the scalar field. As mentioned above, the physics near the core of the brane is in the regime of the short-distance and the high-energy physics, so it would be difficult to understand the physics completely since it is expected that quantum gravity plays an important role in the core physics. However, some low energy effective action might be useful to describe the characteristic behavior of our source functions and insure the stability of a brane under deformations. We wish to report these works in future publication.

Acknowledgement

This work has been partially supported by the grant from the Japan Society for the Promotion of Science, No. 14540277.

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