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Games of Protocol

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1. INTRODUCTION

Many of the decision making processes appearing in economics, engineering, planning, etc., are in nature dynamic. A great deal of work has been done studying them and as a result of many investigations, the discipline of dynamic programming and adaptive processes developed by Bellman in [1–3] has become one of the most powerful techniques to be applied to such processes. However, when in the process there are several groups or players facing competition during the period of time under consideration, some extensions of the theory might be seen to be necessary.

In this paper, we are concerned with dynamic or multistage processes where the competition is taking place at each step and its result will determine the forthcoming steps. One of the easier (but very realistic) dynamic situations occurs when the different groups in the competition perform their actual moves, or make their decisions in an ordered sequence, one after the other. We call such an abstract process a *univalent game of protocol.* The purpose of our first sections is to study such processes. In the first one we introduce an existence theorem regarding stable points which are seen as dynamic solutions of the dynamic competition. They are very general and are related with the concept of competitive structure. The next two sections present some decomposition techniques which are also useful for computational purposes.

More general and complicated situations arise when more than one player at each time determine the continuation of the competitive process. In order to distinguish this more general case from the previous one, we call it *multivalent game of protocol.* In the last section we study two-person games of protocol which of course are bivalent. In order to do this we extend some results on two-person games derived by Shapley in [7].

2. Univalent Games of Protocol

Following the comments made in the introduction we now are going to define a *univalent n-person game of protocol.*

Let $T = \{1, 2,..., t^*\}$ be a discrete *time set* during which the game is taking place and $N = \{1,..., n\}$ be the set of *players*. The *strategy sets*, which are nonempty are given recursively. The first one is Σ^1, and for each $\sigma^1 \in \Sigma^1$, the strategy set at the second step is $\Sigma^2(\sigma^1)$ which depends upon $\sigma^1 \in \Sigma^1$. For $t \in T$ the set $\Sigma^t(\sigma^1, \sigma^2,..., \sigma^{t-1})$ is the strategy set at time t depending on all the previous choices $\sigma(t-1) = (\sigma^1,..., \sigma^{t-1})$ satisfying $\sigma^s \in \Sigma^s(\sigma(s-1))$ for each $s < t$. Given a $\sigma(t)$ we say that t is its *length* which we denote by $|\sigma|$. Let $\mathscr{G}(t)$ be the set of all $\sigma(t)$ having length t. For convenience we can consider the first set of strategies depending on a previous point, that is to say $\Sigma = \Sigma(\sigma^0)$ where $\sigma^0 = \varnothing$ is the empty set. From the sets $\mathscr{G}(t)$ we derive the following useful sets

$$\mathscr{G}^+ = \bigcup_{t=1}^{t^*} \mathscr{G}(t), \qquad \mathscr{G}^- = \bigcup_{t=0}^{t^*-1} \mathscr{G}(t).$$

Consider the shift *operator*

$$\theta\colon \mathscr{G}^+ \to \mathscr{G}^-$$

defined by

$$\theta(\sigma(t)) = \sigma(t-1)$$

for any $\sigma(t) = (\sigma(t-1), \sigma^t)$ in $\mathscr{G}^+$. θ maps $\mathscr{G}(t)$ onto $\mathscr{G}(t-1)$ and its s-composition $\theta_s = \theta \circ \cdots \circ \theta$ (s-times) maps $\mathscr{G}(t)$ onto $\mathscr{G}(t-s)$ when $t \geqslant s$. By definition θ_0 is the identity map.

The actual game structure is obtained from the scheme given above by assigning the strategy set to the players. For this, let $\mathscr{G}_i^-$ with $i \in N$ be a partition of $\mathscr{G}^-$. For each $\sigma \in \mathscr{G}_i^-$ having length $|\sigma| = t$, the strategy sets for the $t+1$ step for player $j \in N$ is given by

$$\Sigma_j^{|\sigma|+1}(\sigma) = \begin{cases} \{\varnothing\} & \text{if} \quad j \neq i, \\ \Sigma^{|\sigma|+1}(\sigma) & \text{if} \quad j = i. \end{cases}$$

Thus, the global strategy set for player $i \in N$ is

$$\Sigma_i = \mathop{\times}_{\sigma \in \mathscr{G}_i^-} \Sigma_i^{|\sigma|+1}(\sigma).$$

Clearly a $\sigma = (\sigma_1,..., \sigma_n) \in \times_{i \in N} \Sigma_i$ determines uniquely one element $\sigma(t^*) \in \mathscr{G}(t^*)$. The payoff functions A_i are real functions defined on $\mathscr{G}(t^*)$.

Hence, we have a global normal n-person game

$$\Gamma = \{\Sigma_i, A_i; i \in N\}$$

corresponding to our protocol game. We refer to both indistinctly since no confusion will arise. For simplicity we assume here that all the strategy sets are finite sets, but we point out that all the results can be obtained with some slight modifications in more general cases.

As an example of a game of protocol, we have the case when an n-person game in a normal form is played during a period of time, where at each time only one player makes the actual choice, having been informed of all the previous choices.

Following the ideas in [4] and [5] let us introduce a *quasistatic simple structure function* $e\colon \mathscr{G}^- \times N \to \mathscr{P}_N$ in the game of protocol, which assigns for each $(\sigma, i) \in \mathscr{G}^- \times N$ a subset $e(\sigma, i) \subset N$ of players, such that $e(\sigma, i) = \varnothing$ if $\sigma \in \mathscr{G}_i^-$ and is either $\varnothing$ or $\{j\}$ if $\sigma \in \mathscr{G}_j^-$ and $j \neq i$.

The intuitive meaning of such a structure function is given by regarding the player $j \in N$ in $\{j\} = e(\sigma, i)$ as a player that player $i \in N$ does not wish to depend on, in the next step after $\sigma \in \mathscr{G}^-$ or, in other words, the player $i \in N$ considers the player $j \in N$ as antagonistic at the step given by $\sigma \in \mathscr{G}^-$ in the dynamic competition.

Thus, for convenience we can write the global strategy set of a player from another player's point of view. Introducing the set

$$e_j(i) = \{\sigma \in \mathscr{G}_j^- : j \in e(\sigma, i)\} \subset \mathscr{G}_j^-,$$

which is empty if $j = i$, then calling

$$\Sigma_j(i) = \mathop{\times}_{\sigma \in e_j(i)} \Sigma^{|\sigma|+1}(\sigma)$$

and

$$\Sigma_j(f, i) = \mathop{\times}_{\sigma \in \mathscr{G}_j^- - e_j(i)} \Sigma^{|\sigma|+1}(\sigma),$$

we have

$$\Sigma_j = \Sigma_j(i) \times \Sigma_j(f, i).$$

The first factor in the last expression takes into account all those places in the game of protocol where player $j \in N$ does not wish to depend upon $i \in N$ and the second are all the remaining strategy sets on those positions where the players are considered to be indifferent by player $j \in N$. The introduction of f indicates the *indifferent coalition* concept's extension of that in [5].

$\Gamma_e = (\Gamma, e)$ is the game of protocol Γ having the quasistatic (q.s.) simple structure function e.

A suitable concept of solution for a game of protocol Γ_e is an e_m-*quasistatic simple stable point*, or concisely e_m-q.s. point, which is defined as a joint strategy $\sigma_1{}^*, \ldots, \sigma_n{}^*$ such that

$$\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{N-\{i\}}(i) \in \Sigma_{N-\{i\}}(i)} A_i(\sigma_i, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i))$$
$$= \min_{\sigma_{N-\{i\}}(i) \in \Sigma_{N-\{i\}}(i)} A_i(\sigma_i{}^*, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i))$$

for each player $i \in N$. Here, the sets involved in the expression are

$$\Sigma_{N-\{i\}}(i) = \bigtimes_{j \in N-\{i\}} \Sigma_j(i), \qquad \Sigma_{N-\{i\}}(f, i) = \bigtimes_{j \in N-\{i\}} \Sigma_j(f, i),$$

which satisfy

$$\Sigma_{N-\{i\}} = \bigtimes_{j \in N-\{i\}} \Sigma_j = \Sigma_{N-\{i\}}(i) \times \Sigma_{N-\{i\}}(f, i).$$

In the case that all the sets $e(\sigma, i)$ are empty an e_m-q.s. point becomes the usual equilibrium point in extensive game in pure strategies.

Next we are concerned with the existence of such an e_m-q.s. point for any given q.s. structure function with an arbitrary labeling. Indeed, the result given below, which is obtained applying a conceptually dynamic programming technique as those very fruitful introduced by Bellman in [1] and [2], implicitly also contributes a constructive algorithm for the construction of such a point.

THEOREM 1. *Any game of protocol Γ_e with any q.s. structure function e, has an e_m-q.s. point $\sigma^* = (\sigma_1{}^*, \ldots, \sigma_n{}^*)$.*

Proof. We proceed by induction on the length of the time set. In order to do this, let i_0 be that player such that $\sigma^0 \in \mathscr{G}^-_{i_0}$, then

$$\Sigma_{i_0} = \Sigma^1 \times \bigtimes_{\sigma \in \mathscr{G}^-_{i_0} - \{\sigma^0\}} \Sigma^{|\sigma|+1}_{i_0}(\sigma).$$

Now for each $\sigma^1 \in \Sigma^1$, we define its truncation game of protocol $_{\sigma^1}\Gamma$, which is given by the time set $_{\sigma^1}T = \{1, \ldots, t^* - 1\}$ and with strategy sets

$$_{\sigma^1}\Sigma^t(\tau(t-1)) = \Sigma^{t+|\sigma^1|}(\sigma^1, \tau^1, \ldots, \tau^{t-1})$$

for each $(\sigma^1, \tau(t-1)) \in \mathscr{G}^+$. Thus $_{\sigma^1}\mathscr{G}(t)$ for $t \in {_\sigma T}$ is the set of τ such that $\theta_t(\sigma^1, \tau) = \sigma^1$ and of course $(\sigma^1, \tau) \in \mathscr{G}(t+1)$. For simplicity we identify in the truncation scheme τ with (σ^1, τ).

Hence we have

$$_{\sigma^1}\mathscr{G}(0) = \{\sigma^1\} \qquad \text{and} \qquad _{\sigma^1}\mathscr{G}^- = \mathscr{G}^- \cap \{\rho \in \mathscr{G}^- : \theta_S(\rho) = \sigma^1,\ s \geqslant 0\}$$

and analogously for $_{\sigma^1}\mathscr{G}^+$. The q.s. structure function in $_{\sigma^1}\Gamma$ is just the restriction of the original one. Then, at once, we have

$$_{\sigma^1}\mathscr{G}_i^- = {}_{\sigma^1}\mathscr{G}^- \cap \mathscr{G}_i^-, \qquad _{\sigma^1}e(\tau, i) = e((\sigma^1, \tau), i)$$

and

$$_{\sigma^1}e_j(i) = {}_{\sigma^1}\mathscr{G}^- \cap e_j(i).$$

Therefore,

$$\begin{aligned} _{\sigma^1}\Sigma_j &= \mathop{\mathsf{X}}\limits_{\rho \in {}_{\sigma}e_j(i)} \Sigma^{|\rho|+1}(\rho) \times \mathop{\mathsf{X}}\limits_{\rho \in {}_{\sigma^1}\mathscr{G}_j^- - {}_{\sigma^1}e_j(i)} \Sigma^{|\rho|+1}(\rho) \\ &= \mathop{\mathsf{X}}\limits_{\rho \in {}_{\sigma^1}\mathscr{G}^- \cap e_j(i)} \Sigma^{|\rho|+1}(\rho) + \mathop{\mathsf{X}}\limits_{\rho \in {}_{\sigma^1}\mathscr{G}^- \cap (\mathscr{G}_i^- - e_j(i))} \Sigma^{|\rho|+1}(\rho) \\ &= {}_{\sigma^1}\Sigma_j(i) \times {}_{\sigma^1}\Sigma_j(f, i). \end{aligned}$$

The payoff function in the truncation are naturally given as

$$_{\sigma^1}A_i(\cdot) = A_i(\sigma^1, \cdot).$$

Clearly we have

$$\Sigma_i = \begin{cases} \Sigma^1 \times \mathop{\mathsf{X}}\limits_{\sigma^1 \in \Sigma^1} {}_{\sigma^1}\Sigma_i & \text{if} \quad i = i_0 \\ \mathop{\mathsf{X}}\limits_{\sigma^1 \in \Sigma^1} {}_{\sigma^1}\Sigma_i & \text{if} \quad i \neq i_0 . \end{cases}$$

It is obvious that if t^* is one then there is an e_m-q.s. point. Now assuming that in $_{\sigma^1}\Gamma$ there is an e_m-q.s. point $_{\sigma^1}\sigma^*, \ldots, {}_{\sigma^1}\sigma_n^*$, define the point $\sigma^*, \ldots, \sigma_n^*$ as

$$\sigma_i^* = ({}_{\sigma^1}\sigma_i^*)_{\sigma^1 \in \Sigma^1} \in \Sigma_i = \mathop{\mathsf{X}}\limits_{\sigma^1 \in \Sigma^1} {}_{\sigma^1}\Sigma_i$$

if $i \neq i_0$, and

$$\sigma_{i_0}^* = (\sigma^{*1}, {}_{\sigma^1}\sigma_{i_0}^*)_{\sigma^1 \in \Sigma^1} \in \Sigma^1 \times \mathop{\mathsf{X}}\limits_{\sigma^1 \in \Sigma^1} {}_{\sigma^1}\Sigma_{i_0}$$

where σ^{*1} is any point in Σ^1 such that

$$\max_{\sigma^1\in\Sigma^1} \min_{\sigma^1\tau_{N-\{i_0\}}(i_0)\in\Sigma_{N-\{i_0\}}(i_0)} {}_{\sigma^1}A_{i_0}[{}_{\sigma^1}\sigma^*_{i_0}, {}_{\sigma^1}\tau_{N-\{i_0\}}(i_0), {}_{\sigma^1}\sigma^*_{N-\{i_0\}}(f, i_0)]$$

$$= \min_{\sigma^{*1}\tau_{N-\{i_0\}}(i_0)\in\Sigma_{N-\{i_0\}}(i_0)} {}_{\sigma^{*1}}A_{i_0}[{}_{\sigma^{*1}}\sigma^*_{i_0}, {}_{\sigma^{*1}}\tau_{N-\{i_0\}}(i_0), {}_{\sigma^{*1}}\sigma^*_{N-\{i_0\}}(f, i_0)].$$

We wish to prove that $\sigma_1^*,..., \sigma_n^*$ is indeed an e_m-q.s. point in the original game of protocol.

Let us begin with player i_0. We have

$$\max_{\sigma_{i_0}\in\Sigma_{i_0}} \min_{\sigma_{N-\{i_0\}}(i_0)\in\Sigma_{N-\{i_0\}}(i_0)} A_{i_0}[\sigma_{i_0}, \sigma_{N-\{i_0\}}(i_0), \sigma^*_{N-\{i_0\}}(f, i_0)]$$

$$= \max_{\sigma^1\in\Sigma^1} \max_{\sigma^1\sigma_{i_0}\in_{\sigma^1}\Sigma_{i_0}} \min_{\sigma^1\sigma_{N-\{i_0\}}(i_0)\in_{\sigma^1}\Sigma_{N-\{i_0\}}(i_0)}$$

$$\times\ {}_{\sigma^1}A_{i_0}[{}_{\sigma^1}\sigma_{i_0}, {}_{\sigma^1}\sigma_{N-\{i_0\}}(i_0), {}_{\sigma^1}\sigma^*_{N-\{i_0\}}(f, i_0)]$$

$$= \max_{\sigma^1\in\Sigma^1} \min_{\sigma^1\sigma_{N-\{i_0\}}(i_0)\in_{\sigma^1}\Sigma_{N-\{i_0\}}(i_0)} {}_{\sigma^1}A_{i_0}[{}_{\sigma^1}\sigma^*_{i_0}, {}_{\sigma^1}\sigma_{N-\{i_0\}}(i_0), {}_{\sigma^1}\sigma^*_{N-\{i_0\}}(f, i_0)]$$

$$= \min_{\sigma^{*1}\sigma_{N-\{i_0\}}(i_0)\in_{\sigma^{*1}}\Sigma_{N-\{i\}_0}(i_0)} {}_{\sigma^{*1}}A_{i_0}[{}_{\sigma^{*1}}\sigma^*_{i_0}, {}_{\sigma^{*1}}\sigma_{N-\{i_0\}}(i_0), {}_{\sigma^{*1}}\sigma^*_{N-\{i_0\}}(f, i_0)]$$

$$= \min_{\sigma_{N-\{i_0\}}(i_0)\in\Sigma_{N-\{i_0\}}(i_0)} A_{i_0}[\sigma^*_{i_0}, \sigma_{N-\{i_0\}}(i_0), \sigma^*_{N-\{i_0\}}(f, i_0)].$$

The first equality is simply derived by splitting the maximum on a Cartesian product set. The second one is due to the definition of e_m-q.s. points in the truncation games ${}_{\sigma^1}\Gamma$. The next is by definition of σ^{*1} and the last one is completely clear.

Now let $i \in N$ be a player such that $i_0 \notin e(\sigma^0, i)$; then σ^{*1} is a component of $\sigma^*_{N-\{i\}}(f, i)$. Therefore,

$$\max_{\sigma_i\in\Sigma_i} \min_{\sigma_{N-\{i\}}\in\Sigma_{N-\{i\}}(i)} A_i[\sigma_i, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i_0)]$$

$$= \max_{\sigma^{*1}\sigma_i\in_{\sigma^{*1}}\Sigma_i} \min_{\sigma^{*1}\sigma_{N-\{i\}}\in_{\sigma^{*1}}\Sigma_{N-\{i\}}(i)} {}_{\sigma^{*1}}A_i[{}_{\sigma^{*1}}\sigma_i, {}_{\sigma^{*1}}\sigma_{N-\{i\}}(i), {}_{\sigma^{*1}}\sigma_{N-\{i\}}(f, i)]$$

$$= \min_{\sigma^{*1}\sigma_{N-\{i\}}(i)\in_{\sigma^{*1}}\Sigma_{N-\{i\}}(i)} {}_{\sigma^{*1}}A_i[{}_{\sigma^{*1}}\sigma_i^*, {}_{\sigma^{*1}}\sigma_{N-\{i\}}(i), {}_{\sigma^{*1}}\sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{\sigma_{N-\{i\}}(i)\in\Sigma_{N-\{i\}}(i)} A_i[\sigma_i^*, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)].$$

Here, the second equality is due to the definition of e_m-q.s. points in the truncation games and the others are different ways to write the expressions.

Finally, it remains to analyse the equality for each player $i \in N$ when $i_0 \in e(\sigma^0, i)$. In this case we can write

$$\Sigma_{N-\{i\}}(i) = \Sigma_{N-\{i_0, i\}}(i) \times \Sigma_{i_0}(i)$$

and Σ^1 is a component of $\Sigma_{i_0}(i)$. But before we go explicitly into the computation, let us deduce a relation in a zero-sum two-person game of protocol Γ_2 which will be of help for the following analysis.

Consider Γ_2 given by $N = \{1, 2\}$, $T = \{1, 2\}$, $G_2^- = \{\sigma^0\}$ and $\mathscr{G}_1^- = \Sigma^1$. All the antagonistic coalitions given by e are empty. The payoff functions $A = A_1 = -A_2$. Define

$$v_- = \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} A(\sigma_1, \sigma_2)$$

and

$$v_+ = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} A(\sigma_1, \sigma_2).$$

We now prove that $v_+ = v_-$.

We remind that

$$\Sigma_2 = \Sigma^1 \qquad \text{and} \qquad \Sigma_1 = \mathop{\times}_{\sigma^1 \in \Sigma^1} \Sigma^2(\sigma^1).$$

Consider the strategy $\bar{\sigma}_1$ given by $\{\bar{\sigma}^2\}_{\sigma^1 \in \Sigma^1}$ such that

$$_{\sigma^1}A(\bar{\sigma}^2) = \max_{\sigma^2 \in \Sigma^2(\sigma^1)} {}_{\sigma^1}A(\sigma^2)$$

for each σ^1. Then, we have that for any $\sigma_2 \in \Sigma_2$

$$\max_{\sigma_1 \in \Sigma_1} A(\sigma_1, \sigma_2) = A(\bar{\sigma}_1, \sigma_2) = {}_{\sigma^1}A(\bar{\sigma}^2)$$

and therefore

$$\begin{aligned} v_+ &\leqslant \min_{\sigma^1 \in \Sigma^1} \max_{\sigma_1 \in \Sigma_1} A(\sigma_1, \sigma^1) = \min_{\sigma^1 \in \Sigma^1} {}_{\sigma^1}A(\bar{\sigma}^2) \\ &= \min_{\sigma_2 \in \Sigma_2} A(\bar{\sigma}_1, \sigma_2) \leqslant v_-, \end{aligned}$$

which implies $v_- = v_+$.

Having this auxiliary result, for any player $i \in N$ with $i_0 \in e(\sigma^0, i)$, we have

$$\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}(i)} A_i[\sigma_i, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)]$$

$$= \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{i_0}(i) \in \Sigma_{i_0}(i)} \min_{\sigma_{N-\{i_0, i\}}(i) \in \Sigma_{N-\{i_0, i\}}(i)} A_i[\sigma_i, \sigma_{i_0}(i), \sigma_{N-\{i_0, i\}}(i), \sigma^*_{N-\{i\}}(f, i)]$$

$$= \max_{\sigma_i \in \Sigma_i} \min_{\sigma^1 \in \Sigma^1} \{ \min_{{}_{\sigma^1}\sigma_{i_0}(i) \in {}_{\sigma^1}\Sigma_{i_0}(i)} \min_{{}_{\sigma^1}\sigma_{N-\{i_0, i\}}(i) \in {}_{\sigma^1}\Sigma_{N-\{i_0, i\}}(i)}$$

$$ {}_{\sigma^1}A_i[\sigma_i, {}_{\sigma^1}\sigma_{i_0}(i), {}_{\sigma^1}\sigma_{N-\{i_0, i\}}(i), {}_{\sigma^1}\sigma^*_{N-\{i_0, i\}}(f, i)]\}$$

$$= \min_{\sigma^1 \in \Sigma^1} \max_{{}_{\sigma^1}\sigma_i \in {}_{\sigma^1}\Sigma_i} \min_{{}_{\sigma^1}\sigma_{N-\{i\}}(i) \in {}_{\sigma^1}\Sigma_{N-\{i\}}(i)} {}_{\sigma^1}A_i[{}_{\sigma^1}\sigma_i, {}_{\sigma^1}\sigma_{N-\{i\}}(i), {}_{\sigma^1}\sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{\sigma^1 \in \Sigma^1} \min_{{}_{\sigma^1}\sigma_{N-\{i\}}(i) \in {}_{\sigma^1}\Sigma_{N-\{i\}}(i)} {}_{\sigma^1}A_i[{}_{\sigma^1}\sigma_i^*, {}_{\sigma^1}\sigma_{N-\{i\}}(i), {}_{\sigma^1}\sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{\sigma_{N-\{i\}}(i) \in \Sigma_{N-\{i\}}(i)} A_i[\sigma_i^*, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)].$$

The first two and the last equalities are clear. The last but one is due to the definition of e_m-q.s. points in the truncation games. The remaining one is nothing else than the application of the previous auxiliary result to the zero-sum game of protocol with length two and players $\{i_0, i\}$ with global strategies Σ^1 and Σ_i respectively and the payoff given inside the parentheses.

By the induction principle, the joint point $\sigma_1^*, \ldots, \sigma_n^*$ is indeed an e_m-q.s. point. Q.E.D.

3. Truncation and Difference Games of Protocol

Having the existence of such points, we would now like to present some properties that they have.

For any given $\sigma \in \mathscr{G}^-$ one can analogously define as in the proof of the theorem the truncation game ${}_\sigma\Gamma$. Calling ${}_\sigma\mathscr{D}_i^- = \mathscr{G}^- - {}_\sigma\mathscr{G}_i$ the *difference* set, then we have that the global strategy set can be written as

$$\Sigma_j = {}_\sigma\Sigma_j(i) \times {}_\sigma\Sigma_j(f, i) \times \bigtimes_{\tau \in e(i) \cap {}_\sigma\mathscr{D}_j} \Sigma^{|\tau|+1}(\tau) \times \bigtimes_{\tau \in {}_\sigma\mathscr{D}_j - [e_j(i) \cap {}_\sigma\mathscr{D}_j]} \Sigma^{|\tau|+1}(\tau)$$

and calling ${}^\sigma\Sigma_j(i)$ and ${}^\sigma\Sigma_j(f, i)$ the third and fourth terms respectively, we concisely have

$$\Sigma_j = {}_\sigma\Sigma_j(i) \times {}_\sigma\Sigma_j(f, i) \times {}^\sigma\Sigma_j(i) \times {}^\sigma\Sigma_j(f, i).$$

On the other hand, for each ${}_\sigma\tau = \tau \in \mathsf{X}_{j\in N}\, {}_\sigma\Sigma_j$ one can define the difference protocol game ${}_\sigma\Gamma(\tau)$, where all the strategy sets are as in the original game except all those $\Sigma^{|\tau|+1}(\tau)$ such that $\theta_s(\tau) = \sigma$ for some $s \geqslant 0$, which now are changed to $\Sigma_{\mathscr{D}}^{|\tau|+1}(\tau) = \{\varnothing\}$. Therefore, the sets

$$^\sigma\Sigma_j = {}^\sigma\Sigma_j(i) \times {}^\sigma\Sigma_j(f, i)$$

represent the global strategy sets in ${}_\sigma\Gamma(\tau)$.

By attaching the payoff value

$${}_{\mathscr{D}}A_i(\sigma, {}_\sigma\tau) = \min_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)} A_i(\sigma, {}_\sigma\tau_i\,, {}_\sigma\sigma_{N-\{i\}}(i), {}_\sigma\tau_{N-\{i\}}(f, i))$$

to the end point represented by σ in ${}_\sigma\Gamma(\tau)$ and all the original values at the remaining end points, the difference game becomes well defined. We emphasize that it depends on ${}_\sigma\tau = \tau$. The new simple structure function is the restriction on ${}_\sigma\mathscr{D}$ and it is empty at σ and all its derivations.

Thus, we have the following result:

THEOREM 2. *Given an* $\sigma \in \mathscr{G}^-$, *if*

$${}_\sigma\sigma^* = ({}_\sigma\sigma_1{}^*, \ldots, {}_\sigma\sigma_n{}^*) \qquad \textit{and} \qquad {}^\sigma\sigma^* = ({}^\sigma\sigma_1{}^*, \ldots, {}^\sigma\sigma_n{}^*)$$

are e_m-*q.s. points in* ${}_\sigma\Gamma$ *and* ${}_\sigma\Gamma({}_\sigma\sigma^*)$ *respectively, then their composition* $\sigma^* = (\sigma_1{}^*, \ldots, \sigma_n{}^*)$ *is an* e_m-*q.s. point in the original game of protocol.*

Proof. By the previous global strategy sets decomposition, we have for player $i \in N$,

$$\begin{aligned}
I_1 &= \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{N-\{i\}}(i) \in \Sigma_{N-\{i\}}(i)} A_i[\sigma_i\,, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)] \\
&= \max_{{}^\sigma\sigma_i \in {}^\sigma\Sigma_i} \min_{{}_\sigma\sigma_i \in {}_\sigma\Sigma_i} \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} \max_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)} \\
&\qquad A_i[{}^\sigma\sigma_i\,, {}_\sigma\sigma_i\,, {}^\sigma\sigma_{N-\{i\}}(i), {}_\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i), {}_\sigma\sigma^*_{N-\{i\}}(f, i)] \\
&= \max_{{}^\sigma\sigma_i \in {}^\sigma\Sigma_i} \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} \{ \max_{{}_\sigma\sigma_i \in {}_\sigma\Sigma_i} \min_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)} \\
&\qquad A_i[{}^\sigma\sigma_i\,, {}_\sigma\sigma_i\,, {}^\sigma\sigma_{N-\{i\}}(i), {}_\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i), {}_\sigma\sigma^*_{N-\{i\}}(f, i)]\},
\end{aligned}$$

where the last equality was obtained by applying in a similar way as in the proof of Theorem 1, the maximim theorem to a zero-sum two-person game of protocol pointed at σ arising naturally from the expression given above.

But, the amount inside the parentheses equals $A_i(\sigma^*(t^*))$ if $\theta_s(\sigma^*(t^*)) \neq \sigma$ for all s and ${}_{\mathscr{D}}A_i(\sigma, {}_\sigma\sigma^*)$ if ${}^\sigma\sigma^*$ determines σ or equivalently if $\theta_s(\sigma^*(t^*)) = \sigma$

for some s, since ${}_\sigma\sigma^*$ is an e_m-q.s. point in ${}_\sigma\Gamma$. Here, $\sigma^*(t^*)$ is the end point determined by $\sigma^* = ({}_\sigma\sigma^*, {}^\sigma\sigma^*)$.

Therefore, using also the definition of ${}^\sigma\sigma^*$, it follows that

$$I_1 = \max_{{}^\sigma\sigma_i \in {}^\sigma\Sigma_i} \; \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} \; \max_{{}_\sigma\sigma_i \in {}_\sigma\Sigma_i} \; \min_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)}$$

$$A_i[{}^\sigma\sigma_i\,, {}_\sigma\sigma_i\,, {}^\sigma\sigma_{N-\{i\}}(i), {}_\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i), {}_\sigma\sigma^*_{N-\{i\}}(f, i)]$$

$$= \max_{{}^\sigma\sigma_i \in {}^\sigma\Sigma_i} \; \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} \; \min_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)}$$

$$A_i[{}^\sigma\sigma_i\,, {}_\sigma\sigma_i{}^*, {}^\sigma\sigma_{N-\{i\}}(i), {}_\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i), {}_\sigma\sigma^*_{N-\{i\}}(f, i)]$$

$$= \max_{{}^\sigma\sigma_i \in {}^\sigma\Sigma_i} \; \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} {}_{\mathscr{D}}A_i[\sigma, {}_\sigma\sigma^*, {}^\sigma\sigma_i\,, {}^\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} {}_{\mathscr{D}}A_i[\sigma, {}_\sigma\sigma^*, {}^\sigma\sigma_i{}^*, {}^\sigma\sigma_{N-\{i\}}(i), {}^\sigma\sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{{}^\sigma\sigma_{N-\{i\}}(i) \in {}^\sigma\Sigma_{N-\{i\}}(i)} \; \min_{{}_\sigma\sigma_{N-\{i\}}(i) \in {}_\sigma\Sigma_{N-\{i\}}(i)}$$

$$A_i[\sigma_i{}^*, {}^\sigma\sigma_{N-\{i\}}(i), {}_\sigma\sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)]$$

$$= \min_{\sigma_{N-\{i\}}(i) \in \Sigma_{N-\{i\}}(i)} A_i[\sigma_i{}^*, \sigma_{N-\{i\}}(i), \sigma^*_{N-\{i\}}(f, i)] = I_2\,,$$

which implies that σ^* is indeed an e_m-q.s. point in the original game of protocol. Q.E.D.

Finally, it is not difficult to prove the property in the next result.

THEOREM 3. *If the restriction ${}_\sigma\sigma^*$ of any e_m-q.s. point σ^* in Γ is an e_m-q.s. point in $\Gamma(\sigma)$, then ${}^\sigma\sigma^*$ is an e_m-q.s. point in the difference game ${}_\sigma\Gamma({}_\sigma\sigma^*)$*

Proof. Indeed from the first expression of I_1 given in the proof of the previous Theorem, the amount in the parentheses equals the corresponding minimum. Therefore since $I_1 = I_2$, we obtain the needed equality in game ${}_\sigma\Gamma({}_\sigma\sigma^*)$ for the payoff of each player. Q.E.D.

It is interesting to point out that the projection ${}_\sigma\sigma^*$ of an e_m-q.s. point does not necessarily lead to an e_m-q.s. point in the truncation game. Indeed, consider the following two-person game of protocol Γ given by $\Sigma^1 = \{1, 2\}$, $\Sigma^2(\sigma^1) = \{1, 2\}$ and payoff functions $A_2 = A_1 = A$:

$$A(1, 1) = 2, \qquad A(1, 2) = A(2, 1) = A(2, 2) = 1,$$

where

$$\mathscr{G}_1 = \{\sigma^0\} \qquad \text{and} \qquad \mathscr{G}_2 = \Sigma^1.$$

The structure function is given by $e(\sigma^1, 1) = \{2\}$ for each $\sigma^1 \in \Sigma^1$ and all the others empty. Thus, the joint strategy $\bar{\sigma}^1 = 1$, $\bar{\sigma}^2(1) = 2$, $\bar{\sigma}^2(2) = 1$ is clearly an e_m-q.s. point in Γ, but $\bar{\sigma}^2(1)$ is not an e_m-q.s. point in the truncation game ${}_{\bar{\sigma}^1}\Gamma$.

4. Bivalent Two-Person Games of Protocol

Having already studied some properties for the univalent games of protocol in the previous sections, we would now like to extend the theory to more general situations where more than one player determine the forthcoming situations in the dynamic competition scheme. However, since we cannot expect to obtain important results in the general case, due to the fact that e_m-q.s. points no longer exist, we draw our attention only to zero-sum two-person *bivalent* games of protocol where both players determine the future of the game. It is clear that in such games the structure function is naturally given and therefore could be disregarded.

The definition of such games is similar to the univalent ones but now the strategy sets are given recursively by $\Sigma_i^t(\sigma(t-1))$ where

$$\sigma(t-1) = (\sigma_1(t-1), \sigma_2(t-1))$$

have both components and

$$\Sigma^t(\sigma(t-1)) = \Sigma_1^t(\sigma(t-1)) \times \Sigma_2^t(\sigma(t-1)).$$

The first strategy sets are

$$\Sigma^1 = \Sigma_1^1 \times \Sigma_2^1.$$

The payoff function A is defined on the set of end points $\sigma(t^*)$.

Even in the most simple case when $t^* = 1$, the minimax theorem does not hold in pure strategies. Nevertheless, under sane restrictive conditions we already know that a saddle point exists. Shapley in [7] has investigated such games and he proved that under the condition that all 2×2 submatrices have a saddle point, then the original game has a saddle point too. We call such a condition *Shapley's condition* or briefly *S-condition*. In the proof he uses the existence of the value in mixed strategies. We are going to present his result with a new proof which does not require any information about the mixed extension. In order to do so, we need the following:

Lemma 4. *The game $\Gamma = \{\Sigma_1, \Sigma_2; A\}$ satisfies the S-condition. Let μ be an arbitrary number. If for each $j \in S \in \Sigma_2$ there is an $i \in \Sigma_1$ such that $A(i,j) > \mu$, then there is an $\bar{\imath} \in \Sigma_1$ such that for all $j \in S$, $a_{\bar{\imath}j} > \mu$.*

Proof. We will prove it by induction on the number of elements of S: $|S|$. If $|S| = 1$, the result is clear. Let $R = S \cup \{j_{s+1}\}$ where $S = \{j_1, ..., j_s\}$. By induction, there is an $\hat{\imath} \in \Sigma_1$ such that for each

$$j \in S\colon A(\hat{\imath}, j) > \mu.$$

Let i_{s+1} be such that $A(i_{s+1}, j_{s+1}) > \mu$. If $i_{s+1} = \hat{\imath}$ or $A(\hat{\imath}, j_{s+1}) > \mu$, choose $\bar{\imath} = \hat{\imath}$; otherwise when $i_{s+1} \neq \hat{\imath}$ and $A(\hat{\imath}, j_{s+1}) \leqslant \mu$ consider for $j \in S$ the 2×2 game

$$\begin{vmatrix} A(i_{s+1}, j) & A(i_{s+1}, j_{s+1}) > \mu \\ A(\hat{\imath}, j) > \mu & A(\hat{\imath}, j_{s+1}) \leqslant \mu \end{vmatrix}.$$

The strategies $(\hat{\imath}, j_{s+1})$ and $(\hat{\imath}, j)$ cannot be saddle points. However, by hypothesis it has a saddle point. If (i_{s+1}, j) is a saddle point then

$$A(i_{s+1}, j) \geqslant A(\hat{\imath}, j) > \mu.$$

On the other hand, if (i_{s+1}, j_{s+1}) is a saddle point, we have

$$A(i_{s+1}, j) \geqslant A(i_{s+1}, j_{s+1}) > \mu,$$

and therefore

$$A(i_{s+1}, j) > \mu$$

for all $j \in R$. Q.E.D.

As a consequence of this result, we now derive the following:

THEOREM 5. *Any game Γ satisfying the S-condition has a saddle point.*

Proof. Let $i_0 \in \Sigma_1$ be a maximum strategy and $M_0 \in \Sigma_1$ the set of $j \in \Sigma_2$ such that $A(i_0, j) = v_i$, where v_1 is the maximum values in pure strategies.

If there is a $j \in M_0$ such that for each $i \in \Sigma_1$:

$$A(i_0, j) \geqslant A(i, j)$$

for all $i \in \Sigma_1$, then (i_0, j_0) is a saddle point. Otherwise, assume that for each $j \in M_0$ there is an $i \in \Sigma_1$ such that

$$A(i, j) > A(i_0, j) = v_1.$$

Therefore, by the previous lemma, applied to M_0, there is an $\bar{\imath} \in \Sigma_1$ such that for any $j \in M_0$

$$A(\bar{\imath}, j) > v_1.$$

On the other hand, by definition of M_0 we have

$$A(i_0, j) > v_1$$

for all $j \notin M_0$. Consider the 2×2 game

$$\begin{vmatrix} A(\bar{\imath}, \bar{\jmath}) & A(\bar{\imath}, j) \\ A(i_0, \bar{\jmath}) = v_1 & A(i_0, j) > v_1 \end{vmatrix}$$

with $j \notin M_0$. Clearly, $(i_0, \bar{\jmath})$ and (i_0, j) cannot be saddle points. If $(\bar{\imath}, \bar{\jmath})$ is a saddle point, then

$$A(\bar{\imath}, j) \geqslant A(\bar{\imath}, \bar{\jmath}) > v_1$$

and on the other hand if $(\bar{\imath}, j)$ is

$$A(\bar{\imath}, j) \geqslant A(i_0, j) > v_1 .$$

In both cases, $A(\bar{\imath}, j) > v_1$, and therefore there is an $\bar{\imath} \in \Sigma_1$ such that $A(\bar{\imath}, j) > v_1$ for all $j \in \Sigma_2$, which is impossible by virtue of the definition of i_0. Q.E.D.

Now we want to go further into our analysis and we will study the dynamic situation. We study the case when $t^* = 2$.

In such a case the global description of our game of protocol can be given by

$$\Gamma = \{\Sigma_1, \Sigma_2; A\}$$

where

$$\Sigma_1 = \Sigma_1{}^1 \times \mathop{\mathsf{X}}_{\sigma^1 \in \Sigma^1 = \Sigma_1{}^1 \times \Sigma_2{}^1} \Sigma_1{}^2(\sigma^1) \qquad \text{and} \qquad \Sigma_2 = \Sigma_2{}^1 \times \mathop{\mathsf{X}}_{\sigma^1 \in \Sigma^1 = \Sigma_1{}^1 \times \Sigma_2{}^1} \Sigma_2{}^2(\sigma^1)$$

and the payoff function becomes defined in a natural way. For any $\sigma^1 \in \Sigma^1$, let $M^-(\sigma^1)$ and $M^+(\sigma^1)$ be the set of maximin and minimax strategies respectively in the truncation of ${}_{\sigma^1}\Gamma$.

For a σ^1, let $\rho^2 \in M^-(\sigma^1) \times M^+(\sigma^1)$, thus (σ^1, ρ^2) determines an end point. Now let $\sigma_1{}^1, \bar{\sigma}_1{}^1$ and $\sigma_2{}^1, \bar{\sigma}_2{}^1$ be corresponding strategies for the first and second player at time $t = 1$ respectively. We have four combinations of the form $\sigma^1 \in \Sigma^1$, namely:

$$(\sigma_1{}^1, \sigma_1{}^1), \qquad (\sigma_1{}^1, \bar{\sigma}_2{}^1), \qquad (\bar{\sigma}_1{}^1, \sigma_2{}^1), \qquad (\bar{\sigma}_1{}^1, \bar{\sigma}_2{}^1).$$

If we choose a $\rho^2 \in M^-(\sigma^1) \times M^+(\sigma^1)$ for each σ^1 of this form we also have four end points. Each one of these end points determines a projection in both players strategy set. Thus we can have four different projections for each player. By combining all these entries we can obtain a matrix which may be

4×4 with entries in terms of the payoff function. We call such a matrix *associated* to $(\sigma_1{}^1, \bar{\sigma}_1{}^1)$, $(\sigma_2{}^1, \bar{\sigma}_2{}^1)$ and the corresponding choices ρ^2.

We say that the game of protocol Γ has the *weak Shapley condition* or shortly *SW-condition* if the associated matrix for any $(\sigma_1{}^1, \bar{\sigma}_1{}^1)$, $(\sigma_2{}^1, \bar{\sigma}_2{}^1)$ possesses the S-condition.

Thus, we present the following result:

THEOREM 6. *If for any σ^1, ${}_{\sigma^1}\Gamma$ satisfies the S-condition and Γ satisfies the SW-condition, then the game of protocol Γ with time length 2 has a saddle point.*

Proof. Each truncation ${}_{\sigma^1}\Gamma$ has a saddle point by virtue of Theorem 5, with value $v(\sigma^1)$. Now given $(\sigma_1{}^1, \bar{\sigma}_1{}^1)$, $(\sigma_2{}^1, \bar{\sigma}_2{}^1)$ consider for any choice of maximin and minimax strategies the associated matrix. Since it satisfies the SW-condition it has a saddle point. Now if this occurs at one point $(\hat{\sigma}^1, \rho^2)$ where ρ^2 was chosen in $M^-(\sigma^1) \times M^+(\sigma^1)$, then

$$v(\tau_1{}^1, \hat{\sigma}_1{}^1) \leqslant v(\hat{\sigma}_1{}^1, \hat{\sigma}_2{}^1) \leqslant v(\hat{\sigma}_1{}^1, \tau_2{}^1) \qquad (*)$$

for

$$\tau_1{}^1 = \sigma_1{}^1, \bar{\sigma}_1{}^1 \qquad \text{and} \qquad \tau_2{}^1 = \sigma_2{}^1, \bar{\sigma}_2{}^1.$$

Indeed, in order to see it, suppose that $\hat{\sigma}^1 = (\sigma_1{}^1, \sigma_1{}^1)$ and ρ^2 corresponding to it is $\rho^2 = (\rho_1{}^2, \rho_2{}^2)$. Consider the row with fixed $\sigma_1{}^1, \rho_1{}^2$ then $v(\sigma_1{}^1, \sigma_2{}^1)$ is the minimum value of this row. In particular

$$v(\sigma_1{}^1, \sigma_2{}^1) \leqslant A(\sigma_1{}^1, \bar{\sigma}_2{}^1, \rho_1{}^2, \bar{\rho}_2{}^2),$$

where $\bar{\rho}_2{}^2$ is the choice in $M^-(\sigma_1{}^1, \bar{\sigma}_2{}^1)$. But by definition of $v(\sigma_1{}^1, \bar{\sigma}_2{}^1)$ in the truncation ${}_{(\sigma_1{}^1, \bar{\sigma}_1{}^2)}\Gamma$, we have

$$A(\sigma_1{}^1, \bar{\sigma}_2{}^1, \rho_1{}^2, \bar{\rho}_2{}^2) \leqslant v(\sigma_1{}^1, \bar{\sigma}_2{}^1)$$

for each $\rho_1{}^2 \in \Sigma_1{}^2(\sigma_1{}^1, \bar{\sigma}_2{}^1)$, and therefore

$$v(\sigma_1{}^1, \sigma_2{}^1) \leqslant v(\sigma_1{}^1, \bar{\sigma}_2{}^1).$$

Similarly for the other inequality. In the case that the saddle point in the associated matrix does not occur at one point $(\hat{\sigma}^1, \rho^2)$ where ρ^2 was not chosen in $M^-(\sigma^1) \times M^+(\sigma^1)$, one can easily observe that the value of the payoff function $A(\hat{\sigma}^1, \rho^2)$ has to be equal to $v(\hat{\sigma}_1{}^1, \hat{\sigma}_2{}^1)$ and again it is easy to prove that the relation (*) holds true. Therefore, we have proven that the game

$$\{\Sigma_1{}^1, \Sigma_2{}^1; v\}$$

where the payoff function is given by $v(\sigma_1{}^1, \sigma_2{}^1)$ satisfies the S-condition.

Consequently it has a saddle point $(\bar{\sigma}_1^1, \bar{\sigma}_2^1)$ or equivalently

$$\max_{\sigma_1^1 \in \Sigma_1^1} \min_{\sigma_2^1 \in \Sigma_2^1} v(\sigma_1^1, \sigma_2^1) = \min_{\sigma_2^1 \in \Sigma_2^1} \max_{\sigma_1^1 \in \Sigma_1^1} v(\sigma_1^1, \sigma_2^2).$$

On the other hand, by replacing $v(\sigma_1^1, \sigma_2^2)$ the first amount has the expression

$$\max_{\sigma_1^1 \in \Sigma_1^1} \min_{\sigma_2^1 \in \Sigma_2^1} \max_{\sigma_1^2 \in \Sigma_1^2(\sigma_1^1, \sigma_2^1)} \{ \min_{\sigma_2^2 \in \Sigma_2^2(\sigma_1^1, \sigma_2^1)} A[\sigma_1^1, \sigma_2^1, \sigma_1^2, \sigma_2^2]\}$$

$$= \max_{\sigma_1^1 \in \Sigma_1^1} \max_{\sigma_1^2 \in \times_{\sigma_2^1 \in \Sigma_2^1} \Sigma_1^2(\sigma_1^1, \sigma_2^1)} \min_{\sigma_2^1 \in \Sigma_2^1} \min_{\sigma_2^2 \in \Sigma_2^2(\sigma_1^1, \sigma_2^1)} A[\sigma_1^1, \sigma_2^1, \sigma_1^2, \sigma_2^2]$$

$$= \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} A[\sigma_1, \sigma_2],$$

where the last equality is the consequence of applying the minimax theorem to a suitable univalent two-person game of protocol having the payoff function described in the parentheses.

Similarly, the second amount equals the minimax in the global game.

Q.E.D.

The general case having t^* arbitrary could be carried out in a similar fashion as that just presented. No difficulties should arise.

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