Solution of differential algebraic equations via semi-analytic method

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Abstract Differential algebraic equations arise in many applications. In this paper, the homotopy perturbation method is applied to solve differential algebraic equations. In this method, the solution is found in the form of a convergent series and usually converges to the exact solution. In addition, the terms of the series can be computed easily. A few examples are presented to illustrate the method. The results show the simplicity and efficiency of the proposed method.

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1. Introduction

Differential algebraic equations (DAEs) have successfully modeled many phenomena in circuit analysis, power systems, chemical process simulations [1], and optimal control problems. The most general form of a DAEs is in the following:

\[ f(t, u(t), u'(t)) = 0, \quad t \in (0, T), \]  

(1.1)

where \( f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) and its partial derivative with respect to the third argument (i.e. \( \frac{\partial}{\partial t} \)) is singular. If \( \frac{\partial}{\partial t} \) is nonsingular, then it is possible to solve (1.1) for \( u'(t) \) in order to obtain a system of ordinary differential equations (ODEs). When \( \frac{\partial}{\partial t} \) is singular, the \( u(t) \) has to satisfy algebraic constrains. Several concepts of index have been developed to describe the structure of DAEs. For instance, the nonlinear DAE (1.1) has index \( \mu \), where \( \mu \) is the minimal number of differentiations while the equations

\[
\begin{align*}
    f(t, u(t), u'(t)) &= 0, \\
    \frac{\partial f(t, u(t), u'(t))}{\partial t} &= 0, \\
    &\vdots \\
    \frac{d^\mu f(t, u(t), u'(t))}{dt^\mu} &= 0,
\end{align*}
\]

(1.2)

allow to extract an explicit ordinary differential system using only algebraic manipulations. For example, consider the following DAEs

\[ u'_1(t) = f(u_1(t), u_2(t)), \]  

(1.3)

\[ 0 = g(u_1(t), u_2(t)). \]  

(1.4)
Eq. (1.4) results in
\[ 0 = \frac{dg(u(t), u(t))}{dt} = g_n(u(t), u(t))u_N(t) + g_{n,0}(u(t), u(t))u_N(t). \] (1.5)

If \( g_{n,0}(u(t), u(t)) \) is nonsingular in a neighborhood of the solution, we obtain:
\[ u_N(t) = g_n^{-1}(u(t), u(t))g_{n,0}(u(t), u(t))u_N(t). \] (1.6)

Consequently, we have
\[
\begin{align*}
[u_N(t) = f(u(t), u(t)), \\
u_N(t) = -g_n^{-1}(u(t), u(t))g_{n,0}(u(t), u(t))f(u(t), u(t)).
\end{align*}
\] (1.7)

Therefore, the index is \( \mu = 1. \)

Now, we consider the following DAEs
\[
\begin{align*}
[u_N(t) = f(u(t), u(t)), \\
u_N(t) = g(u(t)).
\end{align*}
\] (1.8, 1.9)

This problem can be simplified to the system of ODEs in the similar way. The second equation (i.e. (1.9)) results in
\[
\begin{align*}
0 = \frac{dg(u(t))}{dt} = g_n(u(t))u_N(t) \\
= g_n(u(t))f(u(t), u(t)) = h(u(t), u(t)).
\end{align*}
\] (1.10)

Therefore, according to the aforesaid example, if \( h_{u_N}(u(t), u(t)) \) is nonsingular, then the index is \( \mu = 2. \)

Finally we consider
\[
\begin{align*}
[u_N(t) = f(u(t), u(t)), \\
u_N(t) = g(u(t), u(t), u_N(t)), \\
u_N(t) = h(u(t)).
\end{align*}
\] (1.11, 1.12, 1.13)

Similarly by three differentiations, if
\[ h_{u_N}(u(t))f_{u_N}(u(t), u(t))g_{u_N}(u(t), u(t), u_N(t), u_N(t), u_N(t)). \] (1.14)

is nonsingular, then the index is \( \mu = 3. \) These examples have modeled many phenomena. DAEs can be difficult to solve when they have a higher index. Higher index DAEs are isolated especially when the index is greater than two [2]. Straightforward discretization generally does not work well for high index problems. An alternative treatment is the index reduction methods (for further see [3] and references therein).

Many methods such as BDF [2,4,5], implicit Runge-Kutta [2,6,7], Pade approximation [1,8], Adomian decomposition [3], and multiquadric approximation [9] have been presented to solve DAEs. Recently, homotopy perturbation method (HPM) has been applied with a great success to obtain approximate solutions for a large variety of problems in ODEs [10,11], partial differential equations (PDEs) [12–16], and integral equations [17]. In addition, some modifications of HPM have been suggested [18–22]. HPM approximates the solution as an infinite series. This series usually converges to the exact solution. The present paper is devoted to apply HPM for solving DAEs.

The organization of this paper is as follows. In Section 2, for convenience of the reader, a short review of HPM is presented. In Section 3, HPM is applied to solve a few examples of DAEs. Finally, a short conclusion is given in Section 4.

2. The homotopy perturbation method (HPM)

J.H. He presented a homotopy perturbation technique based on the introduction of a homotopy and an artificial parameter for the solution of algebraic and ODEs [23]. To explain HPM, we consider the following nonlinear differential equation:
\[ A(u) - f(r) = 0, \quad r \in \Omega, \] (2.1)

with the boundary conditions
\[ B\left( t, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma, \] (2.2)

where \( A \) is a differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytic function, and \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be divided into parts \( L \) and \( N \), where \( L \) is a linear operator and \( N \) is a nonlinear operator. Therefore, Eq. (2.1) can be rewritten as
\[ L(u) + N(u) - f(r) = 0. \] (2.3)

Now, a homotopy \( v(r, p) : \Omega \times [0,1] \rightarrow R \) is constructed as follows:

\[
H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = L(v) - (1 - p)L(u_0) + p[N(v) - f(r)] = 0, \quad r \in \Omega,
\] (2.4)

where \( p \in [0,1] \) is an embedding parameter and \( u_0 \) is an initial approximation which satisfies the boundary conditions. Obviously, we have:

\[
\begin{align*}
H(v, 0) &= L(v) - L(u_0) = 0, \\
H(v, 1) &= A(v) - f(r) = 0.
\end{align*}
\] (2.5)

Changing the process of \( p \) from zero to unity is just that of \( v(r, p) \) from \( u_0(r) \) to \( v(r) \). We assume that the solution of Eq. (2.4) can be written as a power series in the following equation:
\[ v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots = \sum_{i=0}^{\infty} p^i v_i. \] (2.6)

By setting \( p = 1 \), the solution of (2.1) is obtained. Now, we consider the following system of first-order ODEs:
\[
\begin{align*}
\frac{du_1}{dt} + N_1(u_1, u_2, \ldots, u_n) &= f_1(t), \\
\frac{du_2}{dt} + N_2(u_1, u_2, \ldots, u_n) &= f_2(t), \\
\frac{du_3}{dt} + N_3(u_1, u_2, \ldots, u_n) &= f_3(t), \\
&\vdots \\
\frac{du_n}{dt} + N_n(u_1, u_2, \ldots, u_n) &= f_n(t),
\end{align*}
\] (2.7)

subject to the initial conditions:
\[ u_1(t_0) = c_1, u_2(t_0) = c_2, u_3(t_0) = c_3, \ldots, u_n(t_0) = c_n. \] (2.8)

According to the HPM, the following homotopy is considered:
\[
\begin{align*}
L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1, u_2, \ldots, u_n) - f_1(t)] &= 0, \\
L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1, u_2, \ldots, u_n) - f_2(t)] &= 0, \\
L(u_3) - L(v_3) + pL(v_3) + p[N_3(u_1, u_2, \ldots, u_n) - f_3(t)] &= 0, \\
&\vdots \\
L(u_n) - L(v_n) + pL(v_n) + p[N_n(u_1, u_2, \ldots, u_n) - f_n(t)] &= 0,
\end{align*}
\] (2.9)

where \( \lambda \) is an embedding parameter and \( v_0 \) is an initial approximation which satisfies the boundary conditions. Obviously, we have:
\[
\begin{align*}
L(v_0) &= 0, \\
L(v_1) &= f_1, \\
&\vdots \\
L(v_n) &= f_n.
\end{align*}
\] (2.10)

Changing the process of \( \lambda \) from zero to unity is just that of \( v_0(r, \lambda) \) from \( v_0(r) \) to \( v(r) \). We assume that the solution of Eq. (2.10) can be written as a power series in the following equation:
\[ v(r, \lambda) = v_0(r, \lambda) + \lambda v_1(r, \lambda) + \lambda^2 v_2(r, \lambda) + \cdots = \sum_{n=0}^{\infty} \lambda^n v_n(r), \] (2.11)

By setting \( \lambda = 1 \), the solution of (2.1) is obtained. Now, we consider the following system of first-order ODEs:
\[
\begin{align*}
\frac{du_1}{dt} + N_1(u_1, u_2, \ldots, u_n) &= f_1(t), \\
\frac{du_2}{dt} + N_2(u_1, u_2, \ldots, u_n) &= f_2(t), \\
\frac{du_3}{dt} + N_3(u_1, u_2, \ldots, u_n) &= f_3(t), \\
&\vdots \\
\frac{du_n}{dt} + N_n(u_1, u_2, \ldots, u_n) &= f_n(t),
\end{align*}
\] (2.12)

subject to the initial conditions:
\[ u_1(t_0) = c_1, u_2(t_0) = c_2, u_3(t_0) = c_3, \ldots, u_n(t_0) = c_n. \] (2.13)

According to the HPM, the following homotopy is considered:
\[
\begin{align*}
L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1, u_2, \ldots, u_n) - f_1(t)] &= 0, \\
L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1, u_2, \ldots, u_n) - f_2(t)] &= 0, \\
L(u_3) - L(v_3) + pL(v_3) + p[N_3(u_1, u_2, \ldots, u_n) - f_3(t)] &= 0, \\
&\vdots \\
L(u_n) - L(v_n) + pL(v_n) + p[N_n(u_1, u_2, \ldots, u_n) - f_n(t)] &= 0,
\end{align*}
\] (2.14)
where \( v_{1,0}, v_{2,0}, \ldots, v_{m,0} \) are initial approximations which satisfy the given conditions and \( L = \frac{\partial}{\partial x} \). The initial approximations are usually initial conditions, i.e.:
\[
v_{1,0} = c_1, v_{2,0} = c_2, v_{3,0} = c_3, \ldots, v_{m,0} = c_m.
\]  
(2.10)

Similarly, we assume that the solutions can be written as a power series in the following equations:
\[
egin{align*}
u_1 &= u_{1,0} + p u_{1,1} + p^2 u_{1,2} + p^3 u_{1,3} + \cdots, \\
u_2 &= u_{2,0} + p u_{2,1} + p^2 u_{2,2} + p^3 u_{2,3} + \cdots, \\
u_3 &= u_{3,0} + p u_{3,1} + p^2 u_{3,2} + p^3 u_{3,3} + \cdots, \\
\vdots & \quad \vdots \\
u_m &= u_{m,0} + p u_{m,1} + p^2 u_{m,2} + p^3 u_{m,3} + \cdots
\end{align*}
\]  
(2.11)

By substituting (2.11) in (2.9) and equating the coefficient of \( t \) to zero, we obtain:
\[
egin{align*}
L(u_1) - L(v_{1,0}) + p L(v_{2,0}) + p [1 + t] u_1 - u_2 &= 0, \\
L(u_2) - L(v_{2,0}) + p L(v_{2,0}) + p [(t^2 - 1) u_1 + (1 - t) u_2] &= 0, \\
L(u_3) - L(v_{3,0}) + p L(v_{3,0}) + p [- t^3 u_1 + t^2 u_2 - u_3] &= 0,
\end{align*}
\]  
(3.7)

where \( v_{1,0} = 1, v_{2,0} = 0, \) and \( v_{3,0} = 1. \)

Finally, by equating the coefficients of \( p \) to zero, we obtain:

**Coefficient of \( p^0 \):**
\[
\begin{align*}
L(u_1) - L(v_{1,0}) &= 0, \Rightarrow u_{1,0} = v_{1,0} = 1, \\
L(u_2) - L(v_{2,0}) &= 0, \Rightarrow u_{2,0} = v_{2,0} = 0, \\
L(u_3) - L(v_{3,0}) &= 0, \Rightarrow u_{3,0} = v_{3,0} = 1.
\end{align*}
\]  
(3.8)

**Coefficient of \( p^1 \):**
\[
\begin{align*}
\frac{\partial u_{1,0}}{\partial t} + \frac{\partial u_{2,0}}{\partial t} + [(1 + t) u_{1,0} - u_{2,0}] &= 0, \Rightarrow u_{1,1} = - t - \frac{1}{2}, \\
\frac{\partial u_{2,0}}{\partial t} + [(t^2 - 1) u_{1,0} + (1 - t) u_{2,0}] &= 0, \Rightarrow u_{2,1} = t - \frac{1}{4}, \\
\frac{\partial u_{3,0}}{\partial t} + [- t^3 u_{1,0} + t^2 u_{2,0} - u_{3,0}] &= 0, \Rightarrow u_{3,1} = t + \frac{1}{4}.
\end{align*}
\]  
(3.9)

Generally, \( u_{1,n}, u_{2,n}, \) and \( u_{3,n} \) for \( n = 2,3, \ldots \) are obtained from the following recursive relations:
\[
\begin{align*}
\frac{\partial u_{1,n-1}}{\partial t} + [(1 + t) u_{1,n-1} - u_{2,n-1}] &= 0, \\
\frac{\partial u_{2,n-1}}{\partial t} + [(t^2 - 1) u_{1,n-1} + (1 - t) u_{2,n-1}] &= 0, \\
\frac{\partial u_{3,n-1}}{\partial t} + [- t^3 u_{1,n-1} + t^2 u_{2,n-1} - u_{3,n-1}] &= 0.
\end{align*}
\]  
(3.10)

The following Maple program generates \( u_{1,n}, u_{2,n}, \) and \( u_{3,n} \) for \( n = 2,3, \ldots, 10 \). In addition, \( u_{4,n} \) can be obtained from (3.5).

**Program 1.**

```maple
restart;
k:=10;
u[1][0]:=1;
u[2][0]:=0;
u[3][0]:=1;
u[1][1]:=-t-(t^2)/2;
u[2][1]:=-t-(t^3)/3;
u[3][1]:=t*(t^4)/4;
for i from 2 to k do
   u[1][i]:=int((t+1)*u[1][i-1]-u[2][i-1],t);
u[2][i]:=int((t^2-1)*u[1][i-1]-(1-t)*u[2][i-1],t);
u[3][i]:=int(t*t^2*u[1][i-1]-t*t^2*u[2][i-1],t);
   od;
u[approximation][1]:=sort(add(u[1][i],i=0..k));
u[approximation][2]:=sort(add(u[2][i],i=0..k));
u[approximation][3]:=sort(add(u[3][i],i=0..k));
```

We note that this program generates 10-term approximations of \( u_1(t), u_2(t) \) and \( u_3(t) \). Higher accuracies can be obtained by using more components. Figs. 1–3 show the errors of 10-term.
approximation of \( u_1(t) \), \( u_2(t) \) and \( u_3(t) \), respectively. Since the obtained approximate solution is the Maclaurin’s expansion of the exact solution, the exact solution can be guessed.

**Example 2.** Consider the following DAE [8]:

\[
A(t)u'(t) + B(t)u(t) = C(t), \quad 0 \leq t \leq 1, \tag{3.11}
\]

where

\[
A(t) = \begin{pmatrix} 1 & -t & \hat{r} \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & -t - 1 & \hat{r} + 2t \\ 0 & -1 & t - 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and}
\]

\[
C(t) = \begin{pmatrix} 0 \\ 0 \\ \sin t \end{pmatrix}, \tag{3.12}
\]

with initial conditions \( u_1(0) = 1, u_2(0) = 1 \) and \( u_3(0) = 0 \). The exact solution is in the following equation:

\[
u(t) = \begin{pmatrix} e^{-t} + te^t \\ e^t + t \sin t \\ \sin t \end{pmatrix}.
\tag{3.13}

Therefore, we have:

\[
\begin{aligned}
& u_1' + u_1 - (2t + 1)u_2 + (2\hat{r} + t)u_3 = 0, \\
& u_2' - tu_2' - u_2 + (t - 1)u_3 = 0, \\
& u_3 = \sin t.
\end{aligned}
\tag{3.14}
\]

Hence,

\[
\begin{aligned}
& u_1' + u_1 - (2t + 1)u_2 + (2\hat{r} + t)\sin t = 0, \\
& u_2' - u_2 + (t - 1)\sin t - t\cos t = 0.
\end{aligned}
\tag{3.15}
\]

By using (2.9) we obtain:

\[
\begin{aligned}
& L(u_1) - L(u_1) + pL(u_2) + p[u_1 - (2t + 1)u_2 + (2\hat{r} + t)\sin t] = 0, \\
& L(u_2) - L(u_2) + pL(u_3) + p[-u_2 + (t - 1)\sin t - t\cos t] = 0,
\end{aligned}
\tag{3.16}
\]

where \( \nu_{1,0} = 1 \) and \( \nu_{2,0} = 1 \). By equating the coefficients of \( p \) to zero, we obtain:

Coefficient of \( p^0 \):

\[
\begin{aligned}
& u_{1,0} = \nu_{1,0} = 1, \\
& u_{2,0} = \nu_{2,0} = 1.
\end{aligned}
\tag{3.17}
\]

Coefficient of \( p^1 \):

\[
\begin{aligned}
& \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial \hat{r}} + [u_1 - (2t + 1)u_2 + (2\hat{r} + t)\sin t] = 0, \\
& \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial \hat{r}} + [-u_2 + (t - 1)\sin t - t\cos t] = 0.
\end{aligned}
\tag{3.18}
\]

Now, we approximate \( \sin t \) and \( \cos t \) by their 5-term Maclaurin’s expansions. Therefore,

\[
\begin{aligned}
& u_{1,1} = t^2 - \frac{1}{3}t^3 + \frac{1}{9}t^4 + \frac{1}{6}t^5 + \frac{4}{45}t^6 - \frac{3}{120}t^7 + \frac{1}{480}t^8 + \frac{1}{45360}t^9, \\
& u_{1,2} = t^2 - \frac{1}{2}t^3 + \frac{1}{12}t^4 + \frac{1}{45}t^5 + \frac{1}{15}t^6 + \frac{1}{180}t^7 + \frac{1}{540}t^8 + \frac{1}{5400}t^9, \\
& u_{2,1} = t^2 + \frac{1}{4}t^3 + \frac{1}{12}t^4 + \frac{1}{45}t^5 + \frac{1}{15}t^6 + \frac{1}{180}t^7 + \frac{1}{540}t^8 + \frac{1}{5400}t^9.
\end{aligned}
\tag{3.19}
\]

Generally, \( u_{i,n} \) and \( u_{2,n} \) for \( n = 2, 3, \ldots \) are obtained from the following recursive relations:
The following Maple program generates $u_{1,n}$ and $u_{2,n}$ for $n = 0, 1, \ldots, 10$.

Program 2.

```maple
restart;

k := 10;
u[1,0] := 1;
u[2,0] := 1;
a := convert(series(sin(t), t = 0, k), polynomial);
b := convert(series(cos(t), t = 0, k+1), polynomial);
u[1][1] := int(-2*t + (2*t^2 + t)*a, t);
u[2][1] := int(-1 + (t-1)*a - t*b, t);
for i from 2 to k do
    u[1][i] := int(-u[1][i-1] - (2*t + 1)*u[2][i-1], t);
u[2][i] := int(u[2][i-1], t);
od;

u[approximation][1] := add(u[1][i], i = 0..k);
u[approximation][2] := add(u[2][i], i = 0..k);
```

Figs. 4 and 5 show the errors of 10-term approximation of $u_{1}(t)$ and $u_{2}(t)$, respectively. Since the obtained approximate solution is the Maclaurin’s expansion of the exact solution, the exact solution can be guessed.

4. Conclusion

In this paper, HPM is applied to solve the DAEs. This method is tested on two problems. According to the obtained solutions we infer that the HPM is a powerful tool for solving this kind of problems. In addition, this method avoids the round-off errors. These approximate solutions are found in the form of a convergent series. At last, the terms of the series can be easily computed.

References


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