A note on $W^\gamma_p$-theory of linear stochastic parabolic partial differential systems

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Abstract

In this article we construct a $W^\gamma_p$-theory of linear stochastic parabolic partial differential systems. Here, $p \in [2, \infty)$ and $\gamma \in (-\infty, \infty)$. We also provide an example to show that for stochastic systems we need more restriction than the algebraic condition which ensures that diffusion survives against wild convection.

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1. Introduction

In this article we study the Cauchy problem of the following general stochastic parabolic system:

$$\begin{align*}
du_k &= (a_{kr}^{ij} u_r^{x_i x_j} + b_{kr}^i u_r^x + c_{kr} u_r + f^k) dt \\
&\quad + (\sigma_{kr,m}^{il} u_r^{x_l} + \nu_{kr,m} u_r + \zeta^k_m) dw^m_t, \quad t > 0, \ x \in \mathbb{R}^d \\
u_k(0) &= u_0^k,
\end{align*}$$

(1.1)

where $i, j = 1, 2, \ldots, d, \ k, r = 1, 2, \ldots, d_1, m = 1, 2, \ldots$ and the summation convention on the repeated indices $i, j, r, m$ is used. The system (1.1) models the interactions among $d_1$
diffusive quantities under other physical phenomena like convection, internal source or sink, and randomness caused by lack of information. Moreover, the countable sum of the stochastic integrals against independent one-dimensional Brownian motions \( \{ w_t^m : m = 1, 2, \ldots \} \) in (1.1) enables us to include the stochastic integral against a cylindrical Brownian motion (see Section 8.2 of [8]). The coefficients \( a_{ij}^k, b_{kr}^i, c_{kr}, \sigma_{kr,m}^i, \nu_{kr,m} \) and hence the solution \( u = (u^1, u^2, \ldots, u^d) \) are random functions depending on \((t, x)\).

The concrete motivations of studying (1.1) can be easily found in the literature. If \( d_1 = 1 \), (1.1) is a stochastic partial differential equation (SPDE) of parabolic type. Such equations arise in many applications of probability theory (see [8,18]). For instance, the conditional density in nonlinear filtering problems for a partially observable diffusion process obeys a SPDE, and the density of a super-diffusion process also satisfies a SPDE when the dimension of the space domain is 1. If \( d_1 = 3 \), the motion of a random string with a small mass can be modeled by a stochastic parabolic partial differential system (see [2,16]).

General \( L_p \)-theory with \( p \geq 2 \) for stochastic parabolic equations (not systems) has been studied. An \( L_p \)-theory of SPDEs with space domain \( \mathbb{R}^d \) was first introduced by Krylov in [8] (cf. see [6] for \( L_2 \)-theory), and since then the results were extended for SPDEs defined on arbitrary \( C^1 \)-domains \( \mathcal{O} \) in \( \mathbb{R}^d \) by Krylov, his collaborators and many other mathematicians (see, for instance, [9,10,4,3,13] and references therein). On the contrary \( L_p \)-theory of general systems of type (1.1) is not known much in the literature except \( L_p \)-theory of the system with the Laplace operator (see, for instance, [14,15] and the reference therein). By the way, \( L_2 \)-theory of (1.1) exists; see [5]. Our aim in this article is to construct an \( L_p \)-theory of the system (1.1) and convince the readers that, unlike the theory of single equations, we need more restriction than the algebraic condition, which ensures that overall diffusion survives against extremely wild convection.

To point out the differences between equations and systems, we provide the main steps of \( L_p \)-theory of single equations. Let us consider the Cauchy problem

\[
du = (a^{ij} u_{x^i x^j} + f) dt + (\sigma^i_m u_{x^i} + g_m) dw^m_t, \quad t > 0, \quad x \in \mathbb{R}^d.
\]

Denote \( x^i_t := \int_0^t \sigma^i_m(s) dw^m_s \). By applying the Itô–Wentzell formula (see Lemma 4.7 in [8]) to \( u(t, x) := u(t, x - x_t) \), one can transform (1.2) into

\[
dv = (\bar{a}^{ij} v_{x^i x^j} + \bar{f}) dt + \bar{g}_m dw^m_t, \quad t > 0, \quad x \in \mathbb{R}^d,
\]

where \( \bar{a}^{ij} = a^{ij} - \frac{1}{2} (\sigma^i, \sigma^j) t_{ij}, \quad \bar{g}(t, x) = g(t, x - x_t), \quad \bar{f}(t, x) = f(t, x - x_t) - (\sigma^i(t), \bar{g}_x(t, x)) t_{ij} \). Next, let \( \{ T_t : t \geq 0 \} \) be the semi-group corresponding to the Laplacian, i.e. \( T_t' = \Delta T_t \), and define

\[
v^1(t, x) := \int_0^t (T_{t-s} \bar{g}_m(s, \cdot))(x) dw^m_s,
\]

so that it satisfies the stochastic heat equation \( dv^1 = \Delta v^1 dt + \bar{g}_m dw^m_t \). The \( L_p \)-norms of \( v^1 \) are obtained by the generalized Littlewood–Paley inequality [7]; here the restriction \( p \geq 2 \) is necessary. Also, the estimates of \( L_p \)-norms of \( \nu := v - v^1 \) follow from the theory of the deterministic equations since \( \nu \) satisfies the parabolic equation \( \frac{\partial \nu}{\partial t} = \bar{a}^{ij} v_{x^i x^j} + \bar{f} + (\bar{a}^{ij} - \delta^{ij}) v^1_{x^i x^j} \). By summing up these two estimates, one gets \( L_p \) estimates of \( v \) and hence \( u \). It is clear that we now need the following algebraic condition: there exists a constant \( \delta > 0 \) such
that
\[ \delta |\xi|^2 \leq \bar{a}^{ij} \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^d. \]

In the case of system (1.1) the corresponding algebraic condition is the following (see the Appendix for the necessity of this condition):
\[ \delta |\xi|^2 \leq \bar{\xi}_i^* \left( A^{ij} - \tilde{A}^{ij} \right) \xi_j, \quad (1.4) \]

where \( A^{ij} = (\alpha_{kr}^{ij}) \), \( \tilde{A}^{ij} = (\bar{\alpha}_{kr}^{ij}) \), \( \alpha_{kr}^{ij} = \frac{1}{2} \sum_{l=1}^{d_1} (\sigma^{ij}_{lk}, \sigma^{ij}_{lr})_{ij} \), \( \xi \) is any (real) \( d_1 \times d \) matrix, \( \xi_i \) is the \( i \)th column of \( \xi \), \( * \) denotes the matrix transpose, and again the summations on \( i, j \) are understood. However, it turns out that the condition (1.4) alone is not enough for \( L_p \)–theory of systems. One of the main difficulties is that, after the Itô–Wentzell formula is applied, most of the first derivatives of solutions in the stochastic part still remain unlike as in (1.3). Actually, in this article we show by a simple example that \( L_p \)-theory fails unless some extra conditions are imposed. With an extra condition imposed on Theorem 2.6 or Remark 4.5 we construct an \( L_p \)-theory of the system by adopting the strategy from [8] in which the theory of stochastic partial differential equations is constructed (on the base of the result of deterministic partial differential equations); we will need the result of deterministic systems.

The organization of this article is as follows. In Section 2 we state the main results. Section 3 provides an example which shows that the algebraic condition alone is not sufficient for \( L_p \)-theory of our stochastic system. Finally, in Section 4 we prove the main results. We explain the necessity of the algebraic condition (1.4) in the Appendix.

For any \( m \times n \) real-valued matrix \( C = (c_{kr}) \), we define its norm by \( |C| := \sqrt{\sum_{k=1}^{m} \sum_{r=1}^{n} (c_{kr})^2} \).

2. Main results

Let (\( \Omega, \mathcal{F}, P \)) be a complete probability space and \( \{ \mathcal{F}_t : t \geq 0 \} \) be a filtration such that \( \mathcal{F}_0 \) contains all \( P \)-null sets of \( \Omega \). By \( \mathcal{P} \) we denote the predictable \( \sigma \)-algebra on \( \Omega \times (0, \infty) \). Let \( \{ u_t^{(i)} \}_{i=1}^{\infty} \) be independent one-dimensional \( \{ \mathcal{F}_t \} \)-adapted Wiener processes defined on (\( \Omega, \mathcal{F}, P \)) and \( C_0^{\infty}(\mathbb{R}^d; \mathbb{R}^{d_1}) \) denote the set of all \( \mathbb{R}^{d_1} \)-valued infinitely differentiable functions with compact support in \( \mathbb{R}^d \). By \( \mathcal{D} \) we denote the space of \( \mathbb{R}^d \)-valued distributions on \( C_0^{\infty} \); precisely, for \( u \in \mathcal{D} \) and \( \phi \in C_0^{\infty} \) we define \( (u, \phi) \in \mathbb{R}^d \) with components \( (u, \phi) = (u_k, \phi_k) \), \( k = 1, \ldots, d_1 \). Here, each \( u_k \) is a usual \( \mathbb{R} \)-valued distribution defined on \( C^{\infty}(\mathbb{R}^d; \mathbb{R}) \).

We define \( L_p = L_p(\mathbb{R}^d; \mathbb{R}^{d_1}) \) as the space of all \( \mathbb{R}^{d_1} \)-valued functions \( u = (u^1, \ldots, u^{d_1}) \) satisfying
\[ \|u\|_{L_p}^{p} := \sum_{k=1}^{d_1} \|u^k\|_{L_p(\mathbb{R}^d)}^p < \infty. \]

Let \( p \in [2, \infty) \) and \( \gamma \in (-\infty, \infty) \). We define the space of Bessel potential \( H_p^\gamma = H_p^\gamma(\mathbb{R}^d; \mathbb{R}^{d_1}) \) as the space of all distributions \( u \) such that \((1 - \Delta)^{\gamma/2} u \in L_p \), where we define each component of it by
\[ ((1 - \Delta)^{\gamma/2} u)^k = (1 - \Delta)^{\gamma/2} u^k \]
and the operator \((1 - \Delta)^{\gamma/2}\) is defined by
\[
(1 - \Delta)^{\gamma/2} f = \text{the inverse Fourier transform of } (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(f)(\xi)
\]
with \(\mathcal{F}(f)\) the Fourier transform of \(f\). The norm is given by
\[
\|u\|_{H_\gamma^p} := \|(1 - \Delta)^{\gamma/2} u\|_{L^p}.
\]
Then, \(H_\gamma^p\) equipped with the given norm is a Banach space and \(C_0^\infty\) is dense in \(H_\gamma^p\) (see \cite{19}). For non-negative integer \(\gamma = 0, 1, 2, \ldots\) it turns out that
\[
H_\gamma^p = W_\gamma^p := \{u : D^\alpha u \in L_p, \forall \alpha, |\alpha| \leq \gamma\}.
\]
By \(\ell_2\) we denote the set of all real-valued sequences \(e = (e_1, e_2, \ldots)\) with the inner product \((e, f)_{\ell_2} = \sum_{m=1}^\infty e_m f_m\) and the norm \(\|e\|_{\ell_2} = (e, e)^{1/2}_{\ell_2}\). If \(g = (g^1, g^2, \ldots, g^{d_1})\) and each \(g^k\) is an \(\ell_2\)-valued function, then we define
\[
\|g\|_{H_\gamma^p(\ell_2)} := \sum_{k=1}^{d_1} \|(1 - \Delta)^{\gamma/2} g^k\|_{\ell_2} L^p.
\]
For a fixed time \(T < \infty\), we define the stochastic Banach spaces
\[
\begin{align*}
\mathbb{H}_p^\gamma(T) &= \mathbb{H}_p^\gamma(\mathbb{R}, T) := L_p(\Omega \times (0, T], \mathcal{P}, H_\gamma^p), \\
\mathbb{H}_p^\gamma(T, \ell_2) &= L_p(\Omega \times (0, T], \mathcal{P}, H_\gamma^p(\ell_2)), \\
\mathbb{H}_p^\gamma &= \mathbb{H}_p^\gamma(\infty), \\
\mathbb{L}_p(T) &= \mathbb{H}_p^0(T), \\
\mathbb{L}_p(T, \ell_2) &= \mathbb{H}_p^0(T, \ell_2)
\end{align*}
\]
with the norms given by
\[
\|u\|_{\mathbb{H}_p^\gamma(T)} = \mathbb{E} \int_0^T \|u(t)\|_{H_\gamma^p(T)} dt, \\
\|g\|_{\mathbb{H}_p^\gamma(T, \ell_2)} = \mathbb{E} \int_0^T \|g(t)\|_{H_\gamma^p(\ell_2)} dt.
\]
Finally, we set \(U_\gamma^p := L_p(\Omega, \mathcal{F}_0, H_\gamma^{p-2/p})\) for the initial data of the Cauchy problem. The Banach space \(\mathcal{H}_p^{\gamma+2}(T)\) below is modified from \(\mathbb{R}\)-valued version in \cite{8} to the \(\mathbb{R}^{d_1}\)-valued version.

**Definition 2.1.** For a \(\mathcal{D}\)-valued function \(u = (u^1, \ldots, u^{d_1}) \in \mathbb{H}_p^{\gamma+2}(T)\), we write \(u \in \mathcal{H}_p^{\gamma+2}(T)\) if \(u(0, \cdot) \in U_\gamma^{p+2}\), and there exist \(f \in \mathbb{H}_p^{\gamma+1}(T, \ell_2)\) such that, for any \(\phi \in C_0^\infty\), (a.s.) the equality
\[
(u^k(t, \cdot), \phi) = (u^k(0, \cdot), \phi) + \int_0^t (f^k(s, \cdot), \phi) ds + \sum_{m=1}^\infty \int_0^t (g^k_m(s, \cdot), \phi) dw_s^m
\]
holds for each \(k = 1, \ldots, d_1\) and \(t \in (0, T]\). The norm of \(u\) in \(\mathcal{H}_p^{\gamma+2}(T)\) is defined by
\[
\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} = \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|f\|_{\mathbb{H}_p^1(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(T, \ell_2)} + \|u(0, \cdot)\|_{U_\gamma^{p+2}}.
\]
We also define \(\mathcal{H}_p^{\gamma+2}(0) = \mathcal{H}_p^{\gamma+2}(T) \cap \{u; u(0, \cdot) = 0\}\). We write (2.5) in the following simplified ways,
\[
u(t) = u(0) + \int_0^t f(s) ds + \int_0^t g_m(s) dw_s^m \quad \text{or} \quad du = f dt + g_m dw_t^m, \quad t \in (0, T]
\]
and we say that \(du = f dt + g_m dw_t^m\) holds in the sense of distributions.
**Definition 2.2.** We say that \( u \in \mathcal{H}^{γ + 2}_p(T) \) is a solution of the Cauchy problem (1.1) if \( u \) satisfies the problem in the sense of **Definition 2.1** with \( u(0, \cdot) = u_0 \).

The following result is modified from \( \mathbb{R} \)-valued version in [8, Theorem 3.7], to the \( \mathbb{R}^{d_1} \)-valued version. The proof is identical and we omit it.

**Theorem 2.3.** \( \mathcal{H}^{γ + 2}_p(T) \), \( \mathcal{H}^{γ + 2}_{p,0}(T) \) are Banach spaces, and for any \( u \in \mathcal{H}^{γ + 2}_p(T) \)

\[
E \sup_{t \leq T} \| u(t, \cdot) \| \leq N(d_1, d, p, T)\| u \| \quad \text{for any} \quad \mathcal{H}^{γ + 2}_p(T)
\]

holds. In particular, for any \( t \leq T \)

\[
\| u \|_{\mathcal{H}^{γ + 1}_p(t)} \leq N \int_0^t \| u \|_{\mathcal{H}^{γ + 2}_p(s)} \, ds.
\]

Fix \( ε_0 > 0 \). For \( γ \in \mathbb{R} \) denote \( |γ|_+ = |γ| \) if \( |γ| = 0, 1, 2, \ldots \) and \( |γ|_+ = |γ| + ε_0 \) otherwise.

Then we define

\[
B^{γ|γ|}_+ = \begin{cases} 
B(\mathbb{R}^d) : & γ = 0 \\
C^{γ|γ|−1.1}(\mathbb{R}^d) : & |γ| = 1, 2, \ldots \\
C^{γ|γ|+ε_0}(\mathbb{R}^d) : & \text{otherwise,}
\end{cases}
\]

where \( B \) is the space of bounded functions, and \( C^{γ|γ|−1.1} \) and \( C^{γ|γ|+ε_0} \) are the usual Hölder spaces. The Banach space \( B^{γ|γ|}_+ \) is also defined for \( ℓ_2 \)-valued functions. For instance, if \( g = (g_1, g_2, \ldots) \), then \( |g|_{B^0} = \sup_x |g(x)|_{ℓ_2} \) and

\[
|g|_{C^{γ|γ|−1.1}} = \sum_{|α|≤n−1} |D^α g|_{B^0} + \sum_{|α|=n−1} \sup_{x \neq y} \frac{|D^α g(x) − D^α g(y)|_{ℓ_2}}{|x − y|}.
\]

Throughout the article we assume the following for the system (1.1).

**Assumption 2.4.** (i) The coefficients \( a^{ij}_{kr}, b^i_{kr}, c_k, σ_i^{kr}, \) and \( ν_{kr,m} \) are \( \mathcal{P} \otimes B(\mathbb{R}^d) \)-measurable.

(ii) There exists a constant \( δ > 0 \) such that

\[
δ|ξ|^2 ≤ ξ^*(A^{ij} − A^{ij}) ξ_j,
\]

where

\[
A^{ij} = (a^{ij}_{kr}), \quad A^{ij} = (a^{ij}_{kr}), \quad α^{ij}_{kr} = \frac{1}{2} \sum_{l=1}^{d_1} (σ^{l}_{ik}, σ^{l}_{jr})_{ℓ_2},
\]

\( ξ \) is any \( (\text{real}) \) \( d_1 \times d \) matrix, \( ξ_i \) is the \( i \)th column of \( ξ \), \( * \) denotes the matrix transpose, and again the summations on \( i, j \) are understood.

(iii) There exists a finite constant \( K > 0 \) so that

\[
|A^{ij}|, \quad |A^{ij}| \leq K, \quad i, j = 1, 2, \ldots, d.
\]

(iv) The coefficients \( a^{ij}_{kr}, σ^{i}_{kr} \) are uniformly continuous in \( x \), that is, for any \( ε > 0 \) there exists \( δ = δ(ε) > 0 \) so that for any \( ω, t > 0, i, j, k, r, \)

\[
|a^{ij}_{kr}(ω, t, x) − a^{ij}_{kr}(ω, t, y)| + |σ^{i}_{kr}(ω, t, x) − σ^{i}_{kr}(ω, t, y)|_{ℓ_2} < ε, \quad \text{if} \ |x − y| < δ.
\]
(v) For any $\omega, t > 0, i, j, k, r$,

$$
|a'_{ij}(\omega, t, \cdot)|_{|\gamma|_{+}} + |b'_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} + |c_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} + |\sigma^i_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} + |v_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} + |w_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} + |u_{kr}(\omega, t, \cdot)|_{|\gamma|_{+}} < K.
$$

Remark 2.5. In the Appendix we derive the condition (2.7) which is essential to make even $L^2$-theory possible (cf. [5]). It turns out that this assumption is enough for $L^p$-theory of single stochastic equations [8] with $p \in [2, \infty)$ and $L^p$-theory of deterministic linear parabolic systems with $p \in (1, \infty)$ (see, for instance, [12]). However, this is not the case for stochastic systems. We illustrate this in Section 3. Hence, some extra condition is required.

Denote $\Sigma^i = (\sigma^i_{kr})$ with $\sigma^i_{kr} := (\sigma^i_{kr,m} : m = 1, 2, \ldots) \in \ell_2$ and $\Sigma^i_e := \Sigma^i - \Sigma^i_d$ where $\Sigma^i_d$ is the diagonal part of $\Sigma^i$, i.e., $\Sigma^i_d = (\delta^i_{kr}\sigma^i_{kr})$. Here is the main result of the article.

Theorem 2.6. Let $u_0 \in U_p^{r+2}$, $f \in \mathbb{H}_p^r(T)$ and $g \in \mathbb{H}_p^{r+1}(T, \ell_2)$. Then under Assumption 2.4 there exists a constant $\varepsilon > 0$ depending only on $d_1, d, p, \delta, K, T$ such that if

$$
\sup_{\omega, t, x} |\Sigma^i_e(\omega, t, x)|_{\ell_2} \leq \varepsilon, \quad i = 1, 2, \ldots, d, \quad (2.9)
$$

then the Cauchy problem (1.1) has a unique solution $u \in \mathcal{H}_p^{r+2}(T)$, and for this solution we have

$$
\|u\|_{\mathcal{H}_p^{r+2}(T)} \leq N \left( \|f\|_{\mathbb{H}_p^r(T)} + \|g\|_{\mathbb{H}_p^{r+1}(T, \ell_2)} + \|u_0\|_{U_p^{r+2}} \right),
$$

where $N$ depends only on $d, d_1, p, \delta, K, T$.

Remark 2.7. In Remark 4.5 we will show that Theorem 2.6 can be extended to the case when $\Sigma^i$’s are diagonalizable via an orthogonal matrix $\mathcal{O}(\omega, x)$. That is, the statement of the theorem holds if there is $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^d)$-measurable $d_1 \times d_1$ orthogonal matrix $\mathcal{O}(\omega, x)$ such that $\mathcal{O}^* \Sigma^i \mathcal{O}$ is diagonal for each $i$.

The restriction (2.9) or the condition in Remark 2.7 is an extra condition we mentioned in the introduction. At this point we do not know how sharp the condition is. We place the proof of Theorem 2.6 in Section 4. The main ingredients of the proof are the results for the deterministic counterpart of (1.1) (see, for instance, [12]) and a perturbation technique.

### 3. An example: need for restriction on $\Sigma^i$

In this section we demonstrate that one needs stronger restrictions on the stochastic part than the algebraic condition (2.7). See [1] for a work handling similar issues.

Set $d = 1, d_1 = 2$ and $A = I_{2 \times 2}$. We consider the following simple example:

$$
du = u_{xx}dt + \Sigma u_x dw_t, \quad (3.10)
$$

where $\Sigma = \left( \begin{array}{cc} 0 & \sigma \\ \sigma & 0 \end{array} \right)$, $\sigma > 0$ is a constant and $w_t$ is a one-dimensional Wiener process. We assume $1 - \frac{1}{2}\sigma^2 > 0$; it is equivalent to (2.7). Note that the matrix $\Sigma$ is skew-symmetric. We impose the initial condition $u(0, x) = (u^1(0, x), u^2(0, x))$, where

$$
u^1(0, x) = 0, \quad u^2(0, x) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$$

with a fixed $\epsilon > 0$. The system (3.10) turns into a single equation if we set $v := u^2 + i u^1$:

$$dv = v_{xx} dt + i \sigma v_x dw_t$$

(3.11)

with $v(0, x) = u^2(0, x)$. We will find an explicit solution of (3.11); we do this in a heuristic way and verify it using Itô’s formula.

**Remark 3.1.** We will use the fact that for any continuous semi-martingale $A_t$ the solution of $df(t) = f(t)dA_t$ with $f(0) > 0$ is given by

$$f(t) = f(0)e^{\int_0^t \ln(1 + dA_s)\,ds}$$

(see p. 153 of [8]). Here is an informal explanation of it. By the given differential, for any $s$

$$f(s + ds) - f(s) = f(s)dA_s, \quad f(s + ds) = f(s)(1 + dA_s),$$

$$\frac{f(s + ds)}{f(s)} = 1 + dA_s.$$

By taking the natural logarithm on both sides of the last expression, we get

$$d(\ln f(s)) = \ln(1 + dA_s)$$

(3.12)

and the integration of (3.12) from 0 to $t$ yields

$$\ln f(t) - \ln f(0) = \int_0^t (1 + dA_s), \quad f(t) = f(0)e^{\int_0^t (1 + dA_s)}.$$

In particular, if the differential $dA_t$ involves only $dt$ and $dw_t$, then by the recipe

$$dt\,dt = 0, \quad dt\,dw_t = dw_t\,dt = 0, \quad dw_t\,dw_t = dt$$

and the Taylor expansion

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots,$$

we get

$$f(t) = f(0)e^{\int_0^t (dA_t - \frac{1}{2}(dA_t)^2)}.$$

(3.13)

Going back to (3.11), we take spatial Fourier transform on it and get

$$d\tilde{v}(t, \xi) = -\xi^2 \tilde{v}(t, \xi) dt - \sigma \xi \tilde{v}(t, \xi) dw_t = \tilde{v}(t, \xi)(-\xi^2 dt - \sigma \xi dw_t)$$

with $\tilde{v}(0, \xi) = e^{-\frac{\xi^2}{2}}$. By (3.13) we have

$$\tilde{v}(t, \xi) = \tilde{v}(0, \xi)e^{-\frac{(2 + \sigma^2)\xi^2}{2} - \sigma \xi w_t} = e^{-\frac{\xi^2}{2} + (2 + \sigma^2)\xi^2 - \sigma \xi w_t}.$$

Using the fact $\tilde{f}_{ia}(\xi) = \tilde{f}(\xi)e^{-a\xi}$ with $f_{ia}(x) = f(x + ia)$ for a fast-decaying holomorphic function $f$ and the inverse Fourier transform,

$$v(t, x) = \frac{1}{\sqrt{\epsilon + (2 + \sigma^2)t}}e^{-\frac{(x + i\sigma w_t)^2}{2(\epsilon + (2 + \sigma^2)t)}}.$$

(3.14)
Since we used the result of an informal argument, we have to verify that (3.14) is, indeed, a solution of (3.11). As viewing $v(t, x) = \phi(t, w_t)$ with

$$
\phi(t, y) := \frac{1}{\sqrt{\varepsilon + (2 + \sigma^2)t}} e^{-\frac{(x+i\sigma y)^2}{2(\varepsilon + (2 + \sigma^2)t)}},
$$

one easily gets that

$$
\left( \phi_t + \frac{1}{2} \phi_{yy} \right) (t, w_t) = v_{xx} (t, x), \quad \phi_y (t, w_t) = i\sigma v_x (t, x).
$$

Then by Itô’s formula

$$
dv(t, x) = d\phi(t, w_t) = \left( \phi_t + \frac{1}{2} \phi_{yy} \right) (t, w_t) dt + \phi_y (t, w_t) d w_t
$$

with $v(0, x) = u^2(0, x)$. We have just found the solution of (3.11) and hence the solution of (3.10).

Now, we calculate the $L_p$-norm of $u$. Using the distribution of a Wiener process at time $t$,

$$
E|u(t, x)|^p = E|v(t, x)|^p
$$

$$
= \frac{1}{\sqrt{2\pi t}} \frac{1}{\sqrt{\varepsilon + (2 + \sigma^2)t}} e^{-\frac{\sigma^2}{2(\varepsilon + (2 + \sigma^2)t)}} \int_{\mathbb{R}} e^{\frac{1}{2} \left( \frac{\sigma^2}{\varepsilon + (2 + \sigma^2)t} - \frac{1}{2} \right) y^2} dy.
$$

(3.15)

We notice that if

$$
\sigma^2 > \frac{\varepsilon + 2t}{(p-1)t} =: g(t),
$$

(3.16)

then (3.15) is $+\infty$. We fix $p$ large enough so that $\frac{2}{p-1} < \sigma^2$; this is possible since $\sigma^2 < 2$ by our assumption on $\sigma$. As $g(t)$ is eventually decreasing to $\frac{2}{p-1}$ as $t \to \infty$, we can find some interval of time on which (3.16) holds and therefore

$$
2^p \| u \|^p_{L_p} = 2^p (\| u_1 \|^p_{L_p} + \| u_2 \|^p_{L_p})
$$

$$
\geq \| u_1 \|^p + \| u_2 \|^p_{L_p}
$$

$$
\geq \| u \|^p_{L_p} = \| v \|^p_{L_p} = E \int_0^\infty \int_{\mathbb{R}} |v(t, x)|^p dx dt = +\infty.
$$

Hence, one cannot control the $L_p$-norm of the solution of (3.10).

**Remark 3.2.** Our example indicates that the skew-symmetricity of $\Sigma$ caused this ill-posedness of the problem and suggests that we may have to impose the restriction that $\Sigma$’s are symmetric, or diagonal with small off-diagonal parts, in addition to the condition (2.7). By the way, this phenomenon does not occur in the single equation case. Note that, since our result, for instance Corollary 4.4, includes the case that $\Sigma$ is symmetric when $d = 1$, if $\Sigma$ were given by $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$, then we could control the $L_p$-norm of $u$ in the above example.
4. Proof of the main results

Before we consider the general system (1.1), we prove a $W_p^γ$-theory for the Cauchy problem with the coefficients independent of $x$:

$$du^k = (a^{ij}_{kr}(t)u^r_{x_i} + f^k)dt + (σ^{ij}_{kr,m}(t)u^r_{x_i} + g^{k}_{m})dw^m, \quad u^k(0, ·) = u_0^k(·),$$  \hspace{2cm} (4.17)

where $i, j = 1, 2, \ldots, d, \ k, r = 1, 2, \ldots, d_1, \ m = 1, 2, \ldots$; recall that we are using summation notation on $i, j, r$.

We start with a theorem which easily follows from the results for single equations.

**Theorem 4.1.** Assume that $A^{ij}$ is a $d_1 \times d_1$ diagonal matrix and all entries of $Σ^i$ are zero for each $i, j$. Then for any $v_0 \in U_p^{γ+2}, f \in H_p^γ(T)$ and $g \in H_p^{γ+1}(T, ℓ_2)$ the Cauchy problem (4.17) with initial condition $v(0) = u_0$ has a unique solution $v \in H_p^{γ+2}(T)$. For this solution we have

$$\|v\|_{H_p^{γ+2}(T)} \leq N\left(\|f\|_{H_p^γ(T)} + \|g\|_{H_p^{γ+1}(T, ℓ_2)} + \|u_0\|_{U_p^{γ+2}}\right),$$ \hspace{2cm} (4.18)

where $N$ only depends on $d, d_1, p, γ, δ, K, T$.

**Proof.** Under the given assumptions the system in (4.17) is a set of $d_1$ number of independent single equations. Thus the unique solvability and the estimate (4.18) follow from Theorem 5.1 in [8].

In the next theorem we remove the condition that $A^{ij}$s are diagonal.

**Theorem 4.2.** The claim of Theorem 4.1 holds true even if one drops the assumption that $A^{ij}$ is a diagonal matrix for each $i, j$.

**Proof.** First assume $γ = 0$. As a particular case of Theorem 4.1 we have the unique solution $v \in H_p^2(T)$ of

$$dv = (Δv + f)dt + gdw_t, \quad v(0) ≡ u_0$$

with the estimate

$$\|v\|_{H_p^2(T)} \leq N\left(\|f\|_{L_p(T)} + \|g\|_{H_p^1(T, ℓ_2)} + \|u_0\|_{U_p^2}\right).$$ \hspace{2cm} (4.19)

On the other hand, for each fixed $ω$ the system

$$dw_t = (A^{ij}w_{x_i} + (A^{ij} - δ_{ij}I)v_{x_i}dt, \quad w(0) = 0$$ \hspace{2cm} (4.20)

is a deterministic system. Hence, for instance, by Theorem 1.1 in [12] (cf. see Section 10, Chapter 7 in [11] for the results with anisotropic spaces) it follows that the problem (4.20) has a unique solution $W \in H_p^2(T)$ with

$$\|W\|_{H_p^2(T)} \leq N\|(A^{ij} - δ_{ij}I)v_{x_i}\|_{L_p(T)} \leq N\|v_{x_i}\|_{L_p(T)}.$$

\hspace{2cm} (4.21)
Now, \( u = v + w \) is a solution of our Cauchy problem and one gets the estimate (4.18) for \( \gamma = 0 \) by combining (4.19) and (4.21). The uniqueness follows from the theory for deterministic systems.

The case \( \gamma \neq 0 \) easily follows from the fact that \((1 - \Delta)^{\mu/2} : H_p^\gamma \rightarrow H_p^{\gamma - \mu}\) is an isometry for any \( \gamma, \mu \in \mathbb{R} \) when \( p \in (1, \infty) \); indeed, \( u \in H_p^{\gamma + 2}(T) \) is a solution of (4.17) if and only if \( \tilde{u} := (1 - \Delta)^{\gamma/2} u \in H_p^T(T) \) is a solution of (4.17) with \((1 - \Delta)^{\gamma/2} f, (1 - \Delta)^{\gamma/2} g, (1 - \Delta)^{\gamma/2} u_0\) in places of \( f, \ g, \ u_0 \) respectively. Moreover, we have

\[
\|u\|_{H_p^{\gamma + 2}(T)} = \|\tilde{u}\|_{H_p^2(T)} 
\leq N \left( \|(1 - \Delta)^{\gamma/2} f\|_{L_p(T)} + \|(1 - \Delta)^{\gamma/2} g\|_{H_p^1(T, \ell_2)} + \|(1 - \Delta)^{\gamma/2} u_0\|_{U_2^p} \right) 
= N \left( \|f\|_{H_p^2(T)} + \|g\|_{H_p^1(T, \ell_2)} + \|u_0\|_{U_p^{\gamma + 2}} \right).
\]

The theorem is proved. \( \square \)

Now, we weaken the assumption \( \Sigma^i \equiv 0 \) of Theorem 4.2. Recall \( \Sigma^i_e := \Sigma^i - \Sigma^i_d \) where \( \Sigma^i_d \) is the diagonal part of \( \Sigma^i \), i.e., \( \Sigma^i_d = (\delta_{kr}, \sigma^i_{kr}) \).

**Theorem 4.3.** There exists a constant \( \varepsilon > 0 \) depending on \( d_1, d, p, \delta, K, T \) such that if

\[
\kappa_0 := \sup_{\omega \in \Omega, 1 \leq \ell \leq d, t \in [0, T]} |\Sigma^i_e(\omega, t)| \leq \varepsilon, \tag{4.22}
\]

then the Cauchy problem (4.17) admits a unique solution \( u \in H_p^{\gamma + 2}(T) \), and the estimate (4.18) holds.

**Proof.** As in the proof of Theorem 4.2 we may assume \( \gamma = 0 \). Also, as usual we assume \( u_0 = 0 \) (see the proof of Theorem 5.1 in [8]).

**Case 1.** Assume that \( \Sigma^i \) is diagonal, i.e., \( \Sigma^i_e \equiv 0 \) for each \( i \). Then we have \( \Sigma^i(t) = (\sigma^i_{kr}(t)\delta_{kr}) \) and (4.17) becomes

\[
du^k = (a^i_{kr}u^r + f^k) \, dt + (\sigma^i_{kr,m}u^r + \bar{g}^k_m) \, dw^m_t, \\
u^k(0) = 0; \quad k = 1, 2, \ldots, d_1. \tag{4.23}
\]

Define the process \( x^i_{kr} := \sum_{m=1}^{\infty} \int_0^t \sigma^i_{kr,m}(s) \, dw^m_s \) for each \( i, k \) and \( x^i_k = (x^i_{1k}, x^i_{2k}, \ldots, x^i_{dk}) \).

Also, define \( \tilde{u}^k(t, x) = u^k(t, x - x^i_k) \), and \( \tilde{f}(t, x), \tilde{g}(t, x) \) are defined similarly. Now, we apply the Itô–Wentzell formula (see Lemma 4.7 in [8]) to the equation of each \( u^k \), and we find that (4.23) is equivalent to

\[
d\tilde{u}^k = \left( \left( a^i_{kr} - \frac{1}{2} (\sigma^i_{kr})^2 \right) \tilde{u}^r_{x^i_{kr}} + \tilde{f}^k(t, x^i_{kr}, \tilde{g}^k_m) \right) \, dt + \tilde{g}^k_m \, dw^m_t, \\
\tilde{u}^k(0) = 0; \quad k = 1, 2, \ldots, d_1,
\]

or

\[
d\tilde{u} = \left( \left( A^i_{kr} - \frac{1}{2} (\Sigma^i)^* \Sigma^i \right) \tilde{u}^r_{x^i_{kr}} + \tilde{f} - \tilde{\Sigma}^i \tilde{g}^i_{x^i_{kr}} \right) \, dt + \tilde{g}_m \, dw^m_t, \quad \tilde{u}(0) = 0, \tag{4.24}
\]

where \( \tilde{u} := (\tilde{u}^1, \ldots, \tilde{u}^d_1) \). By Theorem 4.2 the problem (4.24) has a unique solution \( \tilde{u} \in H^2_p(T) \) with

\[
\|\tilde{u}\|_{H^2_p(T)} \leq N \left( \|\tilde{f} - \tilde{\Sigma}^i \tilde{g}^i_{x^i_{kr}}\|_{L_p(T)} + \|\tilde{g}\|_{H^1_p(T, \ell_2)} \right) \leq N \left( \|\tilde{f}\|_{L_p(T)} + \|\tilde{g}\|_{H^1_p(T, \ell_2)} \right).
\]
Hence, (4.23) admits a unique solution \( u \), and the estimate (4.18) follows since the \( L_p \)-norms are translation-invariant.

**Case 2.** By Assumption 2.4, for any \( d_1 \times d \) matrix \( \xi \) we have

\[
2K|\xi|_2^2 \geq \xi_i^* \left( A^{ij} - \frac{1}{2}(\Sigma^i)^* \Sigma^j \right) \xi_j \geq \delta|\xi|_2^2
\]

and on the other hand

\[
\xi_i^* \left( A^{ij} - \frac{1}{2}(\Sigma^i)^* \Sigma^j \right) \xi_j = \xi_i^* \left( A^{ij} - \frac{1}{2}(\Sigma^i_d)^* \Sigma^j_d \right) \xi_j
\]

\[
- \frac{1}{2} \xi_i^* \left( (\Sigma^i_d)^* \Sigma^j_d + (\Sigma^i_c)^* \Sigma^j_c \right) \xi_j.
\]

It is easy to find a constant \( \varepsilon_1 > 0 \) so that if \( \kappa_0 \leq \varepsilon_1 \), then

\[
4K|\xi|_2^2 \geq \xi_i^* \left( A^{ij} - \frac{1}{2}(\Sigma^i_d)^* \Sigma^j_d \right) \xi_j \geq \frac{\delta}{2}|\xi|_2^2
\]

holds. Hence, by the result of Case 1, for each \( u \in \mathcal{H}^2_{p,0}(T) \) we can define \( v := Ru \in \mathcal{H}^2_{p,0}(T) \) as the solution of

\[
dv = \left( A^{ij} v_{x^i} + f \right) dt + \left( \Sigma^i_{d,m} v_{x^i} + \Sigma^i_{m} u_{x^i} + g_m \right) dw^m_t, \quad v(0) = 0. \tag{4.25}
\]

If we assume \( \kappa_0 \leq \varepsilon_1 \), the map \( R : \mathcal{H}^2_{p,0}(T) \to \mathcal{H}^2_{p,0}(T) \) is well defined and bounded. We plan to show that \( R^n \) is a contraction for some large integer \( n \) with a further restriction on \( \kappa_0 \). We note that for \( t \leq T \) and any \( u_1, u_2 \in \mathcal{H}^2_{p,0}(T) \) we have

\[
\| Ru_1 - Ru_2 \|_{\mathcal{H}^2_{p}(t)}^p \leq N_0 \| \Sigma^i_{e}(u_1 - u_2)_{x^i} \|_{\mathcal{H}^2_{p}(t)}^p,
\]

where \( N_0 \) depends only on \( d_1, d, p, \delta, K, T \). By the inequality \( \| u_x \|_1,p \leq N(\| u_x \|_p + \| u \|_p) \), for each \( t \) we have

\[
\| \Sigma^i_{e}(u_1 - u_2)_{x^i} \|_{\mathcal{H}^2_{p}(t)}^p \leq N\kappa_0^p (\| (u_1 - u_2)_{xx} \|_{L_p(t)}^p + \| u_1 - u_2 \|_{L_p(t)}^p)
\]

\[
\leq N\kappa_0^p (\| u_1 - u_2 \|_{\mathcal{H}^2_{p}(t)}^p + \| u_1 - u_2 \|_{L_p(t)}^p).
\]

It follows that by (2.6)

\[
\| Ru_1 - Ru_2 \|_{\mathcal{H}^2_{p}(t)}^p \leq N_0 N\kappa_0^p \| u_1 - u_2 \|_{\mathcal{H}^2_{p}(t)}^p + N_0 N\kappa_0^p \times \int_0^t E \| u_1(s) - u_2(s) \|_{L_p}^p ds
\]

\[
\leq N_0 N\kappa_0^p \| u_1 - u_2 \|_{\mathcal{H}^2_{p}(t)}^p + N_0 N\kappa_0^p N_1 \times \int_0^t \| u_1 - u_2 \|_{L_p(t)}^p ds, \tag{4.26}
\]

where \( N_1 \) depends only on \( d_1, d, p, T \). After this, by following the argument in the proof of Theorem 6.4 in [8], we can find \( \varepsilon \in (0, \varepsilon_1) \) and \( n \) so that for any \( \kappa_0 \leq \varepsilon \) the operator \( R^n \) is a contraction, and thus the existence, the uniqueness and the estimate (4.18) follow. \( \square \)

In some situation we can remove the restriction that \( \Sigma^i \)'s are close to diagonal matrices.
Corollary 4.4. Assume \( \bar{\Sigma}^j(\omega, t) = O^*(\omega) \Sigma^j(\omega, t) O(\omega) \) for some \( F_0 \)-measurable orthogonal matrix \( O \), where \( A^{ij}(\omega, t) \), \( \bar{\Sigma}^j(\omega, t) \) satisfy Assumption 2.4 and \( \Sigma^j \)'s satisfy the condition (4.22) in Theorem 4.3. Then the assertion of Theorem 4.3 holds for the system (4.17) with \( \bar{\Sigma}^j \) in place of \( \Sigma^j \), \( i = 1, 2, \ldots, d \).

**Proof.** Consider the problem

\[
dv = ((O A^{ij}(t) O^*) v_{x_i x_j} + O f) dt + (\Sigma^j m(t)) v_{x_i} + O g_m dw^m_t, \quad v(0) = O u_0.
\]

Notice that \( OA^{ij} O^*, \Sigma^j (i, j = 1, 2, \ldots, d) \) satisfy the same conditions that \( A^{ij}, \bar{\Sigma}^j \) (i, j = 1, 2, \ldots, d) satisfy since \( O \) is orthogonal. Moreover, note that \( \varepsilon \) in (4.22) is independent of the choice of \( O \). Hence, by Theorem 4.3 there exists a unique solution \( v \in H_p^r(\mathcal{T}) \) with the estimate

\[
\|v\|_{H_p^{r+2}(\mathcal{T})} \leq N \left( \|O f\|_{H_p^r(\mathcal{T})} + \|O g\|_{H_p^{r+1}(\mathcal{T}, \ell_2)} + \|O u_0\|_{U_p^{r+2}} \right)
\]

\[
= N \left( \|f\|_{H_p^r(\mathcal{T})} + \|g\|_{H_p^{r+1}(\mathcal{T}, \ell_2)} + \|u_0\|_{U_p^{r+2}} \right). \tag{4.27}
\]

Now, we define \( u = O^* v \). It is clear that \( u \) is the unique solution of

\[
du = (A^{ij}(t) u_{x_i x_j} + f) dt + (\bar{\Sigma}^j(t) u_{x_i} + g_m) dw^m_t, \quad u(0) = u_0
\]

and the estimate (4.18) follows from (4.27). The corollary is proved. \( \square \)

**Proof of Theorem 2.6.** Since we have already proved the theorem for the system with coefficients independent of \( x \), it is enough to follow a standard perturbation argument based on the regularity condition of the coefficients (see Assumption 2.4), localization technique and method of continuity. The details of this argument can be found in the proof of Theorem 5.1 of [8], where the theorem is proved for single equations. The only difference is that one needs to use Theorem 4.3 in this article in place of Theorem 4.10 of [8]. The theorem is proved. \( \square \)

**Remark 4.5.** Using Corollary 4.4 instead of Theorem 4.3 and following the arguments in the proof of Theorem 2.6, we can extend Theorem 2.6 to the case when \( \Sigma^j \)'s are diagonalizable via an orthogonal matrix \( O(\omega, x) \), i.e., if there is \( F_0 \times \mathcal{B}(\mathbb{R}^d) \)-measurable \( d_1 \times d_1 \) orthogonal matrix \( O(\omega, x) \) such that \( O^* \Sigma^j O \) is diagonal for each \( i \).

**Remark 4.6.** Recently [17] also considered a class of stochastic systems driven by cylindrical Brownian motion and used small perturbation arguments.

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Appendix. Algebraic condition (2.7)

We show that the condition (2.7) is necessary even if $p = 2, d = 1$ and $A = (a_{kr})$ is a positive symmetric matrix. We demonstrate this in a heuristic manner, using a simple system. Assume $d = 1$ and consider

$$du(t, x) = \left(a_{kr}(t)u_x^r(t, x) + f^k(t, x)\right)dt + \left(\sigma_{kr,m}(t)u_x^r(t, x) + g_m^k(t, x)\right)dw^m_t, \quad (A.28)$$

or,

$$du = A u_{xx} + f) dt + (\Sigma_m u_x + g_m) dw^m_t, \quad u(0) = u_0$$

where $A = (a_{kr})$ and $\Sigma_m = (\sigma_{kr,m})$. For simplicity, assume $f \equiv 0, u(0) \equiv 0$, and $\Sigma$ is bounded, i.e., we have

$$du = Au_{xx}dt + (\Sigma_m u_x + g_m) dw^m_t, \quad u(0) \equiv 0. \quad (A.29)$$

We aim to show

$$\|u_x\|_{L^2_1} \leq N \|g\|_{H_p^2},$$

which implies (remember $\partial(1 - \Delta)^{-1/2}$ is a bounded operator in $H_p^\gamma$ for any $\gamma$)

$$\|u_{xx}\|_{L^2_1} = \|\partial_x u_x\|_{L^2_1} = \|\partial(1 - \Delta)^{-1/2}(1 - \Delta)^{1/2}u_x\|_{L^2_1} \leq N \|(1 - \Delta)^{1/2}u_x\|_{L^2_1} = N\|u_x\|_{H^2_{\gamma,1}} \leq N \|g\|_{H^2_{\gamma,1}}. \quad (A.30)$$

The norm $\|u_{xx}\|_{L^2_1}$ is a part of $\|u\|_{H^2_{\gamma,1}}$. Since $(1 - \Delta)^{1/2}u, (1 - \Delta)^{1/2}g$ instead of $u, g$ also satisfy the system (A.29), we only need to show

$$\|u_x\|_{L^2_1} \leq N \|g\|_{L^2_1}. \quad (A.30)$$

We take spatial Fourier transform on (A.28) with the following form of Fourier transform,

$$\mathcal{F}M(\xi) = \tilde{M}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot y} M(y)dy,$$

for matrix-valued function $M$ defined on $\mathbb{R}^d$, where $\xi \cdot y$ is the real scalar product of $\xi$ and $y$. We have

$$d\tilde{u}^k(t, \xi) = -\xi^2 a_{kr}(t)\tilde{u}^r(t, \xi)dt + \left(i\xi \sigma_{kr,m}(t)\tilde{u}^r(t, \xi) + \tilde{g}_m^k(t, \xi)\right)dw^m_t$$

for each $k$. The complex conjugate of this equation is

$$d\overline{u}^k(t, \xi) = -\xi^2 a_{kr}(t)\overline{u}^r(t, \xi)dt + \left(-i\xi \sigma_{kr,m}(t)\overline{u}^r(t, \xi) + \overline{g}_m^k(t, \xi)\right)dw^m_t.$$

By Itô’s formula, or stochastic product rule, we have

$$d|\tilde{u}^k(t, \xi)|^2 = d\left(\tilde{u}^k(t, \xi)\overline{u}^k(t, \xi)\right)$$
Now, by taking mathematical expectation and recalling the zero initial condition, we get

\[
E|\tilde{u}(t, \xi)|^2 = E \sum_{k=1}^{d_1} |\tilde{u}^k(t, \xi)|^2
\]

\[
= -E \int_0^t \xi^2 \left( a_{kr}(s)\tilde{u}^k(s, \xi)\tilde{u}'(s, \xi) + a_{kr}(s)\tilde{u}'(s, \xi)\tilde{u}^k(s, \xi) \\
- (\sigma_{kl}(s), \sigma_{kr}(s))\tilde{u}_i(s, \xi)\tilde{u}_i(s, \xi) \right) ds
+ E \int_0^t \left( 2\text{Re} \left( i\xi g_m^k(s, \xi)\sigma_{kr,m}(s)\tilde{u}'(s, \xi) \right) + |\tilde{g}(s, \xi)|^2 \right) ds.
\]

Since \(A\) is symmetric positive and \(E|\tilde{u}(t, \xi)|^2\) is nonnegative, we see

\[
E \int_0^t \left( 2|A^{1/2}u_x(s, \xi)|^2 - \sum_m |\tilde{\Sigma}_m u_x(s, \xi)|^2 \right) ds
\]

\[
\leq E \int_0^t \left( 2\text{Re} \left( i\xi g_m^k(s, \xi)\sigma_{kr,m}(s)\tilde{u}'(s, \xi) \right) + |\tilde{g}(s, \xi)|^2 \right) ds.
\]

We integrate both sides with respect to \(\xi\) and use Parseval’s identity, Young’s inequality, the boundedness of \(\Sigma\) to have

\[
E \int_0^t \int_{\mathbb{R}} (u_x)^*(s, x) (2A(s) - A(s)) u_x(s, x) dx ds
\]

\[
\leq E \int_0^t \int_{\mathbb{R}} \left( 2\text{Re} \left( g_m^k(s, \xi)\sigma_{kr,m}(s)\tilde{u}_x(s, \xi) \right) + |\tilde{g}(s, \xi)|^2 \right) d\xi ds
\]

\[
\leq E \int_0^t \int_{\mathbb{R}} \left( \delta|\tilde{u}_x|^2 + N(\delta)|\tilde{g}(s, \xi)|^2 \right) d\xi ds
\]

\[
\leq \delta\|u_x\|^2_{L^2(t)} + N(\delta)\|g\|^2_{L^2(t)}
\]

with \(\delta > 0\) from (2.7). Now, we see that the condition (2.7) is essential to have

\[
2\delta\|u_x\|^2_{L^2(t)} \leq \delta\|u_x\|^2_{L^2(t)} + N(\delta)\|g\|^2_{L^2}, \quad \delta\|u_x\|^2_{L^2} \leq N(\delta)\|g\|^2_{L^2}.
\]
Remark A.1. In the calculation above we have used Parseval's identity which works when $p = 2$. If one wants to avoid such calculation based on Fourier transform, one starts with

\[ d(u^k)^2 = 2u^kdu^k + du^kdu^k \]

and uses integration by parts, carry on similar arguments, then one gets (A.30) again. □

References