Global smoothness preservation and simultaneous approximation for multivariate general singular integral operators

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

Abstract
In this article we continue with the study of multivariate smooth general singular integral operators over \( \mathbb{R}^N \), \( N \geq 1 \), regarding their simultaneous global smoothness preservation property with respect to the \( L_p \) norm, \( 1 \leq p \leq \infty \), by involving multivariate higher order moduli of smoothness. Also we study their multivariate simultaneous approximation to the unit operator with rates. The multivariate Jackson type inequalities obtained are almost sharp containing elegant constants, and they reflect the high order of differentiability of the engaged function. In the uniform case of global smoothness we prove optimality. At the end we list the multivariate Picard, Gauss–Weierstrass, Poisson–Cauchy and Trigonometric singular integral operators as applicators of our general theory.

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1. Introduction

The main motivation for this work comes from [1–3]. We present here the multivariate simultaneous global smoothness preservation property of multivariate general smooth singular integral operators. We study also the simultaneous \( L_p \), \( 1 \leq p \leq \infty \), approximation of these operators to the unit operator with rates. See Theorems 2 and 4, Proposition 5, Theorem 8, Corollary 9 and Theorems 10–15. At the end we list specific operators that fulfill our theory. One can derive many interesting convergence properties based on our results.

2. Main results

Here \( r \in \mathbb{N} \), \( m \in \mathbb{Z}_+ \), we define

\[
\alpha_{j,r}^{[m]}(x) := \begin{cases} 
(-1)^{r-j}(\frac{r}{j})^j j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\
1 - \sum_{j=1}^{r} (-1)^{r-j}(\frac{r}{j})^j j^{-m}, & \text{if } j = 0,
\end{cases}
\]

and

\[
\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}.
\]
See that
\[ \sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \]  
(3)
and
\[ - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^{r} \binom{r}{0}. \]  
(4)

Let \( \mu_{\xi_n} \) be a probability Borel measure on \( \mathbb{R}^N, N \geq 1, \xi_n > 0, n \in \mathbb{N} \).

We now define the multiple smooth singular integral operators
\[ \theta_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) \, d\mu_{\xi_n} (s), \]  
(5)
where \( s := (s_1, \ldots, s_N) \), \( x := (x_1, \ldots, x_N) \in \mathbb{R}^N; n, r \in \mathbb{N}, m \in \mathbb{Z}_+, f : \mathbb{R}^N \to \mathbb{R} \) is a Borel measurable function, and also \( (\xi_n)_{n \in \mathbb{N}} \) is a bounded sequence of positive real numbers.

Above operators \( \theta_{r,n}^{[m]} \) are not in general positive operators and they preserve constants; see [4].

**Definition 1.** Let \( f \in C (\mathbb{R}^N), N \geq 1, m \in \mathbb{N} \), the \( m \)th modulus of smoothness for \( 1 \leq p \leq \infty \), is given by
\[ \omega_m (f; h)_p := \sup_{\| x \| \leq h} \left\| \Delta_{r}^{m} (f) \right\|_{p,x}, \]  
(6)
where
\[ \Delta_{r}^{m} (f) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f (x + jt). \]  
(7)
Denote
\[ \omega_m (f; h)_{\infty} = \omega_m (f, h). \]  
(8)

Above, \( x, t \in \mathbb{R}^N \).

We present the general global smoothness preservation result

**Theorem 2.** We suppose \( \theta_{r,n}^{[m]} (f; x) \in \mathbb{R}, \forall x \in \mathbb{R} \). Let \( h > 0, f \in C (\mathbb{R}^N), N \geq 1 \).

(i) Assume \( \omega_m (f, h) < \infty \). Then
\[ \omega_m \left( \theta_{r,n}^{[m]} f, h \right) \leq \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^{[m]} \right| \right) \omega_m (f, h). \]  
(9)

(ii) Assume \( f \in (C (\mathbb{R}^N) \cap L_1 (\mathbb{R}^N)) \). Then
\[ \omega_m \left( \theta_{r,n}^{[m]} f, h \right)_1 \leq \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^{[m]} \right| \right) \omega_m (f, h)_1. \]  
(10)

(iii) Assume \( f \in (C (\mathbb{R}^N) \cap L_p (\mathbb{R}^N)) \), \( p > 1 \). Then
\[ \omega_m \left( \theta_{r,n}^{[m]} f, h \right)_p \leq \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^{[m]} \right| \right) \omega_m (f, h)_p. \]  
(11)

**Proof.** We recall \( (x \in \mathbb{R}^N, N \geq 1) \)
\[ \theta_{r,n}^{[m]} (f; x) = \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x + sj) \, d\mu_{\xi_n} (s). \]
We see \((t \in \mathbb{R}^N)\)

\[
\Delta_t^m (\theta_{r,n}^m (f; x)) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \theta_{r,n}^m (f; x + jt)
\]

\[
= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}^N} \left( \sum_{j=0}^{r} \alpha_{j,r}^m f (x + jt + \tilde{jk}) \right) d\mu_{\tilde{\xi}_n} (k)
\]

\[
= \sum_{j=0}^{r} \alpha_{j,r}^m \int_{\mathbb{R}^N} \left( \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f (x + jt + \tilde{jk}) \right) d\mu_{\tilde{\xi}_n} (k)
\]

\[
= \sum_{j=0}^{r} \alpha_{j,r}^m \int_{\mathbb{R}^N} (\Delta_t^m f (x + \tilde{jk})) d\mu_{\tilde{\xi}_n} (k).
\]

I.e. we get

\[
\Delta_t^m (\theta_{r,n}^m (f; x)) = \sum_{j=0}^{r} \alpha_{j,r}^m \int_{\mathbb{R}^N} (\Delta_t^m f (x + \tilde{jk})) d\mu_{\tilde{\xi}_n} (k).
\]

Therefore

\[
|\Delta_t^m (\theta_{r,n}^m (f; x))| \leq \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \int_{\mathbb{R}^N} |\Delta_t^m f (x + \tilde{jk})| d\mu_{\tilde{\xi}_n} (k).
\]

(i) From the previous inequality (16) we derive

\[
\omega_m (\theta_{r,n}^m f, h) \leq \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \int_{\mathbb{R}^N} \omega_m (f, h) d\mu_{\tilde{\xi}_n} (k) = \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \right) \omega_m (f, h),
\]

proving claim (9).

(ii) We see that

\[
\int_{\mathbb{R}^N} \left| \Delta_t^m (\theta_{r,n}^m (f; x)) \right| dx \leq \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \left| \Delta_t^m f (x + \tilde{jk}) \right| dx \right) d\mu_{\tilde{\xi}_n} (k)
\]

\[
= \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \int_{\mathbb{R}^N} \left\| \Delta_t^m f \right\|_1 d\mu_{\tilde{\xi}_n} (k) = \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \right) \left\| \Delta_t^m f \right\|_1.
\]

So we got

\[
\left\| \Delta_t^m (\theta_{r,n}^m (f)) \right\|_1 \leq \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \right) \left\| \Delta_t^m f \right\|_1,
\]

which implies

\[
\omega_m (\theta_{r,n}^m f, h) \leq \left( \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \right) \omega_m (f, h),
\]

proving claim (10).

(iii) Let \(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1\). Then

\[
\left\| \Delta_t^m (\theta_{r,n}^m (f; x)) \right\|_{p,x} \leq \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \left\| \int_{\mathbb{R}^N} \left| \Delta_t^m f (x + \tilde{jk}) \right| d\mu_{\tilde{\xi}_n} (k) \right\|_{p,x}
\]

\[
= \sum_{j=0}^{r} \left| \alpha_{j,r}^m \right| \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \left| \Delta_t^m f (x + \tilde{jk}) \right| d\mu_{\tilde{\xi}_n} (k) \right)^p dx \right)^{\frac{1}{p}}.
\]
Let 

\[
W \text{enote that (for}
\]

\[
\text{Proof.}
\]

is sharp, namely it is attained by any 

Remark 3. Now clearly 

That is 

\[
\| \Delta^m_r (\alpha^{|m|}_r, f) \|_p \leq \left( \sum_{r=0}^\infty \| \alpha^{|m|}_{r, r} \| \right) \| \Delta^m_r f \|_p. \tag{23}
\]

Now clearly (24) proves (11). \qed

Remark 3. Let \( r = 1 \), then \( \alpha^{|m|}_{0, 1} = 0, \alpha^{|m|}_{1, 1} = 1 \). Hence 

\[
\theta^{|m|}_{1, n} (f; x) = \int_{\mathbb{R}^N} f (s + x) \, d\mu^{|m|}_n (s) =: \theta^{|m|}_n (f; x). \tag{25}
\]

By Theorem 2 we obtain the following theorem.

Theorem 4. We suppose \( \theta^{|m|}_n (f; x) \in \mathbb{R}, \forall x \in \mathbb{R} \). Let \( h > 0, f \in C (\mathbb{R}^N) \), \( N \geq 1 \).

(i) Assume \( \omega^{|m|}_m (f, h) < \infty \). Then 

\[
\omega^{|m|}_m (\theta^{|m|}_m f, h) \leq \omega^{|m|}_m (f, h). \tag{26}
\]

(ii) Assume \( f \in (C (\mathbb{R}^N) \cap L_1 (\mathbb{R}^N)) \). Then 

\[
\omega^{|m|}_m (\theta^{|m|}_m f, h) \leq \omega^{|m|}_m (f, h)_1. \tag{27}
\]

(iii) Assume \( f \in (C (\mathbb{R}^N) \cap L_p (\mathbb{R}^N)) \), \( p > 1 \). Then 

\[
\omega^{|m|}_m (\theta^{|m|}_m f, h)_p \leq \omega^{|m|}_m (f, h)_p. \tag{28}
\]

Next we get an optimality result.

Proposition 5. Above inequality (26):

\[
\omega^{|m|}_m (\theta^{|m|}_m f, h) \leq \omega^{|m|}_m (f, h)
\]

is sharp, namely it is attained by any 

\[
f^*_j (x) = x^m_j, \quad j = 1, \ldots, N, \quad x = (x_1, \ldots, x_j, \ldots, x_N) \in \mathbb{R}^N. \tag{29}
\]

Proof. We notice that (for \( x, t \in \mathbb{R}^N \))

\[
\Delta^m_r f^*_j (x) = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} (x_j + it_j)^m
\]

\[
= m!t_j^m, \quad j = 1, \ldots, N,
\]

so that 

\[
\omega^{|m|}_m (f^*_j, h) = \sup_{||t|| \leq h} m! \sup_{||t|| \leq h} ||t_j||^m = m!h^m. \tag{31}
\]

Also 

\[
\Delta^m_r (\theta^{|m|}_n (f^*_j; x)) = \int_{\mathbb{R}^N} (\Delta^m_r f^*_j (x + s)) \, d\mu^{|m|}_n (s)
\]

\[
= \int_{\mathbb{R}^N} m!t_j^m d\mu^{|m|}_n (s) = m!t_j^m, \quad j = 1, \ldots, N, \text{ etc.}, \tag{32}
\]

proving the claim. \qed
We need the following theorem.

**Theorem 6.** Let \( f \in C^l(\mathbb{R}^N) \), \( l, N \in \mathbb{N} \). Here \( \mu_{\xi_n} \) is a Borel probability measure on \( \mathbb{R}^N \), \( \xi_n > 0 \), \( (\xi_n)_{n \in \mathbb{N}} \) a bounded sequence. Let \( \beta := (\beta_1, \ldots, \beta_N) \), \( \beta_i \in \mathbb{Z}^+ \), \( i = 1, \ldots, N \); \( |eta| := \sum_{i=1}^N \beta_i = l \). Here \( f(x+sj) \), \( x, s \in \mathbb{R}^N \), is \( \mu_{\xi_n} \)-integrable w.r.t. \( s \), for \( j = 1, \ldots, r \). There exist \( \mu_{\xi_n} \)-integrable functions \( h_{\beta_1,\beta_2,\ldots,\beta_N}(x) \geq 0 \) \( (j = 1, \ldots, r) \) on \( \mathbb{R}^N \) such that

\[
\begin{align*}
\left| \frac{\partial^{\beta_1} f(x+sj)}{\partial x_1^{\beta_1}} \right| & \leq h_{\beta_1,j}(s), \quad i_1 = 1, \ldots, \beta_1, \\
\left| \frac{\partial^{\beta_1+\beta_2} f(x+sj)}{\partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| & \leq h_{\beta_1,j_2,j}(s), \quad i_2 = 1, \ldots, \beta_2, \\
& \vdots \\
\left| \frac{\partial^{\beta_1+\beta_2+\ldots+\beta_N} f(x+sj)}{\partial x_N^{\beta_N-1} \ldots \partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| & \leq h_{\beta_1,\beta_2,\ldots,\beta_N,j}(s), \quad i_N = 1, \ldots, \beta_N.
\end{align*}
\]

\( \forall x, s \in \mathbb{R}^N \),

Then, both of the next exist and

\[ (\theta_{\beta,n}(f; x))_\beta = \theta_n(f; x). \tag{34} \]

**Proof.** By Bauer [5], pp. 103–104. \( \square \)

**Corollary 7** (To Theorem 6, \( r = 1 \)). It holds

\[ (\theta_n(f; x))_\beta = \theta_n(f; x). \tag{35} \]

We give simultaneous global smoothness results.

**Theorem 8.** Let \( h > 0 \) and assumptions of Theorem 6 are valid. Here \( \gamma = 0, \beta, (0 = (0, \ldots, 0)). \)

(i) Assume \( \omega_m(f, h) < \infty \). Then

\[ \omega_m\left((\theta_{\beta,n}(f))_\gamma, h\right) \leq \left( \sum_{j=0}^r |\alpha_{j,r}^{[\beta]}| \right) \omega_m(f, h). \tag{36} \]

(ii) Additionally assume \( f_r \in L_1(\mathbb{R}^N) \). Then

\[ \omega_m\left((\theta_{\beta,n}(f))_\gamma, h\right)_1 \leq \left( \sum_{j=0}^r |\alpha_{j,r}^{[\beta]}| \right) \omega_m(f, h). \tag{37} \]

(iii) Additionally assume \( f_r \in L_p(\mathbb{R}^N) \), \( p > 1 \). Then

\[ \omega_m\left((\theta_{\beta,n}(f))_\gamma, h\right)_p \leq \left( \sum_{j=0}^r |\alpha_{j,r}^{[\beta]}| \right) \omega_m(f, h). \tag{38} \]

Then we have the following corollary.

**Corollary 9.** Let \( h > 0 \) and assumptions of Corollary 7 are valid. Here \( \gamma = 0, \beta. \)

(i) Assume \( \omega_m(f, h) < \infty \). Then

\[ \omega_m\left((\theta_n(f))_\gamma, h\right) \leq \omega_m(f, h). \tag{39} \]

(ii) Additionally assume \( f_r \in L_1(\mathbb{R}^N) \). Then

\[ \omega_m\left((\theta_n(f))_\gamma, h\right)_1 \leq \omega_m(f, h). \tag{40} \]
(iii) Additionally assume \( f_\gamma \in L_p(\mathbb{R}^N) \), \( p > 1 \). Then
\[
\omega_m ((\theta_n(f))_\gamma, h)_p \leq \omega_m (f_\gamma, h)_p.
\]

Next comes multi-simultaneous approximation.

**Theorem 10.** Let \( f \in C^{m+1}(\mathbb{R}^N) \), \( m, l, N \in \mathbb{N} \). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Assume \( \|f_{\gamma + \alpha}\|_\infty < \infty \) and
\[
\int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s) < \infty,
\]
for all \( \alpha_i \in \mathbb{Z}^+, \, j = 1, \ldots, N, |\alpha| := \sum_{j=1}^{N} \alpha_j = m \), where \( \mu_{\xi_n} \) is a Borel probability measure on \( \mathbb{R}^N \), for \( \xi_n > 0 \). Let \( \xi_n \) be a bounded sequence.

For \( j = 1, \ldots, m \), and \( \alpha := (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, N \), \( |\alpha| := \sum_{i=1}^{N} \alpha_i = j \), call
\[
c_{\alpha, n, j} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s).
\]

Then
\[
\left\| \left( \theta_{\alpha, l}^{[\gamma]}(f; \cdot) \right)_\gamma - f_\gamma(\cdot) - \sum_{j=1}^{m} s_j^{\alpha_l} \left( \sum_{\alpha_1, \ldots, \alpha_N \geq \alpha_l}^{N} \prod_{i=1}^{N} \alpha_i! \right) \right\|_\infty
\leq \sum_{(\alpha_1, \ldots, \alpha_N \geq \alpha_l)} \left( \omega_j(f_{\gamma + \alpha}, \xi_n) \right) \left( \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s) \right).
\]

**Proof.** Based on Theorems 6 and 9 of [4].

**Theorem 11.** Let \( f \in C_l^l(\mathbb{R}^N) \), \( l, N \in \mathbb{N} \) (functions \( l \)-times continuously differentiable and bounded). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Assume
\[
\int_{\mathbb{R}^N} \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s) < \infty.
\]

Then
\[
\left\| \left( \theta_{\alpha, l}^{[0]}(f) \right)_\gamma - f_\gamma \right\|_\infty \leq \left( \int_{\mathbb{R}^N} \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s) \right) \omega_l(f_\gamma, \xi_n).
\]

**Proof.** By Theorems 6 and 11 of [4].

We continue with the following theorem.

**Theorem 12.** Let \( f \in C^{m+1}(\mathbb{R}^N) \), \( m, l, N \in \mathbb{N} \). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Let \( f_{\gamma + \alpha} \in L_p(\mathbb{R}^N) \), \( |\alpha| = m, x \in \mathbb{R}^N \), and \( p, q > 1 \) : \( \frac{1}{p} + \frac{1}{q} = 1 \). Here \( \mu_{\xi_n} \) is a Borel probability measure on \( \mathbb{R}^N \), for \( \xi_n > 0 \). Let \( \xi_n \) be a bounded sequence. Assume for all \( \alpha := (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, N \), \( |\alpha| := \sum_{i=1}^{N} \alpha_i = m \) we have that
\[
\int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_l} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s) < \infty.
\]

For \( j = 1, \ldots, m \), and \( \alpha := (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, N \), \( |\alpha| := \sum_{i=1}^{N} \alpha_i = j \), call
\[
c_{\alpha, n, j} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s).
\]
Then
\[
\left\| \left( \gamma^{[m]} (f ; x) \right)_\gamma - f_x (x) - \sum_{j=1}^m \delta_j^{[m]} \left( \sum_{|\alpha|=j} \prod_{i=1}^N \alpha_i! \right) \right\|_{p,x} \leq \left( \frac{m}{(q (m-1) + 1)^{\frac{1}{q}}} \right) \left( \sum_{|\alpha|=m} \frac{1}{n!} \right) \cdot \int_{\mathbb{R}^N} \left( \prod_{i=1}^N |s_i|^{1/m} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^{\frac{C}{p}} d\mu_{\xi_n}(s) \cdot \omega_\gamma \left( f_{\gamma + \alpha} ; \xi_n \right)_p. \tag{48}
\]

**Proof.** By Theorems 6 and 4 of [6]. □

We also give the following theorem.

**Theorem 13.** Let \( f \in C^1 (\mathbb{R}^N) \), \( 1, N \in \mathbb{N} \). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Let \( f_{\gamma} \in L_p (\mathbb{R}^N) \), \( x \in \mathbb{R}^N \); \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \). Assume \( \mu_{\xi_n} \) probability Borel measures on \( \mathbb{R}^N \), \( (\xi_n)_{n \in \mathbb{N}} > 0 \) and bounded. Also suppose

\[
\int_{\mathbb{R}^N} \left( 1 + \frac{||s||_2}{\xi_n} \right)^p d\mu_{\xi_n}(s) < \infty.
\]

Then

\[
\left\| \left( \gamma^{[m]} (f) \right)_\gamma - f_x \right\|_p \leq \left( \int_{\mathbb{R}^N} \left( 1 + \frac{||s||_2}{\xi_n} \right)^p d\mu_{\xi_n}(s) \right)^{\frac{1}{p}} \omega_\gamma \left( f_{\gamma + \alpha} ; \xi_n \right)_p. \tag{49}
\]

**Proof.** By Theorems 6 and 6 of [6]. □

**Theorem 14.** Let \( f \in C^1 (\mathbb{R}^N) \), \( 1, N \in \mathbb{N} \). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Let \( f_{\gamma} \in L_1 (\mathbb{R}^N) \), \( x \in \mathbb{R}^N \). Assume \( \mu_{\xi_n} \) probability Borel measures on \( \mathbb{R}^N \), \( (\xi_n)_{n \in \mathbb{N}} > 0 \) and bounded. Also suppose

\[
\int_{\mathbb{R}^N} \left( 1 + \frac{||s||_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty.
\]

Then

\[
\left\| \left( \gamma^{[m]} (f) \right)_\gamma - f_x \right\|_1 \leq \left( \int_{\mathbb{R}^N} \left( 1 + \frac{||s||_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_\gamma \left( f_{\gamma + \alpha} ; \xi_n \right)_1. \tag{50}
\]

**Proof.** By Theorems 6 and 8 of [6]. □

**Theorem 15.** Let \( f \in C^{m+\iota} (\mathbb{R}^N) \), \( m, l, N \in \mathbb{N} \). The assumptions of Theorem 6 are valid. Call \( \gamma = 0, \beta \). Let \( f_{\gamma + \alpha} \in L_1 (\mathbb{R}^N) \), \( |\alpha| = m, x \in \mathbb{R}^N \). Here \( \mu_{\xi_n} \) is a Borel probability measure on \( \mathbb{R}^N \) for \( \xi_n > 0 \), \( (\xi_n)_{n \in \mathbb{N}} \) is a bounded sequence. Assume for all \( \alpha := (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \in \mathbb{Z}^+ \), \( i = 1, \ldots, N \), \( |\alpha| := \sum_{i=1}^N \alpha_i = m \), we have that

\[
\int_{\mathbb{R}^N} \left( \prod_{i=1}^N |s_i|^{\alpha_i} \right) \left( 1 + \frac{||s||_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{51}
\]

For \( j = 1, \ldots, m \), and \( \alpha := (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \in \mathbb{Z}^+ \), \( i = 1, \ldots, N \), \( |\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j} \), call

\[
c_{\alpha, \beta, j} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{52}
\]

Then

\[
\left\| \left( \gamma^{[m]} (f ; x) \right)_\gamma - f_x (x) - \sum_{j=1}^m \delta_j^{[m]} \left( \sum_{|\alpha|=j} \prod_{i=1}^N \alpha_i! \right) \right\|_{1,x} \]
\[ \sum_{|\omega|=m} \left( \frac{1}{N} \prod_{i=1}^{N} \alpha_i! \right) \omega_i \left( f_{r+\alpha}, \xi_n \right) \int_{\mathbb{R}^N} \left( \prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left( 1 + \frac{\|s\|_2}{\xi_n} \right)^r \, d\mu_{\xi_n}(s). \]  

\quad \text{(53)}

**Proof.** Based on Theorems 6 and 10 of [6]. \( \square \)

### 3. Applications

Let all entities be as in Section 2. We define the following specific operators.

(i) The general multivariate Picard singular integral operators:

\[ P_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \frac{1}{(2\pi)^N} \sum_{j=0}^{r} \alpha_j^{[m]} \cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) e^{\frac{N}{2n} \left( \sum_{i=1}^{N} |s_i| \right)} \, ds_1 \ldots ds_N. \]  

\quad \text{(54)}

(ii) The general multivariate Gauss–Weierstrass singular integral operators:

\[ W_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \frac{1}{(\sqrt{\pi} \xi_n)^N} \sum_{j=0}^{r} \alpha_j^{[m]} \cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) e^{\frac{N}{2n} \left( \sum_{i=1}^{N} s_i^2 \right)} \, ds_1 \ldots ds_N. \]  

\quad \text{(55)}

(iii) The general multivariate Poisson–Cauchy singular integral operators:

\[ U_{r,n}^{[m]}(f; x_1, \ldots, x_N) := W_n^N \sum_{j=0}^{r} \alpha_j^{[m]} \cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j) \prod_{i=1}^{N} \frac{1}{(2\pi)^{\alpha_i} \xi_n^{2\alpha_i}} \, ds_1 \ldots ds_N, \]  

\quad \text{with } \alpha \in \mathbb{N}, \beta > \frac{1}{2\alpha}, \text{ and } W_n := \frac{\Gamma(\beta) \alpha \xi_n^{2\alpha \beta - 1}}{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right)}. \]  

\quad \text{(56)}

(iv) The general multivariate trigonometric singular integral operators:

\[ T_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_j^{[m]} \cdot \int_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j) \prod_{i=1}^{N} \left( \frac{\sin \left( \frac{s_i}{\xi_n} \right)}{s_i} \right)^{2\beta} \, ds_1 \ldots ds_N, \]  

\quad \text{where } \beta \in \mathbb{N}, \text{ and } \lambda_n := 2^{1-2\beta} \pi (-1)^{\beta} \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta - 1}}{(\beta-k)! (\beta+k)!}. \]  

\quad \text{(58)}

\[ \lambda_n := 2k^{1-2\beta} \pi (-1)^{\beta} \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta - 1}}{(\beta-k)! (\beta+k)!}. \]  

\quad \text{(59)}

One can apply the results of this article to the operators \( P_{r,n}^{[m]}, W_{r,n}^{[m]}, U_{r,n}^{[m]}, T_{r,n}^{[m]} \) (special cases of \( \alpha_j^{[m]} \)) and derive interesting results. We intend to do that in a future article.

**Conclusion:** Our approximation results here imply important convergence properties of operators \( \vartheta_{r,n}^{[m]} \) to the unit operator.

### References


