

Contents lists available at [SciVerse ScienceDirect](http://SciVerse.Sciencedirect.com)

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Pivots, determinants, and perfect matchings of graphs

Robert Brijder^{a,b,*}, Tero Harju^c, Hendrik Jan Hoogeboom^d^a Hasselt University, Belgium^b Transnational University of Limburg, Belgium^c Department of Mathematics, University of Turku, FI-20014 Turku, Finland^d Leiden Institute of Advanced Computer Science, Leiden University, The Netherlands

ARTICLE INFO

Keywords:

Principal pivot transform
 Perfect matchings
 Gene assembly in ciliates
 Local complementation
 Circle graph

ABSTRACT

We consider sequences of local and edge complementations on graphs with loops (we allow local complementation only on looped vertices). We recall that these operations together form the matrix operation of principal pivot transform (or pivot) restricted to graphs. This fact is not well known, and as a consequence various special cases of the properties of pivot have been rediscovered multiple times. In this paper we give a gentle overview of various properties of pivot for local and edge complementations on graphs. Moreover, we relate the pivot operation to perfect matchings to obtain a purely graph-theoretical characterization of the effect of sequences of pivot operations. Finally, we show that two of the three operations that make up a formal graph model of the biological process of gene assembly in ciliates together form the matrix operation of Schur complement restricted to graphs.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

The operation of local complementation for a vertex in a graph takes the neighbourhood of that vertex in the graph and replaces that neighbourhood by its graph complement. The related operation of edge local complementation (or edge complementation for short) takes an edge in the graph and toggles edges in a particular way based on how its endpoints are connected to the endpoints of the given edge.

The operations are related in a natural way to circle graphs (also called overlap graphs). Given a finite set of chords of a circle, the circle graph contains a vertex for each chord, and two vertices are connected if the corresponding chords cross. Taking out a piece of the perimeter of the circle delimited by the two endpoints of a chord, and reinserting it in reverse, changes the way the chords intersect, and hence changes the associated circle graph. The effect of this reversal on the circle graph can be obtained by a local complementation on the vertex corresponding to the chord. Similarly, interchanging two pieces of the perimeter of the circle, each starting at the different endpoints of one common chord and ending at the endpoints of another, can be modelled by edge complementation in the circle graph.

Circle graphs naturally occur in theories of genetic rearrangements [13,8], and also the local and edge complementation operations are applied in various settings, such as in the theories of Eulerian tours in 4-regular graphs [15], equivalence of certain codes [6], rank-width of graphs [17], and quantum graph states [24].

In the present paper we are interested in sequences of local and edge complementations in graphs where loops are possible—we allow local complementation only on looped vertices. In this setting local and edge complementation can be interpreted as the matrix operation of principal pivot transform (called pivot for short) [22,21] applied on the adjacency matrix of the graph [11].

* Correspondence to: Department WET-INF, Hasselt University, Agoralaan, Building D, B-3590 Diepenbeek, Belgium.
 E-mail address: robert.brijder@uhasselt.be (R. Brijder).

The definition of edge complementation in the literature usually distinguishes three disjoint neighbourhoods in the graph, and edges are updated according to the neighbourhoods to which the endpoints belong. Describing the effect of a sequence of edge complementation operations in terms of neighbourhood connections is involved—the number of neighbourhoods to consider grows exponentially in the size of the sequence.

It turns out that by considering local and complementation as the pivot operation we can more effectively describe the effect of sequences of pivot operations. As a consequence, we trivially obtain that the result of a sequence of local and edge complementation only depends on the nodes involved (with multiplicity modulo 2). Various special case of this result have been rediscovered [17,12,1,14]. Also, we find that for any applicable sequence of local and edge complementation there exists an equivalent *reduced* sequence where each node appears at most once in the sequence.

Subsequently, we relate pivot to perfect matchings, a perfect matching is a set of edges that forms a partition of the set of vertices, to obtain a graph-theoretical characterization. We find that the existence of an edge between two given unlooped vertices after a sequence of pivots directly depends on the number (modulo two) of perfect matchings in the subgraph induced by the two vertices and the vertices used in the pivot sequence (with ‘multiplicity’ if vertices occur more than once).

This paper is a revised edition of our arXiv contribution [2]. Recently Pflueger [19] has independently observed the importance of pivots for the theory of gene assembly in ciliates.

2. Preliminaries

We use \oplus to denote both the logical exclusive-or (addition over $GF(2)$) as well as the related set operation of symmetric difference.

Let V be a finite set, and let A be a $V \times V$ matrix (i.e., a matrix where the columns and rows are indexed by V). For a set $X \subseteq V$ we use $A[X]$ to denote the principal submatrix induced by X (i.e., the rows and columns are indexed by X). Moreover, we define $A \setminus X = A[V \setminus X]$.

By *graph* we mean an undirected graph G where loops are allowed, but parallel edges are not allowed. More precisely, $G = (V, E)$ where V is a finite set of vertices and $E \subseteq \{\{x, y\} \mid x, y \in V\}$ (we have $\{x\} \in E$ iff x is a looped vertex). We use $V(G)$ and $E(G)$ to denote its set of vertices V and set of edges E of G , respectively. In fact, if the graph G is clear from the context of considerations, then we also denote $V(G)$ simply by V .

For $X \subseteq V$, we denote the subgraph of G induced by X as $G[X]$. Let $N_G(v) = \{w \in V \mid \{v, w\} \in E, v \neq w\}$ denote the neighbourhood of vertex v in graph G .

With a graph G one associates its adjacency matrix $A(G)$, which is a $V \times V$ matrix $(a_{u,v})$ over $GF(2)$ with $a_{u,v} = 1$ iff $\{u, v\} \in E$. Obviously, for $X \subseteq V$, $A(G[X]) = A(G)[X]$.

We often make no distinction between G and $A(G)$ and so we write, e.g., $\det G$ to denote $\det A(G)$, the determinant of $A(G)$ computed over $GF(2)$. In this way, graphs correspond precisely to symmetric $V \times V$ -matrices over $GF(2)$. By convention, the determinant of the empty matrix (or graph) is 1.

3. Pivot operation

We recall in this section the graph operations of local and edge complementation. As observed in [11], these graph operations arise as a special case of a matrix transformation operation called principal pivot transform.

3.1. Matrices

Let A be a $V \times V$ -matrix (over an arbitrary field), and let $X \subseteq V$ be such that $A[X]$ is nonsingular, i.e., $\det A[X] \neq 0$. The *principal pivot transform* (or *pivot* for short) of A on X , denoted by $A * X$, is defined as follows, see [22]. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with $P = A[X]$. Then

$$A * X = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}.$$

Matrix $(A * X) \setminus X = S - RP^{-1}Q$ is called the *Schur complement* of X in A . Hence, $A * X$ is defined iff $A[X]$ is nonsingular. We will say that $*X$ is *applicable* to A iff $A * X$ is defined.

The pivot is sometimes considered a partial inverse, as A and $A * X$ are related by the following characteristic equality, where the vectors x_1 and x_2 correspond to the elements of X . In fact, this formula defines $A * X$ given A and X [21].

$$A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \text{iff} \quad A * X \begin{pmatrix} x_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}. \tag{1}$$

Note that if $\det A \neq 0$, then $A * V = A^{-1}$. By Eq. (1) we see that a pivot operation is an involution (i.e., operation of order 2), and more generally, if $(A * X) * Y$ is defined, then $A * (X \oplus Y)$ is defined and they are equal.

In order to apply the pivot $*X$ to matrix A it is required that $A[X]$ is nonsingular. The following fundamental result on pivots is due to Tucker [22] (see also [5, Theorem 4.1.1] and [18]). This result allows one to formulate the applicability of the pivot $*Y$ to the resulting matrix $A * X$ in terms of applicability of the pivot $*(X \oplus Y)$ to the original matrix A .

Proposition 1 ([22]). *Let A be a $V \times V$ -matrix, and let $X \subseteq V$ be such that $A[X]$ is nonsingular. Then, for all $Y \subseteq V$, $\det(A * X)[Y] = \det A[X \oplus Y] / \det A[X]$.*

We remark here that Proposition 1 for the case $Y = V \setminus X$ is called the Schur determinant formula, $\det((A * X) \setminus X) = \det A / \det A[X]$, and was shown already in 1917 by Issai Schur, see [20].

It is easy to verify from the definition of pivot that $A * X$ is skew-symmetric whenever A is (a matrix A is skew-symmetric if $A = -A^T$, where A^T denotes the transpose of A).

3.2. Graphs

From now on we restrict our attention in this paper to (undirected, and loops allowed) graphs, i.e., to symmetric matrices over $GF(2)$. Note that if G is a graph, then, since the notions of skew-symmetric and symmetric coincide over $GF(2)$, $G * X$ is also a graph.

We may now trivially specialize Proposition 1 to graphs.

Theorem 2. *Let G be a graph and let $X \subseteq V(G)$ such that $G[X]$ is nonsingular. Then, for all $Y \subseteq V$, $(G * X)[Y]$ is nonsingular iff $G[X \oplus Y]$ is nonsingular.*

It is noted in Little [16] that singularity of G over $GF(2)$ has a combinatorial graph interpretation: $\det(G) = 0$ (i.e., G is singular) iff there exists a non-empty set $S \subseteq V(G)$ such that every $v \in V(G)$ is adjacent to an even number of vertices in S (a vertex is adjacent to itself iff it has a loop). Indeed, S represents a linear dependent set of columns modulo 2.

For a graph G and nonempty $X \subseteq V$, X is called an *elementary* in G if $G[X]$ is nonsingular and for all nonempty $Y \subset X$, $G[Y]$ is singular. Hence, if X is elementary in G , then $G * X$ is defined but $G * Y$ is not defined for any nonempty proper subset of X . It is observed in [11] that if X is elementary in G , either (1) $X = \{u\} \in E(G)$ (i.e., X is a loop) or (2) $X = \{u, v\} \in E(G)$ and $\{u\}, \{v\} \notin E(G)$. A pivot of Case (1) is called *local complementation* and a pivot of Case (2) is called *edge complementation*.

Local complementation. If vertex u has a loop in G , then the matrix $G[\{u\}]$ is equal to the 1×1 identity matrix: $u \begin{pmatrix} 1 \end{pmatrix}$. Hence, $*\{u\}$ is indeed applicable to G and

$$G * \{u\} = \begin{matrix} u & V \setminus \{u\} \\ V \setminus \{u\} & \end{matrix} \begin{pmatrix} 1 & \chi_u^T \\ \chi_u & G[V - u] - \chi_u \chi_u^T \end{pmatrix},$$

where χ_u is the column vector belonging to u without element at position (u, u) . From this description it follows that the local complement $G * \{u\}$ is obtained from G by complementing the neighbourhood $N_G(u)$ of u : $\{x, y\} \in E(G * \{u\})$ iff $(\{x, y\} \in E) \oplus (\{x, u\} \in E \wedge \{y, u\} \in E)$ (we allow $x = y$). Note that, as a consequence of the case $x = y$, for $v \in N_G(u)$, v is a looped vertex of G iff v is not a looped vertex of $G * \{u\}$.

Note that local complementation in this paper is only defined for looped vertices. We remark that the results in this paper do *not* carry over straightforwardly to local complementation for unlooped vertices; in order to interpret the results for local complementation on unlooped vertices one needs to consider loop complementation as well [4].

Edge complementation. If $\{u, v\}$ is an edge in G and u and v are not looped vertices, then the matrix $G[\{u, v\}]$ is equal to

$$\begin{matrix} u & v \\ u & v \end{matrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \text{ Hence, } *\{u, v\} \text{ is indeed applicable to } G \text{ and}$$

$$G * \{u, v\} = \begin{matrix} u & v & V \setminus \{u, v\} \\ u & v & \\ V \setminus \{u, v\} & & \end{matrix} \begin{pmatrix} 0 & 1 & \chi_v^T \\ 1 & 0 & \chi_u^T \\ \chi_v & \chi_u & G[V - u - v] - (\chi_v \chi_u^T + \chi_u \chi_v^T) \end{pmatrix}$$

where χ_u is the column vector of G belonging to u without elements at positions (u, u) and (v, u) (and similarly for χ_v).

Example 3. The (adjacency matrix of) graph G and its pivot $G * \{2, 3\}$ are respectively

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 0 & 1 \\ 6 & 1 & 0 & 1 & 1 & 1 & 0 \end{matrix} \quad \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 1 & 0 \\ 5 & 1 & 0 & 1 & 1 & 0 & 0 \\ 6 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$$

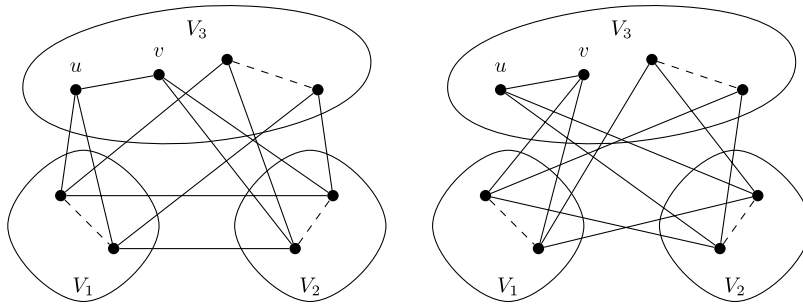


Fig. 1. Pivot on an edge $\{u, v\}$ in a graph. Adjacency between vertices x and y is toggled iff $x \in V_i$ and $y \in V_j$ with $i \neq j$. Note that u and v are adjacent to all vertices in V_3 – these edges are omitted in the diagram. The operation does not affect edges adjacent to vertices outside the sets V_1, V_2, V_3 , nor does it change any of the loops.

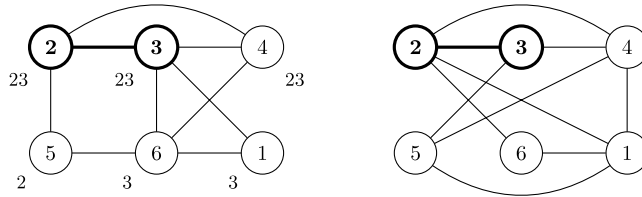


Fig. 2. A graph G and its pivot $G * \{2, 3\}$, cf. Example 5.

The (adjacency matrix of) graph $G * \{2, 3\}$ is obtained from G by swapping the neighbours of 2 and 3, given by $\chi_2^T = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$, and $\chi_3^T = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}$, and by adding the product $\chi_2 \chi_3^T$, and its transpose $\chi_3 \chi_2^T$ to the submatrix of the remaining vertices $G[V - 2 - 3]$, given by respectively

$$\begin{matrix} & \begin{matrix} 1 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \quad \begin{matrix} & \begin{matrix} 1 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{matrix}, \quad \text{and} \quad \begin{matrix} & \begin{matrix} 1 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}.$$

Note that the element at position (x, y) in $(\chi_v \chi_u^T + \chi_u \chi_v^T)$ is 1 iff $((\{x, v\} \in E \wedge \{y, u\} \in E) \oplus (\{x, u\} \in E \wedge \{y, v\} \in E))$. We may now straightforwardly extend this characterization to the full graph $G * \{u, v\}$ by defining $x \sim_G y$ if either $\{x, y\} \in E$ or $x = y$. Thus we obtain an easy extension of the characterization of Oum [17] from simple graphs to graphs where loops are allowed.

Lemma 4 ([17]). *Let G be a graph, and let $\{u, v\} \in E(G)$ where u and v are unlooped vertices. We have $\{x, y\} \in E(G * \{u, v\})$ iff*

$$(\{x, y\} \in E(G)) \oplus ((x \sim_G u) \wedge (y \sim_G v)) \oplus ((x \sim_G v) \wedge (y \sim_G u))$$

for all $x, y \in V(G)$ (we allow $x = y$).

For a vertex x consider its closed neighbourhood $N'_G(x) = N_G(x) \cup \{x\} = \{y \in V_G \mid x \sim_G y\}$. The edge $\{u, v\}$ partitions the vertices of G connected to u or v into three sets $V_1 = N'_G(u) \setminus N'_G(v)$, $V_2 = N'_G(v) \setminus N'_G(u)$, $V_3 = N'_G(u) \cap N'_G(v)$. Note that $u, v \in V_3$. Let $\{u, v\} \in E(G)$. Then $G * \{u, v\}$ is obtained from G by ‘toggling’ all edges between different V_i and V_j : for $\{x, y\}$ with $x \in V_i$ and $y \in V_j$ ($i \neq j$): $\{x, y\} \in E(G)$ iff $\{x, y\} \notin E(G * \{u, v\})$, see Fig. 1. The remaining edges remain unchanged.¹

Example 5. For graph G depicted on the left-hand side of Fig. 2 we have the closed neighbourhoods $N'_G(2) = \{2, 3, 4, 5\}$, and $N'_G(3) = \{1, 2, 3, 4, 6\}$. The neighbourhoods are indicated by ‘labels’ next to the vertices. The graph $G * \{2, 3\}$ (right-hand side of Fig. 2) is obtained from G by toggling the edges between vertices that differ in at least one of these neighbourhoods, i.e., that do not have the same non-empty ‘label’.

Notice that edge complementation is (only) applicable to an edge without loops, and moreover does not introduce or remove loops. Hence edge complementation is a natural graph operation for simple graphs (i.e., graphs without loops) – indeed, most often edge complementation is only considered for simple graphs in the literature.

¹ The description of the definition of edge complementation in the literature usually adds the rule that the vertices u and v are swapped. Here this is avoided by including u and v in the set V_3 .

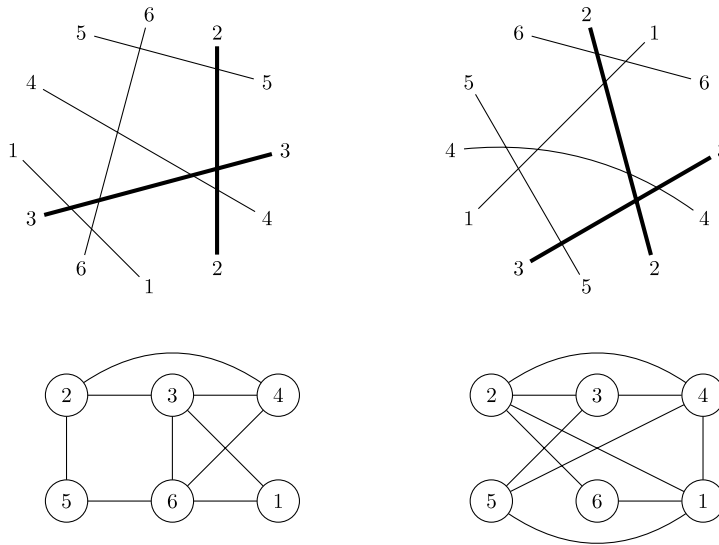


Fig. 3. A circle graph G and its pivot $G * \{2, 3\}$, cf. Example 6, along with circular string representations of the graphs.

3.3. Circle graphs

The operations of local and edge complementation have a natural interpretation for circle graphs (also called overlap graphs), see, e.g., Kotzig [15]. We illustrate this interpretation by an example.

Example 6. We start with six segments, of which the relative positions of the endpoints can be represented by the string **3 5 2 6 5 4 1 3 6 1 2 4**.

The ‘entanglement’ of these intervals can be represented by the circle graph given on the left-hand side of Fig. 3. When we apply edge complementation on $\{2, 3\}$ we obtain the graph on the right-hand side. The resulting graph is the circle graph of **3 6 1 2 6 5 4 1 3 5 2 4**.

4. Sequences of pivots

In this section we study sequences of local and edge complementations that are applied consecutively to a graph. The results in this section follow trivially from the linear algebraic characterization of these graph operations in Section 3. However this perspective is not well known, which is apparent from the fact that various special cases were obtained independently and using non-linear-algebraic approaches. The general result in this section seems hard to obtain if one uses, e.g., purely graph-theoretical arguments.

We assume left associativity of the pivot operation, i.e., $G * X * Y$ denotes $(G * X) * Y$. Let $\varphi = * X_1 * X_2 \cdots * X_n$ be a sequence of pivot operations. The *support* of φ , denoted by $\text{sup}(\varphi)$, is defined as $\bigoplus_i X_i$, i.e., the set of vertices that occur an odd number of times in φ .

Theorem 7. *If φ and φ' are applicable sequences of local and edge complementations for G , then $\text{sup}(\varphi) = \text{sup}(\varphi')$ implies $G\varphi = G\varphi'$.*

Proof. We have recalled that for any matrix A , if $(A * X) * Y$ is defined, then $A * (X \oplus Y)$ is defined and they are equal. Hence we apply this to sequences of operations and obtain $G\varphi = G * \text{sup}(\varphi) = G * \text{sup}(\varphi') = G\varphi'$. \square

As a direct corollary to Theorem 7, if φ is a sequence of local and edge complementations applicable to G , then $\det G[S] = 1$ with $S = \text{sup}(\varphi)$.

Note that by Theorem 7, when calculating the orbit of graphs under the pivot operation, as done in [6], we need not consider every sequence—only those that have different support.

It is easily verified that, $\{x\} \in E(G)$ iff $(\det G[\{x\}]) = 1$, and $\{x, y\} \in E(G)$ iff $(\det G[\{x, y\}]) = 1 \oplus ((\det G[\{x\}]) = 1) \wedge ((\det G[\{y\}]) = 1)$ (for all $x, y \in V(G)$ with $x \neq y$). We thus obtain the following characterization of the resulting graph $G\varphi$.

Theorem 8. *Let φ be an applicable sequence of local and edge complementations for a graph G , and let $S = \text{sup}(\varphi)$. Let $x, y \in V(G)$ with $x \neq y$. Then*

- $\{x\} \in E(G\varphi)$ iff $\det(G[S \oplus \{x\}]) = 1$, and
- $\{x, y\} \in E(G\varphi)$ iff $(\det(G[S \oplus \{x, y\}]) = 1) \oplus ((\det(G[S \oplus \{x\}]) = 1) \wedge ((\det(G[S \oplus \{y\}]) = 1))$.

Note that if G is a simple graph, then [Theorem 8](#) reduces to an easy expression: for all $x, y \in V(G), x \neq y, \{x, y\} \in E(G\varphi)$ iff $\det(G[S \oplus \{x, y\}]) = 1$.

As mentioned above, various special cases of [Theorem 7](#) were obtained independently. In particular, we mention two examples known from the literature: the *triangle equality* (involving three vertices) and commutativity (involving four vertices). Arratia et al. give a proof [[1](#), Lemma 10] of the triangle equality involving certain graphs with 11 vertices. Independently Genest obtains this result in his thesis [[12](#), Proposition 1.3.5]. The cited work of Oum [[17](#), Proposition 2.5] also contains a proof which uses [Lemma 4](#).

Corollary 9. *If u, v, w are three distinct unlooped vertices in graph G such that $\{u, v\}$ and $\{u, w\}$ are edges. Then $G * \{u, v\} * \{v, w\} = G * \{u, w\}$.*

Proof. As v and w are unlooped, by [Theorem 2](#), $\{v, w\}$ is an edge in $G * \{u, v\}$ iff $G * \{u, v\}[\{v, w\}]$ is nonsingular iff $G[\{v, w\} \oplus \{u, v\}] = G[\{w, u\}]$ is nonsingular. The latter holds as $\{u, w\}$ is an edge in G where u and w are unlooped. Hence both sides of the equality are well defined, and thus the result follows from [Theorem 7](#). \square

Another result that fits in this framework is the commutativity of edge complementation on disjoint sets of nodes. It was obtained by Harju et al. [[14](#)] (with its erratum [[3](#)]) studying graph operations modelled after gene rearrangements in organisms called ciliates. The property states that two disjoint edge complementations, when applicable in either order, have a result independent of the order in which they are applied.

The next result is also proved in [[23](#), Corollary 7] using linear fractional transformations. It states that ‘twins’ stay ‘twins’ after pivot. Here we obtain it as a consequence of [Theorem 7](#).

Corollary 10. *Let u, v be unlooped vertices in a graph G such that their closed neighbourhoods are equal, i.e., $N'_C(u) = N'_C(v)$. Then, for each applicable sequence φ of edge complementations, $N'_{G\varphi}(u) = N'_{G\varphi}(v)$.*

Proof. First we observe that, over $GF(2)$, $\det(G[X]) = \det(G[X \oplus \{u, v\}])$ for arbitrary $X \subseteq V(G)$. If X contains exactly one of u and v , then $G[X]$ and $G[X \oplus \{u, v\}]$ are isomorphic. Otherwise, assume X does not contain u, v . $G[X]$ is singular iff there is a nonempty set $S \subseteq V(G)$ such that every vertex in X has an even number of neighbours in S (see the remark below [Theorem 2](#) on a remark by Little). Any such $S \subseteq X$ can be extended to a similar set $S' \subseteq X \oplus \{u, v\}$ for $G[X \oplus \{u, v\}]$ by adding $\{u, v\}$ to S in case u and v have an odd number of neighbours in S . Conversely any ‘even neighbour set’ $S' \subseteq X \oplus \{u, v\}$ for $G[X \oplus \{u, v\}]$ can be restricted to one for $G[X]$ after observing that u, v are either both in S' or both outside S' .

Let $S = \text{sup}(\varphi)$ and $x \in V(G)$. We have $x \in N'_{G\varphi}(u)$ iff $\det(G\varphi[\{u\} \oplus \{x\}]) = 1$ iff (by [Theorem 2](#)) $\det(G[S \oplus \{u\} \oplus \{x\}]) = 1$ iff (by the observation above) $\det(G[S \oplus \{v\} \oplus \{x\}]) = 1$ iff $\det(G\varphi[\{v\} \oplus \{x\}]) = 1$ iff $x \in N'_{G\varphi}(v)$. \square

Note that the result does not hold for sequences of pivot in general, for instance for local complementation with respect to one of the twins u and v .

5. Reduced sequences

We have seen that if we have an applicable sequence of pivots, then the result of that series of operations only depends on the support, the set of vertices occurring an odd number of times as a pivot-vertex. We call a sequence of pivots *reduced* [[12](#)] if no vertex occurs more than once in the pivots.

It turns out that for any applicable sequence of pivots with support S there exists a reduced applicable sequence of pivots with support S . Moreover, this reduced sequence can be obtained using a greedy strategy.

Let G be a graph, and let $S \subseteq V(G)$ be the support of an applicable sequence φ of pivots for G . We will construct a reduced sequence of pivots with support S . Obviously we may assume that S is non-empty. Observe that since $G[S]$ is nonsingular, $*S$ is applicable to G . By [[11](#)] (recall Section 3.2), there is an elementary pivot $*X_1$ applicable to G with $X_1 \subseteq S$. Now, by [Theorem 2](#), $G[S]$ is nonsingular iff $G * X_1[S \setminus X_1]$ is nonsingular. By iteration, we obtain a reduced sequence $\varphi' = *X_1 \cdots *X_n$ of pivots with support equal to the support of φ .

Theorem 11. *Let S be a set of vertices of graph G . Then $\det G[S] = 1$ iff there exists a (reduced) sequence of local and edge complementation φ with support S that is applicable in G .*

The possibility to construct an applicable reduced sequence with given support depends on the fact that there must be at least one edge (which may be a loop) to obtain a non-zero determinant. In fact, every column in the matrix must contain at least one non-zero element. This means we may arbitrarily choose one vertex $u \in X$ of the elementary pivot $*X$.

As an example, we return to the topic of commutativity. It is known that if $*\{u, v\} * \{w, z\}$ is applicable, then we cannot conclude that $*\{w, z\} * \{u, v\}$ is applicable. However, $\det G[u, v, w, z] = 1$, so we can construct an applicable sequence with support $\{u, v, w, z\}$. Fixing z we know that there is an edge adjacent to that vertex, which can be either $\{w, z\}$, $\{v, z\}$ or $\{u, z\}$. When pivoting over this edge, the remaining two vertices must form an edge in the graph.

Hence, we have shown the following fact: if, for different vertices, $*\{u, v\} * \{w, z\}$ is applicable, then either at least one of the pivot sequences $*\{w, z\} * \{u, v\}$, $*\{v, z\} * \{u, w\}$, or $*\{u, z\} * \{v, w\}$ is applicable. This is essentially Lemma 1.2.11 of [[12](#)].

6. Pivots and perfect matchings

To make the transition from the algebraic notion of determinant to equivalent combinatorial terminology we consider the concept of perfect matching for graphs, adapted to the case where loops are allowed. A (generalized) *perfect matching* of graph G is a partition P of $V(G)$ such that $P \subseteq E(G)$. Thus P consists of (disjoint) vertices and edges of G covering $V(G)$.

Like the determinant, the permanent of a square matrix is computed by considering all permutations and choosing matrix elements in positions according this permutation. For the permanent the sign of the permutation is ignored. As we here consider $GF(2)$, permanent and determinant are equal. As each permutation of nonzero elements in the adjacency matrix corresponds to a vertex cycle cover of the graph, the permanent equals the number of these vertex cycle covers. This observation is used by Little [16] to show there is a direct correspondence between (the parity of) the determinant of a graph and (the parity of) the number of perfect matchings in that graph.² The permanent counts the vertex cycle covers, including the perfect matchings, which occur as cycle covers that have only cycles of length one and two (representing disjoint vertices and edges). Little observes that the other cycle covers must occur an even number of times, thus showing the equality between the determinant, permanent, and number of perfect matchings of G over $GF(2)$.

This equality between the determinant and the number of perfect matchings of a graph allows us to reformulate results in terms of graph properties, rather than properties of the associated adjacency matrix. We also give an application, illustrating that the link to perfect matchings gains some useful intuition. Let $\text{pm}(G)$ be the number of perfect matchings of G , modulo 2. Thus, for a graph G we know that $\det(G) = \text{pm}(G)$.

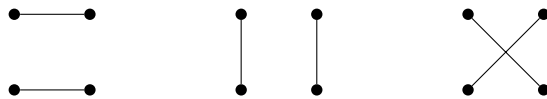
We may now rephrase [Theorem 11](#) by replacing \det by pm .

Theorem 12. *Let S be a set of vertices of a graph G . Then $\text{pm}(G[S]) = 1$ iff there exists a (reduced) sequence of local and edge complementation φ with support S that is applicable in G .*

Similarly, we rephrase [Theorem 8](#). For convenience, we restrict to the case of simple graphs.

Theorem 13. *Let φ be an applicable sequence of local and edge complementations for a simple graph G , and let $S = \text{sup}(\varphi)$. Then, for all $x, y \in V(G)$, $\{x, y\} \in E(G\varphi)$ iff $\text{pm}(G[S \oplus \{x, y\}]) = 1$.*

For small graphs the number of perfect matchings might be easier to determine by hand than the determinant. For instance, for a simple graph G on four nodes there are only three pairs of edges that can be present to contribute to the value $\text{pm}(G)$.



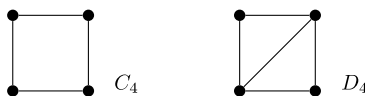
Hence, by [Theorem 8](#), for vertices x and y with $x \neq y$, $\{x, y\} \in E(G * \{u, v\})$ iff $\det(G[\{u, v\} \oplus \{x, y\}]) = 1$ iff $\text{pm}(G[\{u, v\} \oplus \{x, y\}]) = 1$ iff

$$((x \sim_G y) \wedge (u \sim_G v)) \oplus ((x \sim_G u) \wedge (y \sim_G v)) \oplus ((x \sim_G v) \wedge (y \sim_G u))$$

holds. Note that, since $u \sim_G v$ holds and $x \sim_G y$ iff $\{x, y\} \in E(G)$, this is precisely [Lemma 4](#)!

A commutativity result is obtained in [14, Theorem 6.1(iii)]. Assume $\{u, v\}$ and $\{z, w\}$ are edges in G on four different vertices u, v, w, z . Then both $*\{u, v\} * \{w, z\}$ and $*\{w, z\} * \{u, v\}$ are applicable iff the induced subgraph $G[\{u, v, w, z\}]$ has no loops, and is not isomorphic to C_4 or D_4 .

Its proof in [14] is not difficult; a simple case analysis suffices. Here we consider perfect matchings to show the result. Both $*\{u, v\} * \{w, z\}$ and $*\{w, z\} * \{u, v\}$ are applicable iff $\text{pm}(G[u, v, w, z]) = 1$ and u, v, w, z are unlooped. Thus the subgraph $G[u, v, w, z]$ must contain either one or three perfect matchings, where the first $\{\{u, v\}, \{w, z\}\}$ is given. Two perfect matchings occur precisely when the subgraph is isomorphic to C_4 or D_4 .



As is noted just below [Theorem 2](#), we may equivalently look for a non-empty set S such that every $v \in V(G)$ is adjacent to an even number of vertices of S . E.g., for D_4 we can take S to be the set of the two vertices that are not connected by an edge.

Also the observation $\det(G[X]) = \det(G[X \oplus \{u, v\}])$ in the proof of [Corollary 10](#) has an easy argument in terms of perfect matchings. If $u, v \notin X$, then the perfect matchings in $G[X \oplus \{u, v\}]$ are those in $G[X]$ plus the edge $\{u, v\}$, and an even number of perfect matchings without that edge.

² Originally the argumentation is for simple graphs, but it can be used unchanged when loops are allowed in graphs and perfect matchings.

7. Application to gene assembly in ciliates

Local and edge complementation for graph (where loops are allowed) turns out to be very much related to the formal graph model, called GPRS, of intramolecular gene assembly from Ehrenfeucht et al. [7,9,10,8]. The GPRS model consists of three operations called the *graph negative rule*, the *graph positive rule*, and the *graph double rule* defined on *signed graphs*. A signed graph is a simple graph where each vertex is labelled by either $-$ or $+$.

Recall that for a $V \times V$ matrix A and $X \subseteq V$ with $A[X]$ nonsingular, $A * X \setminus X$ is called the Schur complement of X in A . Let G be a graph. If $*\{u\}$ is a local complementation operation applicable to G , then we call the corresponding Schur complementation $*\{u\} \setminus \{u\}$ a *vertex reduction* on $\{u\}$. Similarly, if $*\{u, v\}$ is an edge complementation operation applicable to G , then we call $*\{u, v\} \setminus \{u, v\}$ a *edge reduction* on $\{u, v\}$.

Let us identify a signed graph SG with a graph obtained from G by “encoding” the vertex labelling by loops: label $-$ in SG corresponds to a unlooped vertex in G and label $+$ in SG corresponds to a looped vertex in G . With this identification in place, we observe that vertex reduction on $\{u\}$ is precisely the graph positive rule and edge reduction on $\{u, v\}$ is precisely the graph double rule.

Hence, we obtain the result that any two sequences of graph positive and double rules with equal support obtain the same graph. Finally, we obtain that a signed graph can be transformed into the empty graph by graph positive and double rules iff the determinant of corresponding adjacency matrix is 1 (modulo 2).

Acknowledgements

We thank the anonymous referees for their constructive comments.

References

- [1] R. Arratia, B. Bollobás, G.B. Sorkin, The interlace polynomial of a graph, *Journal of Combinatorial Theory, Series B* 92 (2) (2004) 199–233.
- [2] R. Brijder, T. Harju, H.J. Hoogeboom, Pivots, determinants, and perfect matchings of graphs, 2008 (unpublished). [arXiv:0811.3500](https://arxiv.org/abs/0811.3500).
- [3] R. Brijder, T. Harju, I. Petre, Commutativity of the gdr -operation, 2008 (unpublished).
- [4] R. Brijder, H.J. Hoogeboom, The group structure of pivot and loop complementation on graphs and set systems, *European Journal of Combinatorics* 32 (2011) 1353–1367.
- [5] R.W. Cottle, J.-S. Pang, R.E. Stone, *The Linear Complementarity Problem*, Academic Press, San Diego, 1992.
- [6] L.E. Danielsen, M.G. Parker, Edge local complementation and equivalence of binary linear codes, *Designs, Codes and Cryptography* 49 (2008) 161–170.
- [7] A. Ehrenfeucht, T. Harju, I. Petre, D.M. Prescott, G. Rozenberg, Formal systems for gene assembly in ciliates, *Theoretical Computer Science* 292 (2003) 199–219.
- [8] A. Ehrenfeucht, T. Harju, I. Petre, D.M. Prescott, G. Rozenberg, *Computation in Living Cells – Gene Assembly in Ciliates*, Springer Verlag, 2004.
- [9] A. Ehrenfeucht, T. Harju, I. Petre, G. Rozenberg, Characterizing the micronuclear gene patterns in ciliates, *Theory of Computing Systems* 35 (2002) 501–519.
- [10] A. Ehrenfeucht, I. Petre, D.M. Prescott, G. Rozenberg, String and graph reduction systems for gene assembly in ciliates, *Mathematical Structures in Computer Science* 12 (2002) 113–134.
- [11] J.F. Geelen, A generalization of Tutte’s characterization of totally unimodular matrices, *Journal of Combinatorial Theory, Series B* 70 (1997) 101–117.
- [12] F. Genest, *Graphes eulériens et complémentarité locale*, Ph.D. Thesis, Université de Montréal, 2002. Available online: [arXiv:math/0701421v1](https://arxiv.org/abs/math/0701421v1).
- [13] S. Hannenhalli, P.A. Pevzner, Transforming cabbage into turnip: polynomial algorithm for sorting signed permutations by reversals, *Journal of the ACM* 46 (1) (1999) 1–27.
- [14] T. Harju, C. Li, I. Petre, G. Rozenberg, Parallelism in gene assembly, *Natural Computing* 5 (2) (2006) 203–223.
- [15] A. Kotzig, Eulerian lines in finite 4-valent graphs and their transformations, in: *Theory of graphs, Proceedings of the Colloquium, Tihany, Hungary, 1966*, Academic Press, New York, 1968, pp. 219–230.
- [16] C.H.C. Little, The parity of the number of 1-factors of a graph, *Discrete Mathematics* 2 (1972) 179–181.
- [17] S. Oum, Rank-width and vertex-minors, *Journal of Combinatorial Theory, Series B* 95 (1) (2005) 79–100.
- [18] T.D. Parsons, Applications of principal pivoting, in: H.W. Kuhn (Ed.), *Proceedings of the Princeton Symposium on Mathematical Programming*, Princeton University Press, 1970, pp. 567–581.
- [19] N. Pflueger, Graph reductions, binary rank, and pivots in gene assembly, *Discrete Applied Mathematics* 159 (17) (2011) 2117–2134.
- [20] J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, *Journal für die reine und angewandte Mathematik* 147 (1917) 205–232.
- [21] M.J. Tsatsomeros, Principal pivot transforms: properties and applications, *Linear Algebra and its Applications* 307 (1–3) (2000) 151–165.
- [22] A.W. Tucker, A combinatorial equivalence of matrices, in: *Combinatorial Analysis*, in: *Proceedings of Symposia in Applied Mathematics*, vol. X, American Mathematical Society, 1960, pp. 129–140.
- [23] M. van den Nest, B. de Moor, Edge-local equivalence of graphs, 2005. [arXiv:math/0510246](https://arxiv.org/abs/math/0510246).
- [24] M. Van den Nest, J. Dehaene, B. De Moor, Graphical description of the action of local clifford transformations on graph states, *Physical Review A* 69 (2) (2004) 022316.