Variations and Generalizations of Bohr's Inequality

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Submitted by J. L. Brenner

Received June 8, 1991

In this paper we provide an account of various results that have been obtained concerning Bohr's inequality with emphasis on several generalizations. © 1993 Academic Press, Inc.

If z_1 and z_2 are complex numbers and if c is a positive number, then

$$|z_1+z_2|^2 \leqslant (1+c)\,|z_1|^2 + (1+1/c)\,|z_2|^2, \tag{1}$$

with equality iff $z_2 = cz_1$.

This inequality is due to H. Bohr [1, p. 78].

In the book by J. W. Archbold [2] the following generalization of (1) is given: If $a_1, ..., a_n$ are positive numbers such that $\sum_{k=1}^{n} 1/a_k = 1$, then

$$|z_1 + \cdots + z_n|^2 \leqslant a_1\,|z_1|^2 + \cdots + a_n\,|z_n|^2. \tag{2}$$

A. Makowski [3] proved the following inequalities which are in connection to Bohr's inequality (1) in the case when z_1 and z_2 are real numbers: If a, b, α are real numbers and $c > 0$, then

$$(a-b)^2 \sin\alpha + (a+b)^2 \cos\alpha \leqslant (1 + c\,|\cos 2\alpha|)\,a^2 + \left(1 + \frac{|\cos 2\alpha|}{c}\right) b^2$$

$$\leqslant (1+c)\,a^2 + (1+1/c)b^2.$$

0022-247X/93 $5.00

In connection to the previous results is the following result of H. Bergström [4]:

Let z_1 and z_2 be complex numbers and let u and v be real numbers such that $u \neq 0$, $v \neq 0$, $u+v \neq 0$. Then

$$\frac{|z_1+z_2|^2}{u+v} \leqslant \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \qquad \text{for} \quad \frac{1}{u}+\frac{1}{v} > 0,$$

and (3)

$$\frac{|z_1+z_2|^2}{u+v} \geqslant \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \qquad \text{for} \quad \frac{1}{u}+\frac{1}{v} < 0,$$

with equalities iff $vz_1 = uz_2$.

Proof. Inequalities (3) are simple consequences of the identity

$$\frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} - \frac{|z_1+z_2|^2}{u+v} = \frac{|vz_1 - uz_2|^2}{uv(u+v)}.$$

Note that the previous results are given in the well-known book of D. S. Mitrinović [5, pp. 312–313, 315]. Moreover, on pp. 338–339 of the same book we can find the following result: If a and b are real or complex numbers and $r \geqslant 0$, then

$$|a+b|^r \leqslant C_r(|a|^r + |b|^r), \tag{4}$$

where $C_r = 1$ for $r \leqslant 1$, and $C_r = 2^{r-1}$ for $r > 1$ (see also [6, 7]).

A further generalization of (2) is given by P. M. Vasić and J. D. Kečkić [8]: Let $z_1, ..., z_n$ be complex numbers, and $p_1, ..., p_n$ be positive numbers. Then, for $r > 1$, we have

$$\left|\sum_{i=1}^{n} z_i\right|^r \leqslant \left(\sum_{i=1}^{n} p_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^{n} p_i |z_i|^r, \tag{5}$$

with equality iff

$$p_1 |z_1|^{r-1} = \cdots = p_n |z_n|^{r-1} \qquad \text{and} \qquad z_k \bar{z}_j \geqslant 0 \ (k, j = 1, ..., n).$$

A new proof of this result is given by P. S. Bullen [9].

Th. M. Rassias [10, 11] has generalized Bohr's inequality (1) in the following form: If $a > 0$ and $z_1, ..., z_{n+1}$ are complex numbers, then

$$(1+na)|z_1|^2 + \left(1+(n-1)a+\frac{1}{a}\right)|z_2|^2 + \left(1+(n-2)a+\frac{2}{a}\right)|z_3|^2 + \cdots$$
$$+ \left(1+a+\frac{n-1}{a}\right)|z_n|^2 + \left(1+\frac{n}{a}\right)|z_{n+1}|^2 \geqslant |z_1+\cdots+z_{n+1}|^2. \tag{6}$$

Rassias [10, 11] also gave several inequalities similar to Bohr's, and he also proved (2).

The following generalization of (5) is given in [12].

Let E be a nonempty set and let L be a linear class of real functions $g: E \to R$ such that the following properties are valid:

(L1) $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in R$;

(L2) $\mathbb{1} \in L$, i.e., if $f(t) \equiv 1$ $(t \in E)$, then $f \in L$.

Let us consider linear functionals $A: L \to R$, i.e., functionals which satisfy the following conditions:

(A1) $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in R$;

(A2) $f \in L$, $f(t) \geqslant 0$ on $E \Rightarrow A(f) \geqslant 0$.

Further, let us consider a class of functions

$$\bar{L} = \{f: E \to C \mid \operatorname{Re} f \in L, \operatorname{Im} f \in L\}$$

and a function $\bar{A}: \bar{L} \to C$ defined by

$$\bar{A}(f) = A(\operatorname{Re} f) + iA(\operatorname{Im} f) = \operatorname{Re}(\bar{A}(f)) + i \operatorname{Im}(\bar{A}(f)).$$

If $f: E \to C$ and $p: E \to [0, \infty)$ are such functions that for $r > 1$ we have $p^{1/(1-r)}$, $|f|$, $p|f|^r \in L$, and $f \in \bar{L}$, then

$$|\bar{A}(f)|^r \leqslant A(p^{1/(1-r)})^{r-1} A(p\,|f|^r).$$

The following generalization of (4) is given in [13]:

THEOREM 1. *Let $(X, \|\cdot\|)$ be a linear normed vector space and let r be an arbitrary nonnegative real number. Then for every n-tuple $x = (x_1, ..., x_n)$, where $x_i \in X$, $i = 1, ..., n$, we have*

$$\|x_1 + \cdots + x_n\|^r \leqslant C_{r,n}(\|x_1\|^r + \cdots + \|x_n\|^r),$$

where $C_{r,n} = n^{r-1}$ $(r \geqslant 1)$ and $C_{r,n} = 1$ $(0 \leqslant r < 1)$ is the best possible constant.

In [14] it was shown that Theorem 1 is a simple consequence of the triangle inequality and of Jensen's and Petrović's inequalities for convex functions. Similarly, we have [16]:

THEOREM 2. (a) *Let $f: R_+ \to R_+$ be a nondecreasing convex function. Then for every $x_i \in X$, $p_i \geqslant 0$ $(i = 1, ..., n)$ such that $P_n = \sum_{i=1}^n p_i > 0$, we have*

$$f\left(\frac{1}{P_n}\left\|\sum_{i=1}^n p_i x_i\right\|\right) \leqslant \frac{1}{P_n} \sum_{i=1}^n p_i f(\|x_i\|). \tag{8}$$

(b) *If f is a nondecreasing concave function such that $f(0)=0$ and $p_i \geqslant 1$ $(i=1, ..., n)$, then*

$$f\left(\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) \leqslant \sum_{i=1}^{n} p_i f(\|x_i\|). \tag{9}$$

COROLLARY 2a. *If $r>1$ and $q_i>0$ $(i=1, ..., n)$, then* [15, 16]

$$\left\|\sum_{i=1}^{n} x_i\right\|^r \leqslant \left(\sum_{i=1}^{n} q_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^{n} q_i \|x_i\|^r. \tag{10}$$

Proof. By substitutions,

$$f(t)=t^r, \qquad x_i \to x_i/p_i, \qquad p_i \to q_i^{1/(1-r)},$$

we get (10) from (8).

Remark. This is a generalization of (5).

COROLLARY 2b. *If $0 \leqslant r<1$ and $q_i \geqslant 1$ $(i=1, ..., n)$, then* [16]

$$\left\|\sum_{i=1}^{n} x_i\right\|^r \leqslant \sum_{i=1}^{n} q_i \|x_i\|^r. \tag{11}$$

This is a similar consequence of (9).

In a special case if c is a positive number and $x_1, x_2 \in X$, we have

$$\|x_1+x_2\|^2 \leqslant (1+c)\|x_1\|^2+(1+1/c)\|x_2\|^2. \tag{12}$$

By substitutions $q_i \to 1/p_i$ $(i=1, ..., n)$, and since

$$\left(\sum_{i=1}^{n} p_i^{1/(r-1)}\right)^{r-1} \leqslant \sum_{i=1}^{n} p_i, \qquad 1<r \leqslant 2,$$

we get from (10),

$$\left\|\sum_{i=1}^{n} x_i\right\|^r \Big/ \left(\sum_{i=1}^{n} p_i\right) \leqslant \sum_{i=1}^{n} \|x_i\|^r/p_i \qquad (1 \leqslant r \leqslant 2), \tag{13}$$

where $p_i>0$ $(i=1, ..., n)$. (The case $r=1$ is obvious.)

THEOREM 3. *Let $f\colon R_+ \to R_+$ be a nondecreasing convex function, $p_1>0$, $p_i \leqslant 0$ $(2 \leqslant i \leqslant n)$, and $P_n>0$. Then*

$$f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) \geqslant \frac{1}{P_n} \sum_{i=1}^{n} p_i f(\|x_i\|). \tag{14}$$

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Proof. This is a consequence of Theorem 2 if we use substitutions

$$p_1 \to P_n, \qquad p_i \to -p_i, \qquad i = 2, ..., n,$$

$$x_1 \to \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \qquad x_i \to x_i, \qquad i = 2, ..., n.$$

Then (8) becomes

$$f\left(\frac{\|P_n \cdot (1/P_n) \sum_{i=1}^{n} p_i x_i - p_2 x_2 - \cdots - p_n x_n\|}{P_n - p_2 - \cdots - p_n}\right)$$

$$\leqslant \frac{P_n f(\|(1/P_n) \sum_{i=1}^{n} p_i x_i\|) - p_2 f(\|x_2\|) - \cdots - p_n f(\|x_n\|)}{P_n - p_2 - \cdots - p_n}$$

which is equivalent to (14).

If we set in (14)

$$f(t) = t^r \ (1 < r \leqslant 2), \qquad x_i \to x_i/p_i, \qquad p_i \, |p_i|^{-r} \to q_i$$

we get

$$\left\|\sum_{i=1}^{n} x_i\right\|^r \geqslant \left(\sum_{i=1}^{n} q_i \, |q_i|^{r/(1-r)}\right)^{r-1} \sum_{i=1}^{n} q_i \, \|x_i\|^r,$$

where

$$0 < q_1 \leqslant \left(\sum_{i=2}^{n} |q_i|^{1/(1-r)}\right)^{1-r} \qquad \text{and} \qquad q_i \leqslant 0 \qquad (2 \leqslant i \leqslant n).$$

If we set $q_i \to 1/p_i$ $(1 \leqslant i \leqslant n)$ and use the following inequality from [17]

$$\left(p_1^{1/(r-1)} - \sum_{i=2}^{n} |p_i|^{1/(r-1)}\right)^{r-1} \geqslant p_1 - \sum_{i=2}^{n} |p_i| = \sum_{i=1}^{n} p_i$$

we get the case $1 < r \leqslant 2$ of

$$\left\|\sum_{i=1}^{n} x_i\right\|^r \Big/ \left(\sum_{i=1}^{n} p_i\right) \geqslant \sum_{i=1}^{n} \|x_i\|^r / p_i \qquad (1 \leqslant r \leqslant 2), \tag{15}$$

where $p_1 > 0$, $p_i < 0$, $i = 2, ..., n$, $P_n > 0$. (The case $r = 1$ we get if we set $r \to 1$.)

Remark. Using substitutions, $p_i \to -p_i$ $(i = 1, ..., n)$ we can obtain further results from (13) and (15) in the case when $P_n < 0$.

From (13) and (15) for $n=2$, and from results noted in the previous remark, we get [16]

$$\frac{\|x_1+x_2\|^r}{u+v} \leqslant \frac{\|x_1\|^r}{u} + \frac{\|x_2\|^r}{v} \qquad \text{if} \quad uv(u+v)>0,$$

and (16)

$$\frac{\|x_1+x_2\|^r}{u+v} \geqslant \frac{\|x_1\|^r}{u} + \frac{\|x_2\|^r}{v} \qquad \text{if} \quad uv(u+v)<0,$$

where $x_1, x_2 \in X$, $1 \leqslant r \leqslant 2$.

This is a generalization of (3).

By using Jensen's inequality for convex functions we can obtain another generalization of (5) (see [15]):

THEOREM 4. *Let f be a strictly convex function on I ($=[0, +\infty)$) and let*

$$f(uv) \leqslant f(u)f(v) \ (u, v \in I), \qquad \lim_{t \to 0+} \frac{f(t)}{t} = 0, \qquad \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty.$$

If $x_i \in X$ (X is a normed vector space) and p_i are positive numbers for $i = 1, ..., n$, then

$$f\left(\left\|\sum_{i=1}^{n} x_i\right\|\right) \leqslant g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(\|x_i\|), \tag{17}$$

where $g(t) = f(t)/t$.

Remark. From the hypotheses of the theorem it follows directly that $f(0)=0$ and that the function g, defined by $g(t)=f(t)/t$, is increasing for $t>0$. It means that there exists the function g^{-1}, inverse to g. Therefore, since $\lim_{t\to 0+} g(t)=0$ and $\lim_{t \to +\infty} g(t) = +\infty$, we conclude that equality $g(x)=y$ has a unique solution with respect to x for every $y>0$. (See [15].)

If $f\colon R_+ \to R_+$ is nondecreasing, convex, and submultiplicative, $g(t) = f(t)/t$ is strictly increasing, and $p_i > 0$, then (17) holds. The proof is as follows.

Let $q_i > 0$ and $Q = \sum q_i$ where $\sum$ denotes $\sum_{i=1}^{n}$; then

$$f\left(\left\|\sum q_i x_i\right\|\right) \leqslant f(Q)\, f\left(Q^{-1}\left\|\sum q_i x_i\right\|\right) \leqslant Qg(Q)\, Q^{-1} \sum q_i f(\|x_i\|)$$

by Theorem 2. Replacing x_i by $q_i^{-1}x_i$,

$$f\left(\left\|\sum x_i\right\|\right) \leqslant g(Q) \sum q_i f(q_i^{-1}\|x_i\|) \leqslant g(Q) \sum q_i f(q_i^{-1}) f(\|x_i\|)$$

$$= g(Q) \sum g(q_i^{-1}) f(\|x_i\|).$$

Let $q_i = 1/g^{-1}(p_i)$; then $p_i = g(q_i^{-1})$, and this inequality becomes (17).

Now, we prove the following generalization of (6).

THEOREM 5. *Let x_i $(i = 1, ..., n)$ be elements of an unitary vector space X, and a_{ij} $(1 \leqslant i < j \leqslant n)$ be positive numbers. Then*

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 \leqslant \sum_{k=1}^{n} \|x_k\|^2 \left(1 + \sum_{j=k+1}^{n} a_{kj} + \sum_{i=1}^{k-1} 1/a_{ik}\right). \tag{18}$$

Proof. D. D. Adamović [18] proved the following identity for $x_i \in X$ $(i = 1, ..., n)$:

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 - \sum_{k=1}^{n} \|x_k\|^2 = \sum_{1 \leqslant i < j \leqslant n} (\|x_i + x_j\|^2 - (\|x_i\| + \|x_j\|)^2)$$

which is equivalent to

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 - \sum_{i=1}^{n} \|x_i\|^2 = \sum_{1 \leqslant i < j \leqslant n} (\|x_i + x_j\|^2 - \|x_i\|^2 - \|x_j\|^2).$$

Applying (12) to $\|x_i + x_j\|^2$ we obtain

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 - \sum_{k=1}^{n} \|x_k\|^2$$
$$\leqslant \sum_{1 \leqslant i < j \leqslant n} \left((1 + a_{ij}) \|x_i\|^2 + \left(1 + \frac{1}{a_{ij}}\right) \|x_j\|^2 - \|x_i\|^2 - \|x_j\|^2\right),$$

i.e.,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 - \sum_{k=1}^{n} \|x_k\|^2 \leqslant \sum_{1 \leqslant i < j \leqslant n} \left(a_{ij} \|x_i\|^2 + \frac{1}{a_{ij}} \|x_j\|^2\right)$$

which is equivalent to (18).

Similarly one can use the well-known complementary triangle inequality and its generalizations. Such results for complex numbers are given in

[19, 12], but these results can be improved by using results from [20] instead of a result from [21].

Here we use a generalization of a complementary triangle inequality given in [22]:

THEOREM 6. *Let a be a unit vector in the Hilbert space H. Suppose that the vectors* $x_1, ..., x_n$, *in the case when* $x_i \neq 0$, *satisfy the condition*

$$0 \leqslant r \leqslant \operatorname{Re}(x_i, a)/\|x_i\|, \qquad i = 1, ..., n.$$

Then

$$r(\|x_1\| + \cdots + \|x_n\|) \leqslant \|x_1 + \cdots + x_n\|,$$

with equality if and only if

$$x_1 + \cdots + x_n = r(\|x_1\| + \cdots + \|x_n\|)a.$$

As a consequence of this result, J. B. Diaz and F. T. Metcalf [22] proved:

THEOREM 7. *Let the "weights"* $q_1, ..., q_n$ *be real and positive such that* $q_1 + \cdots + q_n = 1$. *If the conditions of Theorem* 6 *are valid then we have*

$$r\ \|x_1\|^{q_1} \cdots \|x_n\|^{q_n} \leqslant \|q_1 x_1 + \cdots + q_n x_n\| \tag{19}$$

and

$$r(q_1\ \|x_1\|^p + \cdots + q_n\ \|x_n\|^p)^{1/p} \leqslant \|q_1 x_1 + \cdots + q_n x_n\|, \tag{20}$$

where $p < 1$ *and* $p \neq 0$. *Equality holds in* (19) (*or* (20)) *iff*

$$q_1 x_1 + \cdots + q_n x_n = r(q_1\ \|x_1\| + \cdots + q_n\ \|x_n\|)a \tag{21}$$

and

$$\|x_1\| = \cdots = \|x_n\|. \tag{22}$$

In fact these results of Diaz and Metcalf are simple consequences of the following results.

Let the conditions of Theorem 6 be fulfilled. If f is a strictly concave and increasing function, then

$$f\left(\frac{1}{r}\left\|\sum_{i=1}^{n} q_i x_i\right\|\right) \geqslant \sum_{i=1}^{n} q_i f(\|x_i\|), \tag{23}$$

and if f is a strictly convex and decreasing function then we have the reverse inequalities. In both cases equality is valid iff (21) and (22) are valid.

Of course, as in [15] we can prove results which are related to Theorem 4.

ACKNOWLEDGMENT

We express our thanks to the referee for helpful suggestions.

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