Note

Comments on the paper
“Weak almost-convergence theorem without opial’s condition”

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Let $E$ be a Banach space with dual $E^*$. Recall that the duality map $J : E \to E^*$ is defined by

$$J(x) := \{x^* \in E^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in E.$$ 

It is known that $E$ is uniformly smooth [1] if and only if $J$ is single-valued and uniformly continuous on bounded sets in $E$.

The normal structure coefficient [2] of $E$ is defined as the number

$$N(E) := \inf \{d(C)/r(C): C \text{ convex bounded subset of } E \text{ with } d(C) > 0\},$$

where

$$d(C) := \sup \{\|x - y\|: x, y \in C\} \quad \text{and} \quad r(C) := \inf_{x \in C} \sup_{y \in C} \|x - y\|$$

are the diameter of $C$ and, respectively, the Chebyshev radius of $C$ relative to itself. A Banach space $E$ is said to have uniformly normal structure [2] if $N(E) > 1$. It is known that $N(H) = \sqrt{2}$, where $H$ is a Hilbert space.
Given real numbers $\lambda, \alpha, \beta > 0$, a Banach space $E$ is said to satisfy property $(U, \lambda, \alpha, \beta)$ if
\[ \|x + y\|^{\alpha} + \lambda \|x - y\|^{\alpha} - 2^{\beta} (\|x\|^{\alpha} + \|y\|^{\alpha}) \geq 0, \quad x, y \in E. \]

It is known that a Hilbert space satisfies $(U, 1, 1)$ and an $l^p$ (or $L^p$) satisfies $(U, p - 1, 2, 1)$ for $2 \leq p \leq \infty$.

Let $D$ be a nonempty convex subset of a Banach space $E$ and $T : D \to D$ be a mapping.

**Definition 1** [3]. $T$ is said to be **asymptotically pseudocontractive** if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n \to \infty} k_n = 1$ such that
\[ \langle T^n x - T^n y, j \rangle \leq k_n \|x - y\|^2 \quad \text{for all } n \in \mathbb{N}, \ j \in J(x - y), \ x, y \in D, \]
where $\mathbb{N}$ is the set of positive integers.

**Definition 2** [4,5]. $T$ is said to be **asymptotically hemicontractive** if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n \to \infty} k_n = 1$ such that
\[ \langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 \]
for all $n \in \mathbb{N}, \ j(x - p) \in J(x - p), \ x \in D, \ p \in F(T),$
where $F(T)$ is the fixed point set of $T$ (i.e., $F(T) = \{x \in D: Tx = x\}$).

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ and let a point $x_0 \in D$ be given. The Ishikawa iteration process [6] is defined as
\[ \begin{align*}
  x_{n+1} &:= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\
  y_n &:= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \in \mathbb{N}.
\end{align*} \]

Schu [3] proposed a generalized Ishikawa iteration process as
\[ \begin{align*}
  x_{n+1} &:= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\
  y_n &:= (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \in \mathbb{N}. \quad (I)
\end{align*} \]

The concept of almost-convergence, due to Lorentz [7], is as follows.

**Definition 3.** A sequence $\{x_n\}$ in a Banach space $E$ is **weakly almost-convergent** to a point $x \in E$ if and only if
\[ \text{weak-} \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} x_{i+j} = x \quad \text{uniformly in } i = 0, 1, 2, \ldots. \]

The following is the main result of Sharma et al. [4].
**Theorem 1** (p. 641, Theorem 3.1). Let $E$ be a Banach space with property $(U, \lambda, m + 1, m)$, $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$, and a uniformly Gateaux differentiable norm; $D$ a nonempty closed convex bounded subset of $E$; and $T : D \to D$ a uniformly $L$-Lipschitzian asymptotically hemicontractive mapping with sequence $\{k_n\}$ and $L < N(E)^{1/2}$. Suppose in addition that $T$ satisfies the conditions:

\[
\|x - T^n y\|^2 \leq \langle x - T^n y, J(x - y) \rangle
\]

for all $x, y \in D$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. (\star)

Then, for the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (I), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $0 < a \leq \alpha_n < \alpha < 1$ and $0 < b \leq \beta_n < \beta < 1$ for all $n \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} (v_n c - m) < \infty$, where $v_n = (m + 1)k_n - m$ and $c = \lambda/(2^m - 1)$,

(iii) $(1 - 2\beta^m c - \beta^m + 1 L^m + 1 c) c + 1 - \beta^m c - c^2 > 0$ and $1 - \alpha^m c - (1 - mb)c^2 > 0$,

there exists a point $v \in F(T)$ such that $\{J(x_n - v)\}$ weakly almost-converges to zero.

The following are my comments to Theorem 1.

(a) Condition (\star) actually implies that the result is trivial. As a matter of fact, taking $x = y$ and $n = 1$ in condition (\star), one gets $Tx = x$ for all $x \in D$; that is, $T$ coincides with the identity operator on $D$.

(b) The property $(U, \lambda, m + 1, m)$ implies that the Banach space $E$ is uniformly smooth. So the Gateaux differentiable norm assumption is superfluous.

(c) Condition (ii) is satisfied in a Hilbert space if and only if

\[
\sum_{n=1}^{\infty} (k_n - 1) < \infty.
\]

Indeed, since a Hilbert space is $(U, 1, 2, 1)$, one sees that $\lambda = 1$, $m = 1$, $c = 1$, $v_n = 2k_n - 1$, and $v_n c - m = 2(k_n - 1)$. Therefore, $\sum_{n=1}^{\infty} (v_n c - m) < \infty$ if and only if $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.

(d) Neither $l^p$ nor $L^p$ (for $2 < p < \infty$) satisfies condition (ii). Indeed, since both $l^p$ and $L^p$ (for $2 < p < \infty$) satisfy $(U, p - 1, 2, 1)$, we have $\lambda = p - 1$, $m = 1$, $c = p - 1$, $v_n = 2k_n - 1$, and $v_n c - m = (2k_n - 1)(p - 1) - 1$. But $k_n \to 1$, we have $v_n c - m \to p - 2 > 0$ as $n \to \infty$. Therefore $\sum_{n=1}^{\infty} (v_n c - m) = \infty$.

(e) In the Hilbert space case, condition (iii) becomes

\[
\beta^2 L^2 + 3\beta - 1 < 0
\]

and

\[
\sum_{n=1}^{\infty} (k_n - 1) < \infty.
\]
\( b > \alpha \). \hspace{1cm} (2)

Condition (1) holds provided \( \beta \) is small enough; precisely, provided

\[
\beta < \frac{-3 + \sqrt{9 + 4L^2}}{2L^2} < \frac{1}{3}.
\]

Condition (2) holds if and only if there exists a positive number \( \delta (= b - \alpha) \) with the following property:

\[
\beta_n \geq \delta + \alpha_m \text{ for all } m, n \geq 1.
\]

(f) If \( E = l^p \) (or \( L^p \)) for \( 2 < p < \infty \), condition (iii) becomes

\[
(1 + 2\beta + \beta^2L^2)(p - 1)^2 - (1 - \beta)(p - 1) - 1 < 0, \quad (3)
\]

and

\[
(1 - b)(p - 1)^2 + \alpha(p - 1) - 1 < 0. \quad (4)
\]

Clearly, if \( p > 2 \) is big enough, neither (3) nor (4) is satisfied. Precisely, if

\[
p \geq 1 + \frac{1 - \beta + \sqrt{(1 - \beta)^2 + 4(1 + 2\beta + \beta^2L^2)}}{2(1 + 2\beta + \beta^2L^2)},
\]

then (3) is not satisfied; while, if

\[
p \geq 1 + \frac{-\alpha + \sqrt{\alpha^2 + 4(1 - b)}}{2(1 - b)},
\]

then (4) is not satisfied.

(g) The proof is incomplete. What the authors proved is this (see p. 643, line 2):

\[
\lim_n \langle z, J(x_n - v) \rangle = 0 \text{ for all } z \in D, \text{ where } \lim \text{ is a Banach limit and } v \text{ is some point in } F(T). \] But this is obviously not enough to imply that the sequence \( \{J(x_n - v)\} \) is weakly almost-convergent to zero (under the sense of Definition 3).

The second and the last result of the paper [4] is the following

**Theorem 2** (p. 643, Theorem 4.1). Let \( E \) be a Banach space with property \((U, \lambda, m + 1, m), \lambda \in \mathbb{R}, n \in \mathbb{N}, \) and a uniformly Gateaux differentiable norm; \( J^{-1} : E^* \to E \) weakly sequentially continuous at zero; \( D \) a nonempty closed convex bounded subset of \( E \); and \( T : D \to D \) a uniformly \( L \)-Lipschitzian and asymptotically hemicontractive mapping with sequence \( \{k_n\} \) and \( L < N(E)^{1/2} \). Suppose, in addition, \( T \) satisfies the condition (*) Then the sequence \( \{x_n\} \) generated by (I), where \( \{k_n\}, \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy the conditions (i), (ii) and (iii), converges weakly to a fixed point of \( T \).
The comments (a)–(g) above apply to Theorem 2 as well. Moreover, in their Remark 4.1 (p. 643), the authors say that “Theorem 4.1 can be applied to all $L_p$, $p \geq 2$ spaces. Hence it is a good improvement of [5, Theorem 4.1].” This is not true since for any $p \in (1, \infty)$, $p \neq 2$, the duality map $J$ of $L_p$ is not weakly sequentially continuous at zero ($L_p$ even fails to satisfy the weaker Opial’s property [8]).

References


