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A CHARACTERISTIC POLYNOMIAL

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The $n \times n$ matrix

$$A = \begin{pmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix}$$

occurs occasionally in applications (see, for example, [1]). In particular, it is of interest to know when this matrix is positive definite. One way to determine this is to calculate the eigenvalues explicitly and to test whether they are positive. In this note, we therefore calculate the characteristic polynomial

$$D_n(\lambda) = \det(A - \lambda I)$$

for the above matrix.

For simplicity, we calculate first

$$D_n = \det \begin{pmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix}.$$

If we can find D_n explicitly, then we obtain the characteristic polynomial by replacing a by $a - \lambda$.

Let also

$$C_n = \det \begin{pmatrix} b & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix},$$

that is the determinant of the matrix obtained by replacing the element a in the top left corner of A by b.

Expanding D_n by minors in the first row, (see, for example, [2]) we get

$$D_n = aD_{n-1} - b \det \begin{pmatrix} b & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix} + b \det \begin{pmatrix} b & a & b & \cdots & b & b \\ b & b & b & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix}$$

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$$+\cdots+(-1)^{n-1}\begin{pmatrix} b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & b & b \\ b & b & b & \cdots & b & a \end{pmatrix} + b(-1)^n \begin{pmatrix} b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & & \ddots & & & \vdots \\ b & b & b & \cdots & b & a \\ b & b & b & \cdots & b & b \end{pmatrix}$$

$$= aD_{n-1} - (n-1)bC_{n-1}. \tag{1}$$

We have the initial values $D_1 = a$ and $D_2 = a^2 - b^2$. Using the recurrence (1), we get $D_3 = 2b^3 + a^3 - 3ab^2 = (a-b)^2(a+2b)$, which is easily verified directly; and replacing a by $a - \lambda$ in D_3, we get $D_3(\lambda) = (a-\lambda)^3 + 2b^3 - 3(a-\lambda)b^2$, which has the factors $\lambda_{1,2} = a - b$ and $\lambda_3 = a + 2b$.

Similarly, expanding C_n by minors in the first row, we get

$$C_n = bD_{n-1} - (n-1)\,bC_{n-1}, \tag{2}$$

where we have $C_1 = b$, $C_2 = b(a-b)$ and $C_3 = b(a-b)^2$ by direct calculation.

From (1) and (2), we get

$$C_n - D_n = (b-a)D_{n-1}. \tag{3}$$

Multiplying (2) by a and (1) by b and subtracting, we get

$$aC_n - bD_n = b(b-a)(n-1)\,C_{n-1}. \tag{4}$$

Increasing the index in (2) by 1, we get

$$bD_n = C_{n+1} + nbC_n. \tag{5}$$

Equations (4) and (5) together give

$$C_{n+1} + (nb-a)C_n + b(b-a)(n-1)C_{n-1} = 0. \tag{6}$$

Increasing the index in (6) by 1, we get

$$C_{n+2} + ((n+1)b - a)C_{n+1} + b(b-a)n)C_n = 0, \tag{7}$$

which can further be written as

$$(E^2 + ((n+1)b - a)E + b(b-a)n)C_n = 0 \tag{8}$$

using the shift operator, E, defined by $ED_n = D_{n+1}$. Equation (8) can now be factored in a similar manner to the example in [3, Section 6.2.3],

$$E^2 + (b(n+1) - a)E + bn(b-a) = (E + bn)(E + (b-a)) \tag{9}$$

taking care that the noncommutativity of the shift operator with a non-constant coefficient is not violated. We now have two first order recurrences:

$$(E + bn)Y_n = 0 \tag{10}$$

and

$$(E + (b-a))C_n = Y_n. \tag{11}$$

The solution of (10) is clearly

$$Y_n = (-1)^{n-1}b^{n-1}(n-1)!Y_1. \tag{12}$$

Plugging this into (11), we get the equation

$$C_{n+1} = (a-b)\,C_n + (-1)^{n-1}\,b^{n-1}(n-1)!\,Y_1. \tag{13}$$

This is a first order linear recurrence, as given in [3, Section 5.2]. The solution is

$$\begin{aligned} C_n &= C_1 \prod_{2\le i\le n} (a-b) + \sum_{2\le i\le n} (-1)^i b^i (i-2)!\,Y_1 \prod_{i<j\le n} (a-b) \\ &= C_1(a-b)^{n-1} + Y_1 \sum_{1\le i\le n-1} i!(-b)^i (a-b)^{n-1-i}. \end{aligned} \tag{14}$$

Using the values for C_1 and C_2, we get $Y_1 = 0$ so that $C_n = b(a-b)^{n-1}$. This could also have been obtained directly from (10) and (11) using the boundary conditions C_1 and C_2. From (5), we then obtain

$$D_n = (a-b)^n + nb(a-b)^{n-1} = (a-b)^{n-1}(a+(n-1)b). \tag{15}$$

The eigenvalues of D_n are therefore $\lambda_{1,2,\dots,n-1} = (a-b)$ and $\lambda_n = a+(n-1)b$.

Gershgorin's theorem (see [2]) tells us that all of the eigenvalues lie in or on the disk $|z-a| \le (n-1)|b|$ in the complex plane. In this example, we see that $(n-1)$ eigenvalues lie in the interior of the disk and one eigenvalue is on the boundary.

The answer to the question on whether the matrix A is positive or not is now obvious since we have explicit values for the eigenvalues.

References

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