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1D nonlinear Fokker–Planck equations for fermions and bosons

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Abstract

We obtain equilibration rates for nonlinear Fokker–Planck equations modelling the relaxation of fermion and boson gases. We show how the entropy method applies for quantifying explicitly the exponential decay towards Fermi–Dirac and Bose–Einstein distributions in the one-dimensional case.

Keywords: Nonlinear Fokker–Planck equations; Nonlinear diffusions; Entropy methods

1. Introduction

The main aim of this work is to analyze the large-time behavior of solutions of the Cauchy problem

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial v^2} f + \frac{\partial}{\partial v}[vf(1+kf)], \quad v \in \mathbb{R}, t > 0, k = \pm 1, \tag{1}$$

with initial data

$$f(v,0) = f_0(v). \tag{2}$$

These nonlinear Fokker–Planck equations have been proposed in [8,7,4] and the references therein, as kinetic models for the relaxation to equilibrium for bosons ($k = 1$) and fermions ($k = -1$). These models have been introduced as a simplification with respect to Boltzmann-based models as in [9,5]. Here, we will show that entropy methods apply in a direct way for analyzing the equilibration rate for the one-dimensional case.

Let us finally remark that some of these formal computations can be generalized to the fermion case in any dimension. However, the extensions to several dimensions both for fermions and for bosons are relevant open problems. In the remainder, we will assume that we are dealing with smooth positive fast-decaying solutions of Eq. (1). The well-posedness of the Cauchy problem (1) and (2), the properties of their solutions and the rigorous proof of the convergence in the entropy sense will be developed elsewhere.

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2. Reckoning the stationary distributions and the entropy form

We first give the explicit form of integrable stationary solutions for Eq. (1):

Lemma 2.1. *Let F_∞ be an integrable, strictly positive, stationary solution for Eq. (1), with $F_\infty < 1$ in the fermion case. Then*

$$F_\infty(v) = \frac{1}{\beta e^{\frac{v^2}{2}} - k}.$$

Moreover, for each value of the mass $M > 0$, there exists a unique $\beta = \beta(M) \geq 0$ such that $F_\infty(v)$ has mass M.

Proof. We consider the stationary version of Eq. (1):

$$\frac{\partial^2}{\partial v^2} f + \frac{\partial}{\partial v}[vf(1+kf)] = 0,$$

that can be written in the form

$$\frac{\partial}{\partial v}\left\{f(1+kf)\left[\frac{1}{f(1+kf)}\frac{\partial f}{\partial v} + \frac{\partial}{\partial v}\left(\frac{v^2}{2}\right)\right]\right\} = 0,$$

or, equivalently,

$$\frac{\partial}{\partial v}\left\{f(1+kf)\frac{\partial}{\partial v}\left[\log\left(\frac{f}{1+kf}\right) + \frac{v^2}{2}\right]\right\} = 0.$$

Since the solution is smooth fast-decaying and less than 1 in the fermion case, then the previous equation implies that

$$\frac{\partial}{\partial v}\left[\log\left(\frac{f}{1+kf}\right) + \frac{v^2}{2}\right] = 0$$

from which we analytically obtain the stationary solution to Eq. (1):

$$F_\infty(v) = \frac{1}{\beta e^{\frac{v^2}{2}} - k}$$

with $\beta \geq 0$. Now, it is easy to check that these stationary solutions are integrable for all $\beta > 0$ in the fermion case and for $\beta > 1$ in the boson case, and moreover, in the boson case the map $M(\beta) : \beta \in (1, \infty) \longrightarrow (0, \infty)$ given by

$$M(\beta) = \int_{\mathbb{R}} \frac{1}{\beta e^{\frac{v^2}{2}} - k}\, dv$$

is decreasing, surjective and invertible. In the fermion case, $M(\beta)$ has the same properties defined on $\beta \in (0, \infty)$. □

Remark 2.2. Since the stationary states depend on M through β, we shall write $F_{\infty,M}(v)$ instead of $F_\infty(v)$. This family of stationary states corresponds to the classical Fermi–Dirac ($k = -1$) and Bose–Einstein ($k = 1$) distributions. Lemma 2.1 can be generalized to any dimension in the fermion case and to 2D in the boson case. However, in the 3D boson case, the stationary solutions $F_{\infty,\beta}(v)$ converge as $\beta \to 1^+$ to an integrable singular solution, and thus we have the well-known critical mass for Bose–Einstein equilibrium distributions.

Following ideas similar to those in [2,1,3], we can define the entropy of f as

$$H(f) = \int_{\mathbb{R}} \left[\frac{v^2}{2} f + \Phi(f)\right] dv \tag{3}$$

where

$$\Phi(f) = f\log(f) - k(1+kf)\log(1+kf) \tag{4}$$

which acts as a Lyapunov functional for the system, namely:

Proposition 2.3 (*H-Theorem*). *The functional H defined on the set of positive integrable functions with given mass M attains its unique minimum at $F_{\infty,M}(v)$. Moreover, given any solution to (1) with initial data f_0 of mass M, we have*

$$H(F_{\infty,M}) \le H(f(t)) \le H(f_0) \tag{5}$$

for all $t \ge 0$.

Proof. We first remark that the entropy functional coincides with the one introduced in [1] for the nonlinear diffusion equation

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial x}\left\{g \frac{\partial}{\partial x}\left[x + h(g)\right]\right\} \tag{6}$$

for the function $g(x, t)$, $x \in \mathbb{R}$, $t > 0$, where

$$h(g) = \log\left(\frac{g}{1 + kg}\right). \tag{7}$$

We leave the readers to check that the nonlinear diffusion defining previous equation verifies all hypotheses needed [1, Proposition 5], which implies the first statement of this proposition. Let us remark that the minimizing character of the Fermi–Dirac and Bose–Einstein distributions for this entropy is also a consequence of the results of [5,10]. As regards the second part, we can compute the evolution of the entropy functional along solutions getting

$$-D_k(f) := \frac{\partial}{\partial t} H(f) = -\int_{\mathbb{R}} f(1 + kf)\left[v + \frac{\partial}{\partial v} h(f)\right]^2 \mathrm{d}v \le 0 \tag{8}$$

where $D_k(f)$ is by definition the entropy dissipation for Eq. (1). □

Let us point out that the entropy dissipation for the nonlinear diffusion equation (6) is given by

$$-D_0(g) = \frac{\partial}{\partial t} H(g) = -\int_{\mathbb{R}} g\left[x + \frac{\partial}{\partial x} h(g)\right]^2 \mathrm{d}x.$$

The relation between the entropy dissipations for the solutions of the nonlinear diffusion equations (6) and (1) will be the basis of our results.

3. A priori estimates

In order to get rates of decay towards equilibrium states for this problem, we sketch the proof of some comparison properties between solutions of the equation that are obtained by classical arguments; see [6] for instance.

Lemma 3.1. *Let f be a solution of the Cauchy problem (1) and (2). If $f_0 \in L^1(\mathbb{R})$, then the L^1-norm of f is non-increasing for $t \ge 0$.*

Proof. Let us consider a regularized increasing approximation of the sign function $\operatorname{sign}_\varepsilon(z)$, $z \in \mathbb{R}$, and let us define the regularized approximation $|f|_\varepsilon(z)$ of $|f|(z)$ via the primitive of $\operatorname{sign}_\varepsilon(f)(z)$. We now multiply Eq. (1) by $\operatorname{sign}_\varepsilon(z)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |f|_\varepsilon \,\mathrm{d}v = -\int_{\mathbb{R}} \operatorname{sign}'_\varepsilon(f)|\partial_v f|^2 \,\mathrm{d}v + \int_{\mathbb{R}} \operatorname{sign}_\varepsilon(f)\partial_v\left(vf(1 + kf)\right) \mathrm{d}v. \tag{9}$$

We integrate by parts the last term deducing

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |f|_\varepsilon \,\mathrm{d}v = -\int_{\mathbb{R}} \operatorname{sign}'_\varepsilon(f)|\partial_v f|^2 \,\mathrm{d}v - \int_{\mathbb{R}} v \operatorname{sign}'_\varepsilon(f) f(1 + kf)\partial_v f \,\mathrm{d}v.$$

Since $\operatorname{sign}'_\varepsilon(f) f \partial_v f = \partial_v[f \operatorname{sign}_\varepsilon(f) - |f|_\varepsilon]$ and $\operatorname{sign}'_\varepsilon(f) f^2 \partial_v f = \partial_v[f^2 \operatorname{sign}_\varepsilon(f) - f|f|_\varepsilon]$, we obtain, after another integration by parts in the last term of the right-hand side of Eq. (9), that, in the limit $\varepsilon \to 0$, such a term vanishes, and deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |f|\,\mathrm{d}v \le 0,$$

that is the L^1-norm is non-increasing in time. □

A simple consequence of the previous lemma is given by the following corollary:

Corollary 3.2. *Let f be a solution of the Cauchy problem (1) and (2), with initial condition $f_0 \in L^1(\mathbb{R})$. If f_0 is non-negative a.e. in $\mathbb{R}$, then f is also non-negative a.e. in $\mathbb{R}$ for any $t > 0$.*

Proof. We consider the time evolution of $f_\varepsilon^-(v,t) = (|f|_\varepsilon - f)/2$. By the conservation of mass in Eq. (1) and the proof of Lemma 3.1, we have

$$\int_{\mathbb{R}} |f|_\varepsilon^- \,\mathrm{d}v \le \int_{\mathbb{R}} |f_0|_\varepsilon^- \,\mathrm{d}v + O(\varepsilon) \quad \forall t > 0.$$

Taking the limit $\varepsilon \to 0$, the thesis follows easily. □

The main arguments of Lemma 3.1 lead to comparison results, between positive solutions.

Lemma 3.3 (L^1-*Contraction*). *Let f and g be two solutions of the Cauchy problem (1) and (2), with non-negative a.e. initial conditions f_0 and $g_0 \in L^1(\mathbb{R})$ respectively. Then*

$$\|f(v,t) - g(v,t)\|_1 \le \|f_0(v) - g_0(v)\|_1$$

for all $t > 0$. Moreover, if $f_0(v) \le g_0(v)$ a.e., then $f(v,t) \le g(v,t)$ a.e. for all $t > 0$.

Proof. Since both f and g are solutions of Eq. (1), we deduce

$$\frac{\partial}{\partial t}(f-g) = \frac{\partial^2}{\partial v^2}(f-g) + \partial_v\left(v(f-g)\right) + k\partial_v(v(f^2-g^2)).$$

We multiply this equation by $\operatorname{sign}_\varepsilon(f-g)$ and integrate with respect to $v \in \mathbb{R}$. The same procedure as for Lemma 3.1, the observation that $\operatorname{sign}_\varepsilon'(f-g)(f^2-g^2)\partial_v(f-g) = (f+g)\partial_v[(f-g)\operatorname{sign}_\varepsilon(f-g) - |f-g|_\varepsilon]$ and the limiting procedure $\varepsilon \to 0$ finish the proof.

The order-preserving property of the equation is an immediate consequence of the time evolution of the quantity

$$[f(v,t) - g(v,t)]_\varepsilon^- = |f(v,t) - g(v,t)|_\varepsilon - (f-g).$$

Since the initial conditions are of class $L^1(\mathbb{R})$, from the conservation of mass and the L^1-contraction principle, we deduce immediately that the condition $f_0(v) \le g_0(v)$ a.e. in $\mathbb{R}$ implies that $f(v,t) \le g(v,t)$ a.e. for all $t > 0$. □

As a consequence, we can compare solutions to the stationary states $F_{\infty,M}$.

Corollary 3.4. *Let f be a solution of (1) and (2) with initial condition f_0 such that $f_0(v) \le F_{\infty,M}(v)$ a.e. Then $f(v,t) \le F_{\infty,M}(v)$ a.e. for all $t > 0$.*

4. Entropy dissipation and convergence rates towards equilibria

Theorem 4.1. *Let f be a solution for (1) and $F_{\infty,M}$ be the stationary state of the solution with the same mass M. In the fermion case, $k = -1$, we additionally assume that the initial data f_0 is below a given Fermi–Dirac distribution F_{∞,M^*}, i.e., $f_0 \le F_{\infty,M^*}$ a.e. Then*

$$H(f) - H(F_{\infty,M}) \le (H(f_0) - H(F_{\infty,M}))\mathrm{e}^{-2Ct} \tag{10}$$

for all $t \ge 0$, where $C = 1$ for the boson case, $k = 1$, and C depends on M^* in the fermion case, $k = -1$.

Proof. We leave the readers to check that $h(f)$ given by (7) verifies in one dimension the hypotheses of the Generalized Log–Sobolev Inequality [1, thm 17]; in the fermion case we must restrict to $f \in (0, 1)$. The Generalized Log–Sobolev Inequality asserts in our case that

$$H(g) - H(F_{\infty,M}) \leq \frac{1}{2} D_0(g) \tag{11}$$

for all integrable positive g with mass M for which the right-hand side is well defined and finite. We can now compare the entropy dissipation $D_k(f)$ of Eq. (1) and the one $D_0(f)$ of Eq. (6) in each case:

- Bosons: convergence to the Bose–Einstein distribution, $k = 1$:

$$D_1(f) = \int_{\mathbb{R}} \left(f + f^2\right) \left[v + \frac{\partial}{\partial v} h(f)\right]^2 \mathrm{d}v \geq \int_{\mathbb{R}} f \left[v + \frac{\partial}{\partial v} h(f)\right]^2 \mathrm{d}v. \tag{12}$$

- Fermions: convergence to the Fermi–Dirac distribution, $k = -1$: Thanks to Corollary 3.4 we have $f(v, t) \leq F_{\infty,M^*}(v) \leq (\beta^* + 1)^{-1}$ a.e. in $\mathbb{R}$, and thus

$$D_{-1}(f) = \int_{\mathbb{R}} f(1 - f) \left[v + \frac{\partial}{\partial v} h(f)\right]^2 \mathrm{d}v \geq R \int_{\mathbb{R}} f \left[v + \frac{\partial}{\partial v} h(f)\right]^2 \mathrm{d}v \tag{13}$$

  where $R = 1 - (\beta^* + 1)^{-1}$.

Applying the Generalized Log–Sobolev Inequality (11) to the solution $f(t)$ and taking into account previous estimates, we conclude that

$$H(f(t)) - H(F_{\infty,M}) \leq (2C(k))^{-1} D_k(f(t)) \tag{14}$$

where $C(k) = 1$ if $k = 1$ and $C(k) = R$ if $k = -1$. Finally, coming back to the entropy evolution:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[H(f(t)) - H(F_{\infty,M})\right] = -D_k(f(t)) \leq -2C(k) \left[H(f(t)) - H(F_{\infty,M})\right],$$

and the result follows from Gronwall's lemma. □

Now, we can try to give more accurate convergence properties by reckoning rates of decay for the entropy dissipation:

$$D_k(f) = \int_{\mathbb{R}} f(1 + kf)\xi^2 \, \mathrm{d}v$$

where $\xi = v + \partial_v h(f)$. Computing the evolution of the dissipation of the entropy in time, we deduce

$$DD_k(f) = \frac{\mathrm{d}}{\mathrm{d}t} D_k(f) = \int_{\mathbb{R}} (1 + 2kf) \frac{\partial f}{\partial t} \xi^2 \, \mathrm{d}v + 2 \int_{\mathbb{R}} f(1 + kf)\xi \frac{\partial \xi}{\partial t} \, \mathrm{d}v = \text{(I)} + \text{(II)}.$$

Integrating (II) by parts, we obtain that

$$\text{(II)} = -2 \int_{\mathbb{R}} \frac{1}{f(1 + kf)} \left(\frac{\partial}{\partial v}[f(1 + kf)\xi]\right)^2 \mathrm{d}v.$$

Using again integration by parts with (I) and repeating the process for the term with $\frac{\partial}{\partial v}(1 + 2kf)$ we obtain

$$\begin{aligned}
\text{(I)} &= -2 \int_{\mathbb{R}} \left(f + \frac{3}{2} k f^2 + f^3\right) \xi^2 \frac{\partial \xi}{\partial v} \, \mathrm{d}v \\
&= -2 \int_{\mathbb{R}} \varphi_1(f) \xi^2 \, \mathrm{d}v + 2 \int_{\mathbb{R}} \varphi_2'(f) \left(\frac{\partial f}{\partial v}\right)^2 f(1 + kf)\xi^2 \, \mathrm{d}v \\
&\quad + 4 \int_{\mathbb{R}} \varphi_2(f) \xi \frac{\partial f}{\partial v} \frac{\partial}{\partial v}[f(1 + kf)\xi] \, \mathrm{d}v
\end{aligned}$$

where we have considered

$$\varphi_1(f) = f + \frac{3}{2}kf^2 + f^3 \quad \text{and} \quad \varphi_2(f) = \frac{\varphi_1(f)}{(f(1+kf))^2}.$$

Finally, we have

$$DD_k(f) = -2\int_{\mathbb{R}} \varphi_1(f)\xi^2 \,\mathrm{d}v - 2\int_{\mathbb{R}} (B - \varphi_2(f)A)^2 \,\mathrm{d}v + 2\int_{\mathbb{R}} \left[\varphi_2(f)^2 + \varphi_2'(f)\right] A^2$$

where

$$A := \xi \frac{\partial f}{\partial v}\sqrt{f(1+kf)} \quad \text{and} \quad B := \frac{\frac{\partial}{\partial v}[f(1+kf)\xi]}{\sqrt{f(1+kf)}}.$$

For $k = 1$, it is easy to show that $\left[\varphi_2(f)^2 + \varphi_2'(f)\right] \le 0$, so the last term in $DD_k(f)$ is negative, and we get

$$DD_1(f) \le -2\int_{\mathbb{R}} \varphi_1(f)\xi^2 \,\mathrm{d}v \le -2D_1(f) \tag{15}$$

since for $k = 1$, we have $\varphi_1(f) \ge f(1+f)$. We conclude:

Proposition 4.2 (*Entropy Dissipation Decay for Bosons*). *Let f be a solution for (1) with $k = 1$; then, for all $t \ge 0$,*

$$D_1(f(t)) \le D_1(f_0)\mathrm{e}^{-2t}.$$

Finally, we will remark on the consequences of the entropy convergence on L^1 spaces. Due to mass conservation and positivity of the stationary states $F_{\infty,M}$, we have

$$H(f|F_{\infty,M}) := \int_{\mathbb{R}} \left[\Phi(f) - \Phi(F_{\infty,M}) - \Phi'(F_{\infty,M})(f - F_{\infty,M})\right] \mathrm{d}v = H(f) - H(F_{\infty,M}).$$

Corollary 4.3. *Under the assumptions of Theorem 4.1, then*

$$\|f(t) - F_{\infty,M}\|_{L^1(\mathbb{R})} \le C_2(H(f_0|F_{\infty,M}))^{1/2}\mathrm{e}^{-Ct} \tag{16}$$

for all $t \ge 0$, where C_2 depends only on the mass M.

This is a consequence of a direct application of Taylor theorem to the relative entropy $H(f(t)|F_{\infty,M})$ obtaining

$$H(f|F_{\infty,M}) \ge \frac{1}{2}\int_{\mathbb{R}} \Phi''(\xi(v,t))(f - F_{\infty,M})^2 \,\mathrm{d}v \ge \frac{1}{2}\int_{S_\infty} \Phi''(\xi(v,t))(f - F_{\infty,M})^2 \,\mathrm{d}v$$

where $\xi(v,t)$ lies on the interval between $f(v,t)$ and $F_{\infty,M}(v)$ and $S_\infty = \{v \in \mathbb{R} \text{ such that } f(v,t) \le F_{\infty,M}(v)\}$. Now, a direct Cauchy–Schwartz inequality gives

$$\begin{aligned} \|f - F_{\infty,M}\|^2_{L^1(S_\infty)} &\le \left(\int_{S_\infty} \frac{1}{\Phi''(\xi(v,t))}\,\mathrm{d}v\right)\left(\int_{S_\infty} \Phi''(\xi(v,t))(f - F_{\infty,M})^2 \,\mathrm{d}v\right) \\ &\le 2\gamma\left(\int_{S_\infty} F_{\infty,M}(v)\,\mathrm{d}v\right) H(f|F_{\infty,M}) \le 2\gamma M H(f|F_{\infty,M}) \end{aligned} \tag{17}$$

where $\gamma = 1 + (\beta(M) - 1)^{-1}$ for bosons and $\gamma = 1$ for fermions. Taking into account that $f(v,t)$ and $F_{\infty,M}(v)$ have equal mass, then

$$\|f - F_{\infty,M}\|_{L^1(\mathbb{R})} = 2\|f - F_{\infty,M}\|_{L^1(S_\infty)}. \tag{18}$$

Corollary 4.3 is obtained putting together (17) and (18).

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