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DISTANCE TRANSITIVE DIGRAPHS

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A digraph is said to be distance-transitive if for all vertices u, v, x, y such that $d(u, v) = d(x, y)$ there is an automorphism π of the digraph such that $\pi(u) = x$ and $\pi(v) = y$. Some examples of distance-transitive digraphs are given in Section 2. Section 3 defines the intersection matrix and gives some of its properties. Section 4 proves that every distance-transitive digraph with diameter 2 is a balanced incomplete block design with the Hadamard parameters. Intersection matrices of distance-transitive digraphs with diameter 3 and valency not larger than 20 are generated. There are 28 possibilities and each can be realized by the constructions of Section 2.

1. Introduction

A *digraph* Γ is a pair (V, E) where V is a finite set of *vertices* and $E \subseteq V^2$ is a set of *directed edges*. If $e = (u, v) \in E$, then we say that e is an edge from u to v. A *path* of length r in Γ, joining u to v is a finite sequence of vertices

$$u = w_0, w_1, \ldots, w_r = v$$

such that $(w_{t-1}, w_t) \in E$ for $t = 1, 2, \ldots, r$. In this paper, we will only consider digraphs that are *strongly connected*, meaning that every pair of vertices is joined by a path. The number of edges traversed in the shortest path joining u to v is called the *distance* in Γ from u to v and is denoted by $d(u, v)$. The maximum value of the distance function in Γ is called the *diameter* of Γ.

An *automorphism* of a digraph Γ is a permutation π of V which has the property that $(u, v) \in E$ if and only if $(\pi(u), \pi(v)) \in E$. The set of all automorphisms of Γ, with the operation of composition, forms a group. This group $G(\Gamma)$ is called the *automorphism group* of Γ.

A connected digraph Γ is said to be *distance-transitive* if for any vertices u, v, x and y of Γ, satisfying $d(u, v) = d(x, y)$, there is an automorphism $\pi \in G(\Gamma)$ which takes u to x and v to y.

References [1] and [2] present an excellent treatment of undirected graphs with the distance-transitive property. Much of this material is relevant to distance-transitive digraphs as well. Since the automorphism groups of some distance-transitive undirected graphs are related to "sporadic" simple groups, the directed graphs ought to be of interest was well.

First of all, we note that there is a distance-transitive digraph associated with every undirected distance-transitive graph. If vertices u and v are adjacent in the undirected graph, then in the digraph, we put in two edges (u, v) and (v, u). One can check that the distance function, the automorphism group and the distance-transitive property are preserved. Conversely, we can construct undirected distance-transitive graphs from certain distance-transitive digraphs. If in the digraph, all edges come in pairs, from u to v and v *to* u, then we can collapse all the pairs into undirected edges and obtain an undirected distance-transitive graph. Distance transitivity in the digraph implies that if two of the edges are of the form (u, v) and (v, u), then all the edges come in pairs. Hence, in order to study something new, we will restrict ourselves to distance-transitive digraphs with the property that if $(u, v) \in E$, then $(v, u) \notin E$. In other words, we consider only digraphs with girth at least 3, where the *girth* of a digraph is the minimum length of a closed path with at least two vertices. We will also restrict ourselves to loopless digraphs, because distance-transitivity implies that either the digraph has no loop, or it has a loop at every vertex and eliminating all the loops will not affect the distance-transitive property. From now on, we will use the word "digraph" to mean a strongly connected loopless digraph with girth at least 3.

In Section 2, we give examples of distance-transitive digraphs. In Section 3, we introduce the intersection matrix of a distance-transitive digraph and study some of its properties. In Section 4, we prove that all distance-transitive digraphs of diameter 2 are balanced incomplete block designs with parameters $(v, k, \lambda) = (4t-1, 2t-1, t-1)$. We also give the results of a search for distance-transitive digraphs of diameter 3 with valency $k \leq 20$.

2. Examples

We now give another characterization of distance-transitive digraphs. First of all, we give several definitions. If S is a subset of the set of vertices V and H is a subgroup of $G(\Gamma)$, then we say that H is *transitive on the set* S if for all $u, v \in S$, there exists $\pi \in H$ such that $\pi(u) = v$. The digraph Γ is *vertex-transitive* if $G(\Gamma)$ is transitive on V. The *vertex-stabilizer* G_v of a vertex v is the subgroup of $G(\Gamma)$ which fixes v. For each vertex u we define

$$\Gamma_i(u) = \{v \in V \mid d(u, v) = i\}.$$

The proof of the next proposition is similar to the one in the undirected case [2, Lemma 20.1].

Proposition 2.1. *A digraph Γ, with diameter d and automorphism group $G = G(\Gamma)$, is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer G_v is transitive on the sets $\Gamma_i(v)$, for fixed $i = 0, 1, \ldots, d$ and for each $v \in V$.*

Example 1. The simplest of distance-transitive digraphs is a *directed cycle*.

Example 2. The second example is a *quadratic residue digraph.* When n is a prime power congruent to 3 modulo 4, we define a digraph Γ by letting $V=\{0, 1, \dots, n-1\}$ and

$$E=\{(i, j) \mid i \neq j \text{ and } (j-i) \text{ is a quadratic residue modulo } n\}.$$

It is now simple to prove that Γ is a distance-transitive digraph with $g=3$ and $d=2$. These are the digraphs analogous to the Paley graphs [3, p. 14].

The above two classes of examples have the property that $g=d+1$. Starting from a distance-transitive digraph with $g=d+1$, we can construct more distance-transitive digraphs by the following process.

Theorem 2.2. *If Γ is a distance-transitive digraph with $g=d+1$, then for any integer $t \geq 2$, we can construct a distance transitive digraph Γ' with the vertex set*

$$\Gamma'=\{\langle v, i\rangle \mid v \in V \text{ and } 0 \leq i \leq t-1\}$$

and the edge set

$$E'=\{(\langle u, i\rangle, \langle v, j\rangle) \mid (u, v) \in E\}.$$

The digraph Γ' has girth $g'=d+1$ and diameter $d'=d+1$.

Proof. Clearly the distance function between any pair of vertices in Γ' is given by

$$d(\langle u, i\rangle, \langle v, j\rangle)=\begin{cases} d(u, v) & \text{if } u \neq v; \\ 0 & \text{if } u=v \text{ and } i=j; \\ g & \text{if } u=v \text{ and } i \neq j. \end{cases}$$

For fixed u, any permutation π_u on the set of vertices $\{\langle u, i\rangle \mid 0 \leq i < t\}$ and fixing all the other vertices, is an automorphism of Γ'. It is now easy to see that the new digraph is distance-transitive.

These three methods to construct distance-transitive digraphs cover all the cases known to the author.

3. Intersection matrix

In this section, we derive several necessary conditions for the existence of a distance-transitive digraph. The treatment parallels that of the undirected case [2, Chapters 20, 21].

For any digraph Γ, with diameter d and $V=\{v_1, \dots, v_n\}$, we define a set $\{A_0, A_1, \dots, A_d\}$ of $n \times n$ matrices as follows:

$$(A_h)_{rs}=\begin{cases} 1 & \text{if } d(v_r, v_s)=h; \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A_1 is usually simply denoted as A. It is the *adjacency matrix* of Γ. Clearly $A_0 = I$, the identity matrix and $A_0 + A_1 + \cdots + A_d = J$, the matrix of all 1's.

For any digraph Γ and any vertices $u, v \in V$, we let

$$s_{ih}(u, v) = |\{w \in V \mid d(u, w) = i \text{ and } d(w, v) = h\}|.$$

In a distance-transitive digraph, the numbers $s_{ih}(u, v)$ depend only on the distance $d(u, v)$. We write $s_{ihj} = s_{ih}(u, v)$, where $j = d(u, v)$. These $(d+1)^3$ numbers s_{ihj}, where $0 \leq i, h, j \leq d$, are the *intersection numbers* of a distance-transitive digraph.

Proposition 3.1. *If Γ is a distance-transitive digraph, then*

$$A_i A_h = \sum_{j=0}^{d} s_{ihj} A_j.$$

Proof. The value of $(A_i A_h)_{rs}$ counts the number of vertices w with the property that $d(v_r, w) = i$ and $d(w, v_s) = h$. Therefore $(A_i A_h)_{rs} = s_{ih}(v_r, v_s) = s_{ihj}$ if $d(v_r, v_s) = j$.

We now define a set $\{C_0, C_1, \ldots, C_d\}$ of $(d+1)\times(d+1)$ matrices by letting $(C_h)_{ij} = s_{ihj}$. The matrix C_1 is usually denoted simply as C and it is called the *intersection matrix* of Γ. To construct a distance-transitive digraph, one often starts by constructing its intersection matrix. The next result gives some of the necessary conditions for a given matrix to be an intersection matrix. We let c_{ij} denote the (i, j)-th element of C and we let k_i denote the size of $\Gamma_i(v)$ for some $v \in V$. For a distance-transitive digraph, the k_i's are independent of the vertex v. Furthermore, we write k for k_1. Since the in-valency of any vertex is equal to the out-valency, we will simply call k the *valency* of Γ.

Proposition 3.2. *If Γ is a distance-transitive digraph with girth g, diameter d and valency k, then its intersection matrix C satisfies*

(i) $c_{ij} = 0$ *for* $j > i + 1$;

(ii) $c_{01} = 1$;

(iii) $c_{i0} = \begin{cases} 0 & \text{if } i \neq g-1; \\ k & \text{if } i = g-1; \end{cases}$

(iv) $\sum_{j=0}^{d} c_{ij} = k$;

(v) $\sum_{i=0}^{d} c_{ij} k_i = k k_j$.

Proof. If $c_{ij} \geq 1$, then there exist vertices u, v and w such that $d(u, w) = i$, $d(w, v) = 1$ and $d(u, v) = j$. Since the vertex sequence u, w and v gives a path of length $i+1$ from u to v, we have $j \leq i+1$. Hence, we have now proved (i).

Condition (ii) is clear for the definition of c_{01}.

If $(w, u) \in E$, then $w \in \Gamma_{g-1}(u)$. Hence $c_{i0} = 0$ for $i \neq g-1$ and $c_{i0} = k_{g-1}$ for $i = g-1$.

The left-hand side of (iv) counts the number of edges going into a fixed vertex in $\Gamma_j(u)$, which is the valency k. Condition (iv) also implies that $c_{g-1,0} = k$.

Condition (v) counts the number of edges coming out of $\Gamma_i(u)$. There are k_i vertices and each has k edges coming out, for a total of kk_i. For each vertex v in $\Gamma_j(u)$, there are c_{ij} edges from $\Gamma_i(u)$ to v. Thus there are $c_{ij}k_j$ edges from $\Gamma_i(u)$ to $\Gamma_j(u)$. The left-hand side of (v) is obtained by summing over all j. Thus, we proved Proposition 3.2.

Condition (iv) states that the column sums of the matrix C are all equal to k, the valency. Since C is a non-negative integral matrix, there are only a finite number of intersection matrices for any given values of g, d and k. Conditions (i) to (iii) give the first row and first column of C. Condition (v) states that k is an eigenvalue of the matrix C with the column vector $(1, k_1, \ldots, k_d)^{\mathrm{T}}$ being a right eigenvector.

To derive further properties of the intersection matrix, we first give the following definition. Let $v \in \Gamma_i(u)$ for some fixed u. We define $\beta_{ij} = |\{w \in \Gamma_j(u) : d(v, w) = 1\}|$.

By the distance-transitive property, β_{ij} is only a function of i and j and is independent of the vertex v.

Proposition 3.3. *The intersection matrix C of a distance-transitive digraph with girth ≥ 3 satisfies*

(i) $\beta_{ij} = c_{ij}k_j/k_i$;

(ii) $2c_{ii} + 1 \leq k_i$;

(iii) $c_{i,i+1} + \beta_{i+j,i} \leq k_i$ *for* $1 \leq j < g-1$.

Proof. Condition (i) is obtained from counting the number of edges between vertices in $\Gamma_i(u)$ and vertices in $\Gamma_j(u)$.

Let $x \in \Gamma_i(u)$. There are c_{ii} predecessors and c_{ii} successors of x in $\Gamma_i(u)$. Since $g \geq 3$, the predecessors and successors are all distinct. Hence we have counted at least $2c_{ii} + 1$ vertices in $\Gamma_i(u)$.

Let $w \in \Gamma_{i+1}(u)$. There exist $c_{i,i+1}$ vertices $v \in \Gamma_i$ with $(v, w) \in E$. Choose a $z \in \Gamma_{i+j}(u)$ such that $d(w, z) = j-1$. There are $\beta_{i+j,i}$ vertices $x \in \Gamma_i(u)$ with $(z, x) \in E$. Since $j < g-1$, the sets of vertices $\{v\}$ and $\{x\}$ have no common vertex. Hence we have counted at least $c_{i,i+1} + \beta_{i+j,i}$ vertices in $\Gamma_i(u)$.

One should remark that condition (i) of Proposition 3.3 implies that the expression on the right hand side is an integer. This is a very restrictive condition when $k_j \neq k_i$. Further existence conditions can be derived by considering the eigenvalues and eigenvectors of C.

We define the *adjacency algebra* $Z(\Gamma)$ of a digraph Γ as the algebra of polynomials, with complex coefficients, in the adjacency matrix A of Γ. The *dimension* of $Z(\Gamma)$ is the maximum size of a linearly independent set in $Z(\Gamma)$. A linearly independent set whose size is maximal is a *basis* for $Z(\Gamma)$.

Theorem 3.4. *Let Γ be a distance-transitive digraph. Then dimension of $Z(\Gamma)$ is $d+1$ and $\{A_0, A_1, \ldots, A_d\}$ is a basis for $Z(\Gamma)$.*

Proof. Since

$$A_iA = \sum_{j=0}^{i+1} c_{ij}A_j. \tag{3.1}$$

we see that A_i is a polynomial $p_i(A)$ for $i = 2, \ldots, d$. The rest of the proof is the same as [2, Theorem 20.7].

Another basis for $Z(\Gamma)$ is $\{I, A, A^2, \ldots, A^d\}$ where A^i is the matrix A raised to its i-th power. With respect to this basis, we can define a $(d+1)\times(d+1)$ matrix B by

$$A^iA = \sum_{j=0}^{d} b_{ij}A^j.$$

The matrix B is actually the companion matrix for the minimal polynomial of A. Equation (3.1) implies that the matrix C is a representation of A with respect to the basis $\{A_0, A_1, \ldots, A_d\}$. Hence C and B are related by a similarity transformation and they have the same characteristic polynomial. Thus every eigenvalue of A is an eigenvalue of C. If C has no repeated eigenvalues, then we can compute the multiplicity $m(\lambda)$ when λ is taken as an eigenvalue of A.

We introduce a column vector $\bar{v}(\lambda) = [v_0(\lambda), v_1(\lambda) \ldots, v_d(\lambda)]^T$. If we put $v_0(\lambda) = 1$, then we can solve the system $C\bar{v}(\lambda) = \lambda\bar{v}(\lambda)$ using one row of C at a time. This process works because of the special form of C imposed by condition (i) of Proposition 3.2. Thus, for each eigenvalue λ_i of C, we have defined a right eigenvector $\bar{v}_i$ corresponding to λ_i with component $(\bar{v}_i)_j = v_j(\lambda_i)$. Similarly we introduce a set of left eigenvectors $\bar{u}$ corresponding to λ_i by solving the system $\bar{u}(\lambda)C = \lambda\bar{u}(\lambda)$ where $\bar{u}(\lambda) = [u_0(\lambda), \ldots, u_d(\lambda)]$. The system is solved by letting $u_d(\lambda) = 1$ and using a column of C at a time, starting from the d-th column. If the eigenvalues of C are distinct, one can easily prove that the inner product $(\bar{u}_i, \bar{v}_j) = 0$ if $i \neq j$. The proof of the next result is similar to [2, Theorem 21.4]. One should note that if we have put $v_0(\lambda) = 0$ or $u_d(\lambda) = 0$, then we would have obtained an all zero vector.

Theorem 3.5. *Let $m(\lambda_i)$ be the multiplicity of λ_i in A, the adjacency matrix of a* *distance-transitive digraph with n vertices. With the notation above, if the eigenvalues of C are distinct, then*

$$m(\lambda_i) = n(\bar{u}_i)_0/(\bar{u}_i, \bar{v}_i),$$

where $(\bar{u}_i)_0$ denotes the first component of the left eigenvector $\bar{u}_i$ corresponding to the eigenvalue λ_i.

4. Diameters 2 and 3

In this section, we consider the distance-transitive digraphs of diameters 2 and 3. We first prove the following result which is supplied by the referee.

Proposition 4.1. $A_r = (A_{g-r})^{\mathrm{T}}$ *if* $0 < r < g$.

Proof. We first observe that if $d(u, v) = r < g$ and $u \neq v$, then $d(v, u) = g - r$. If we label the vertices $\{v_1, \ldots, v_n\}$, then we have

$$(A_r)_{ij} = 1 \quad \Leftrightarrow \quad d(v_i, v_j) = r \quad \Leftrightarrow \quad d(v_j, v_i) = g - r \quad \Leftrightarrow \quad (A_{g-r})_{ji} = 1.$$

Theorem 4.2. *If Γ is a distance-transitive digraph with n vertices, diameter* 2 *and valency k, then its adjacency matrix A satisfies*

(i) $A + A^{\mathrm{T}} = J - I$, *and*
(ii) $A^{\mathrm{T}}A = (k - \alpha)I + \alpha J$ *where* $k = 2\alpha + 1$ *and* $n = 4\alpha + 3$.

Proof. By definition, a digraph has $g \geq 3$, and since $g \leq d + 1$, we have $g = 3$. Condition (i) follows from the fact that $I + A + A_2 = J$ and that $A_2 = A^{\mathrm{T}}$. From (3.1) we have $A^{\mathrm{T}}A = c_{20}I + c_{21}A + c_{22}A_2$. We now calculate the coefficients c_{20}, c_{22}. Conditions (iii) and (iv) of Proposition 3.2 imply that $c_{20} = k = k_2$, and hence the intersection matrix has 4 undetermined entries. From (ii), (iv) and (v) of Proposition 3.2, we derive the relationships

$$1 + c_{11} + c_{21} = k, \tag{4.1}$$

$$c_{12} + c_{22} = k, \tag{4.2}$$

$$k(c_{11} + c_{12}) = k^2, \tag{4.3}$$

$$k(1 + c_{21} + c_{22}) = k^2. \tag{4.4}$$

Since $k \neq 0$, Eqs. (4.1) to (4.4) imply that $(c_{11}, c_{12}, c_{21}, c_{22}) = (\alpha, k - \alpha, k - 1 - \alpha, \alpha)$. since $B_{21} = c_{21}$, Proposition 3.3 (ii) and (iii) imply $k = 2\alpha + 1$. Hence $A^{\mathrm{T}}A = kI + \alpha A + \alpha A_2$. Since $I + A + A_2 = J$, we have $A^{\mathrm{T}}A = (k - \alpha)I + \alpha J$. Since $n = 1 + k_1 + k_2$, we have $n = 4\alpha + 3$.

Theorem 4.2 states that the adjacency matrix of every distance-transitive digraph of diameter 2 is a balanced incomplete block design with the Hadamard parameters [4, p. 107]. However, it satisfies the extra condition that $A + A^{T} = J - I$. Example 2 gives us digraphs of this type for n a prime power. The smallest n congruent to 3 modulo 4 and not a prime power is 15. An exhaustive search yields various matrices of size 15 satisfying the conditions of Theorem 4.2. However, none of them corresponds to a distance-transitive digraph. One should note that the case $\alpha = 0$ corresponds to the directed cycle and a degenerate Hadamard design.

In the case of distance-transitive digraphs of diameter 3, a computer programme was written to generate all the possible intersection matrices C for a given k where $2 \leq k \leq 20$. The programme generated, first of all, the matrices satisfying Proposition 3.2 and conditions (i) and (ii) of Proposition 3.3. A total of 6810 matrices were generated. Conditions (iii) of Proposition 3.3 reduced the number down to 3200. The number was further reduced to 2082 by the following existence conditions.

Proposition 4.3. *A distance-transitive digraph with diameter* 3 *satisfies the following conditions:*

(i) *if the girth* $g = 3$, *then*

$$k_2 \geq (k - \beta_{21}) + \max(0, c_{33} - \beta_{23} + 1) + \max(0, \beta_{32} - c_{22}, \delta),$$

where

$$\delta = \begin{cases} 0 & \textit{when } c_{21} = 0; \\ \beta_{12} - c_{22} & \textit{when } c_{21} > 0; \end{cases}$$

(ii) *if the girth* $g = 4$, *then*

$$k_2 \geq (k - \beta_{31}) + \max(0, \gamma) + \max(0, \eta)$$

where

$$\gamma = \begin{cases} 0 & \textit{when } c_{31} = 0; \\ \beta_{12} - \beta_{32} & \textit{when } c_{31} > 0; \end{cases}$$

and

$$\eta = \begin{cases} 0 & \textit{when } c_{32} = 0; \\ \beta_{23} - c_{33} & \textit{when } c_{32} > 0. \end{cases}$$

Proof. The basic idea of this result is to derive a lower bound for the number of elements at distance 2 from a vertex to be specified later and compare this lower bound to k_2.

In the case when $g = 3$, we take a vertex $v \in \Gamma_2$. Among the vertices in Γ_1, β_{21} of them are at a distance 1 from v and $(k - \beta_{21})$ of them are at a distance 2. Next we consider Γ_3. Take $u \in \Gamma_3$ such that $(v, u) \in E$. Vertex u has c_{33} successors in Γ_1 and at least $c_{33}-(\beta_{23}-1)$ of these vertices are at a distance 2 from v. From the same u, there are β_{32} edges going to vertices in Γ_2. Since $g \neq 2$, v is not one of these vertices. Hence, there are at least $\beta_{32}-c_{22}$ vertices in Γ_2 that are at a distance 2 from v. If $c_{21}>0$, then we can further consider vertices at distance 2 from v and going through a vertex in Γ_1.

The case for $g=4$ is similar. We take a $v \in \Gamma_3$. In Γ_1, there are at least $k-\beta_{31}$ vertices at distance 2 from v. The values of $\max(0, \gamma)$ and $\max(0, \eta)$ count the minimum number of vertices at distance 2 in Γ_2 and Γ_3 respectively.

For each of the remaining 2082 matrices, the programme calculated its characteristic polynomial and the polynomial's discriminant. If the discriminant was nonzero, then the matrix did not have a repeated eigenvalue and Theorem 3.5 was applied. In the range of our search, every matrix had distinct eigenvalues. Theorem 3.5 reduced the number of matrices to 36. It took 160 seconds of computing time on a CDC Cyber 172 computer to generate these 36 matrices. Among them, 34 are of the form

$$C=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ k & c_{21} & c_{22} & k \\ 0 & t-1 & 0 & 0 \end{bmatrix}, \tag{4.5}$$

where t divides k, $k=k_1=k_2$, $k_3=(t-1)$ and $t \geq 2$. It was noticed that intersection matrices obtained from the construction of Theorem 2.2 were of the same form. We will next prove that if the intersection matrix is of the form (4.5), then the digraph is constructed from a distance-transitive digraph with diameter 2.

Lemma 4.4. *Let Γ' be a distance-transitive degraph whose intersection matrix C is of the form* (4.5). *If u and v are vertices in Γ' and $d(u, v)=3$, then $d(v, u)=3$.*

Proof. It suffices to let $u \in \Gamma_0'$ and $v \in \Gamma_3'$. Since the diameter is 3, we have $d(v, u) \leq 3$. However, from the form of C, every path from v to u must pass through a vertex in Γ_1'. Hence $d(v, u)=3$.

Moreover, if v and $w \in \Gamma_3'$ and $v \neq w$, then $d(v, w)=3$. Hence the relation, u is related to v if $u=v$ or $d(u, v)=3$, is an equivalence relation. The vertex set of Γ' can be divided into equivalence classes. We form a subset by taking one vertex from each of the equivalence classes. We define a new digraph Γ as the subgraph of Γ' induced by the set of vertices we have chosen.

Proposition 4.5. *Let Γ' be a distance-transitive digraph whose intersection matrix C is of the form* (4.5). *The digraph Γ defined above is a distance-transitive digraph with diameter* 2 *and Γ' can be constructed from Γ using Theorem* 2.2.

Proof. We first show that if $u, v \in \Gamma'$ and $d(u, v)=3$, then u and v have the same adjacency properties. It suffices to consider $u \in \Gamma'_0$ and $v \in \Gamma'_3$. From (4.5), it is simple to show that the predecessors of u and v are all the vertices in Γ'_2. Similarly, the successors of u and v are all the vertices in Γ'_1. Hence they have the same adjacency property.

Now, any permutation of the vertices in the same equivalence class is an automorphism of Γ'. If follows easily that the induced subgraph Γ has the distance-transitive property and that Γ' can be constructed from Γ using Theorem 2.2.

Corollary 4.6. *If Γ' exists, then (k/t) is odd.*

Proof. The valency of the new digraph Γ is (k/t). Theorem 4.2 shows that it must be odd.

Corollary 4.6 shows that 5 of the 34 matrices do not exist. The matrix corresponding to $k = 14$ and $t = 2$ requires the existence of a distance-transitive digraph of diameter 2 with $n = 15$ which does not exist, as we have mentioned. The remaining 28 cases all exist. There are 19 extensions of the directed cycle using Theorem 2.2 with $t = 2, 3, \ldots, 20$. There are 5 extension of the quadratic residue digraph with $n = 7$, using $t = 2, 3, 4, 5$ and 6. There are 3 extensions of the quadratic residue digraph with $n = 11$, using $t = 2, 3$ and 4. There is one extension of the quadratic residue digraph with $n = 19$ and $t = 2$.

There were 2 matrices remaining, one with 21 vertices and the other 64 vertices. The author showed the nonexistence of the digraph with 21 vertices in an *ad hoc* manner. The referee supplied the nonexistence proof of the digraph with 64 vertices.

In the search for distance-transitive digraph, if a solution is found for a particular intersection matrix, then no effort was made to find all inequivalent solutions. Because of Proposition 4.5, for $k \leq 20$, the question of inequivalent solutions with $d = 3$ reduces to the problem of inequivalent solutions with $d = 2$.

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