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Improved Poincaré inequalities with weights ☆

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A B S T R A C T

In this paper we prove that if $\Omega \in \mathbb{R}^n$ is a bounded John domain, the following weighted Poincaré-type inequality holds:

$$\inf_{a\in\mathbb{R}} \left\| f(x) - a \right\|_{L^q(\Omega, w_1)} \leqslant C \left\| \nabla f(x) d(x)^\alpha \right\|_{L^p(\Omega, w_2)}$$

where f is a locally Lipschitz function on Ω, $d(x)$ denotes the distance of x to the boundary of Ω, the weights w_1, w_2 satisfy certain cube conditions, and $\alpha \in [0,1]$ depends on p, q and n. This result generalizes previously known weighted inequalities, which can also be obtained with our approach.

1. Introduction

The purpose of this paper is to present a simple unified approach to prove weighted Poincaré-type inequalities in John domains.

The class of John domains was first introduced in [11] and named after the author of that paper by Martio and Sarvas [14]. It contains Lipschitz domains as well as other domains with very non-regular boundaries, and it has played an important role in several problems in analysis. In particular, as it has been made clear in [2], it is closely connected to the improved Poincaré inequalities we are interested in.

The Sobolev–Poincaré inequality

$$\inf_{a\in\mathbb{R}} \left\| f(x) - a \right\|_{L^{\frac{np}{n-p}}(\Omega)} \leqslant C \left\| \nabla f(x) \right\|_{L^p(\Omega)} \tag{1.1}$$

with $\Omega \subseteq \mathbb{R}^n$ being a John domain, and f locally Lipschitz in Ω, was proved in the case $1 < p < n$ in [13], and later extended to the case $p = 1$ in [3]. See also [8] for proofs, other references and a nice account on the history of this problem.

Moreover, it was proved in [2] that John domains are essentially the largest class of domains for which this inequality can hold, more precisely, if $\Omega \subseteq \mathbb{R}^n$ is a domain of finite volume that satisfies a separation property (cf. [2]) and $1 \leqslant p < n$, then Ω satisfies the Sobolev–Poincaré inequality if and only if it is a John domain.

The Sobolev–Poincaré inequality can be seen as a special case of a much wider family of so-called improved Poincaré inequalities. Indeed, it was proved in [9] that if $\Omega \subseteq \mathbb{R}^n$ is a bounded John domain, and $f \in L^1_{\mathrm{loc}}(\Omega)$ is such that $\nabla f(x) d(x)^\alpha \in L^p(\Omega)$, then

$$\inf_{a\in\mathbb{R}} \left\| f(x) - a \right\|_{L^q(\Omega)} \leqslant C \left\| \nabla f(x) d(x)^\alpha \right\|_{L^p(\Omega)} \tag{1.2}$$

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whenever $1 < p \leqslant q \leqslant \frac{np}{n-p(1-\alpha)}$ with $p(1-\alpha) < n$, and $\alpha \in [0,1]$, with $d(x)$ being the distance of a point x to the boundary of Ω (the same inequality holds for unbounded John domains with $1 \leqslant p \leqslant q = \frac{np}{n-p(1-\alpha)}$). Letting $\alpha = 0$ in (1.2) one clearly obtains inequality (1.1).

Recently, inequality (1.2) was extended to more general domains, namely s-John and ϕ-John domains, allowing also a power of the distance to the boundary on the left-hand side of the inequality (see [6] and references therein).

A different kind of generalization on bounded John domains was made in [5], extending inequality (1.2) to weighted spaces. Indeed, it was shown in that paper that under certain cube conditions on the weights w_1, w_2, the following inequality holds:

$$\inf_{a\in\mathbb{R}} \left\| f(x) - a \right\|_{L^q(\Omega,w_1)} \leqslant C \left\| \nabla f(x) \right\|_{L^p(\Omega,w_2)} \tag{1.3}$$

whenever f is a Lipschitz function and $1 < p \leqslant q < \infty$. Notice that the author of [5] refers to domains satisfying the Boman chain condition, but for connected domains in $\mathbb{R}^n$ this is exactly the same class as that of John (see [4] for proof of this inequality even in a much more general context).

Inequality (1.3) can also be extended to unbounded John domains as it was done in [9] for the case of (1.2) (see [10]). Both results rely heavily on the main theorem of [20], which states that an unbounded John domain can be written as an increasing union of bounded John domains in a way that allows to pass to the limit using the dominated convergence theorem.

As we did for inequality (1.1), we could also think of inequality (1.3) as a special case of a wider family of inequalities explicitly involving powers of the distance to the boundary. Indeed, we will prove in this paper that if f is a locally Lipschitz function on Ω

$$\inf_{a\in\mathbb{R}} \left\| f(x) - a \right\|_{L^q(\Omega,w_1)} \leqslant C \left\| \nabla f(x) d(x)^{\alpha} \right\|_{L^p(\Omega,w_2)} \tag{1.4}$$

for suitable weights w_1, w_2, and with α depending on p, q as in inequality (1.2), thus extending the results in [5].

It is worth noting that the technique we will use for the proof of this inequality differs completely from the one used in that paper for the case $\alpha = 0$. Instead of relying on chains of cubes and cube-by-cube inequalities, we recover the simpler classical ideas which relate Sobolev–Poincaré inequalities with fractional integrals (see, e.g., [8] and references therein). Similar ideas were previously used for John domains in [13] to prove the Sobolev–Poincaré inequality, but the fact that they can also be used in connection with the distance to the boundary seems to be new.

We will use a representation formula proved in [1] that essentially allows us to recover f from its gradient (an alternative proof of inequality (1.1) can also be found in that paper). It has, as mentioned before, the advantage of allowing us to introduce the distance to the boundary without recurring to Whitney cubes, and it will allow us to reduce the proof of inequalities (1.2) and (1.4) to known continuity results for fractional integrals and the Hardy–Littlewood maximal function.

Although inequality (1.2) can be seen as a special case of (1.4) taking $w_1 = w_2 = 1$, we have chosen to present them separately for the sake of clarity and because the hypotheses needed are weaker than those we require for the more general cases. We will also split inequality (1.4) into the cases $w_1 = w_2$ and $w_1 \neq w_2$. We shall refer to the first case as 'one-weighted' case (notice, however, that the weight is raised to different powers in each side of the inequality) and to the second one as 'two-weighted' case. Once the ideas are made clear in the simpler cases, we shall be somewhat sketchy to indicate how they can be adapted to the more general case.

The paper is organized as follows. In Section 2 we recall some definitions, obtain the representation formula that we will be using in the remainder of the paper, and show how it relates to the distance to the boundary. Section 3 is devoted to the unweighted and one-weighted cases. We obtain a simpler proof of the results in [9] and, following the technique presented in [8], we extend inequality (1.4) for $w_1 = w_2$ to the previously unknown case $p = 1$ (Theorem 3.4). Finally, in Section 4 we show how our arguments can be used to generalize the results in [5] and obtain new inequalities in the two-weighted case (Theorems 4.1 and 4.2).

2. Preliminaries

The notation used in this paper is rather standard. By C we will denote a general constant which can change its value even within a single string of estimates. We will write $C(*,\dots,*)$ to emphasize that the constant depends on the quantities appearing in the parentheses only.

By a weight function we mean a nonnegative measurable function on $\mathbb{R}^n$. We will write

$$\|f\|_{L^p(\Omega,w)} = \left\| f(x) w(x)^{\frac{1}{p}} \right\|_{L^p(\Omega)} = \left(\int_{\Omega} \left| f(x) \right|^p w(x)\, dx \right)^{\frac{1}{p}}.$$

Given a domain $\Omega \subset \mathbb{R}^n$ and any $x \in \Omega$, we let $d(x)$ denote the distance of x to the boundary of Ω. A bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if for a fixed $x_0 \in \Omega$ and any $y \in \Omega$ there exists a rectifiable curve, called John curve, given by

$$\gamma(\cdot, y) : [0,1] \to \Omega$$

such that $\gamma(0,y)=y$ and $\gamma(1,y)=x_0$, and there exist constants δ and K, depending only on the domain Ω and on x_0, such that

$$d\big(\gamma(s,y)\big)\geqslant \delta s \tag{2.1}$$

and

$$\big|\dot\gamma(s,y)\big|\leqslant K \tag{2.2}$$

where $\dot\gamma(s,y):=\frac{\partial\gamma}{\partial s}(s,y)$.

In what follows, we will be using that $\gamma(s,y)$ and $\dot\gamma(s,y)$ are measurable functions. This property need not be fulfilled if we take $\gamma(\cdot,y)$ to be an arbitrary John curve for each fixed $y\in\Omega$, but it can be obtained by means of a slight technical modification (see [1, Lemma 2.1] for details). Moreover, to simplify notation we will assume, without loss of generality, that $x_0=0$.

Let $\varphi\in C_0^\infty$ such that $\int_\Omega\varphi=1$ and $\operatorname{supp}\varphi\subset B(0,\delta/2)$. Given a locally Lipschitz function f, we denote by f_φ the weighted average of f, namely, $f_\varphi=\int_\Omega f\varphi$.

The following lemmas of this section will be fundamental for the remainder of this paper. They were proved in [1] but we have chosen to reproduce their proofs here for the sake of completeness.

Lemma 2.1. *With the above notations, if $\Omega\subseteq\mathbb{R}^n$ is a bounded John domain and $y\in\Omega$,*

$$f(y)-f_\varphi=\int_\Omega G(x,y)\cdot\nabla f(x)\,dx \tag{2.3}$$

with

$$G(x,y):=-\int_0^1\left(\dot\gamma(s,y)+\frac{x-\gamma(s,y)}{s}\right)\varphi\left(\frac{x-\gamma(s,y)}{s}\right)\frac{1}{s^n}\,ds. \tag{2.4}$$

Proof. In view of (2.1), for any $y\in\Omega$ and $z\in B(0,\delta/2)$ the curve given by

$$\gamma(s,y)+sz,\quad s\in[0,1],$$

which joins y and z, is contained in Ω. Then

$$f(y)-f(z)=-\int_0^1\nabla f\big(\gamma(s,y)+sz\big)\cdot\big(\dot\gamma(s,y)+z\big)\,ds$$

where $\dot\gamma(s,y)$ indicates the derivative with respect to s. Multiplying by $\varphi(z)$ and integrating in z we obtain

$$f(y)-f_\varphi=-\int_\Omega\int_0^1\nabla f\big(\gamma(s,y)+sz\big)\cdot\big(\dot\gamma(s,y)+z\big)\varphi(z)\,ds\,dz.$$

Making the change of variable $x=\gamma(s,y)+sz$ we have

$$f(y)-f_\varphi=-\int_0^1\int_\Omega\nabla f(x)\cdot\left(\dot\gamma(s,y)+\frac{x-\gamma(s,y)}{s}\right)\varphi\left(\frac{x-\gamma(s,y)}{s}\right)\frac{1}{s^n}\,dx\,ds$$

as we wanted to prove. □

Lemma 2.2. *There exists a constant $C=C(n,\delta,K)$ such that*

$$\big|G(x,y)\big|\leqslant C\frac{\|\varphi\|_\infty}{|x-y|^{n-1}}. \tag{2.5}$$

Proof. If $(x-\gamma(s,y))/s\in\operatorname{supp}\varphi$ then $|x-\gamma(s,y)|<(\delta/2)s$. Therefore, using (2.2) and $\gamma(0,y)=y$ we have

$$|x-y|\leqslant\big|x-\gamma(s,y)\big|+\big|\gamma(s,y)-\gamma(0,y)\big|\leqslant(\delta/2)s+Ks. \tag{2.6}$$

Therefore,

$$G(x,y)=-\int_{C|x-y|}^1\left\{\dot\gamma(s,y)+\frac{x-\gamma(s,y)}{s}\right\}\varphi\left(\frac{x-\gamma(s,y)}{s}\right)\frac{1}{s^n}\,ds.$$

And, since

$$\left|\dot{\gamma}(s,y)+\frac{x-\gamma(s,y)}{s}\right|\leqslant K+\delta/2,$$

the above estimate follows easily. □

Lemma 2.3. *There exists a constant $C=C(\delta,K)$ such that, if $\varphi(\frac{x-\gamma(s,y)}{s})\neq 0$, then $|x-y|\leqslant C(\delta,K)d(x)$.*

Proof. Given $x,y\in\Omega$, let $\bar{x}\in\partial\Omega$ be such that $d(x)=|x-\bar{x}|$.

By (2.6) and property (2.1), we have

$$|x-y|\leqslant\left(\frac{\delta}{2}+K\right)s\leqslant\left(\frac{1}{2}+\frac{K}{\delta}\right)\delta s\leqslant\left(\frac{1}{2}+\frac{K}{\delta}\right)d\big(\gamma(s,y)\big).$$

But,

$$d\big(\gamma(s,y)\big)\leqslant\big|\gamma(s,y)-\bar{x}\big|\leqslant\big|\gamma(s,y)-x\big|+|x-\bar{x}|\leqslant\frac{\delta s}{2}+d(x)\leqslant\frac{d(\gamma(s,y))}{2}+d(x),$$

whence

$$|x-y|\leqslant\left(1+\frac{2K}{\delta}\right)d(x).\qquad\square$$

3. The unweighted and one-weighted cases

Theorem 3.1. *If $\Omega\subseteq\mathbb{R}^n$ is a bounded John domain,*

$$\inf_{a\in\mathbb{R}}\|f-a\|_{L^q(\Omega)}\leqslant C\big\|\nabla f(x)d(x)^\alpha\big\|_{L^p(\Omega)}\tag{3.1}$$

whenever $f\in L^q(\Omega)$ is a locally Lipschitz function, $\nabla f(x)d(x)^\alpha\in L^p(\Omega)$, $\alpha\in[0,1]$, $p(1-\alpha)<n$, and $1<p\leqslant q\leqslant\frac{np}{n-p(1-\alpha)}$.

Proof. By duality,

$$\|f-f_\varphi\|_{L^q(\Omega)}=\sup_{g\in L^{q'}(\Omega)}\frac{\int_\Omega(f-f_\varphi)g}{\|g\|_{L^{q'}(\Omega)}},$$

with q' being the dual exponent of q, $1/q+1/q'=1$. Therefore, it suffices to obtain a bound for $\int_\Omega(f-f_\varphi)g$ for $g\in L^{q'}(\Omega)$.

Using the representation formula (2.3), we can write

$$\int_\Omega\big(f(y)-f_\varphi\big)g(y)\,dy=\int_\Omega\int_\Omega G(x,y)\cdot\nabla f(x)\,dx\,g(y)\,dy.$$

Interchanging the order of integration and using Lemmas 2.2 and 2.3, we obtain

$$\int_\Omega\big|\big(f(y)-f_\varphi\big)g(y)\big|\,dy\leqslant C\int_\Omega\int_{|x-y|\leqslant Cd(x)}\frac{|g(y)|}{|x-y|^{n-1}}\,dy\big|\nabla f(x)\big|\,dx.\tag{3.2}$$

We consider separately the cases $\alpha\in[0,1)$ and $\alpha=1$.

In the case $\alpha\in[0,1)$, if we denote $I_\beta g(x)=\int g(y)|x-y|^{\beta-n}\,dy$, we can bound the above expression by

$$C\int_{\mathbb{R}^n}\big|\nabla f(x)\big|d(x)^\alpha I_{1-\alpha}g(x)\,dx\tag{3.3}$$

where we have assumed that $|\nabla f|$ and g are extended by zero outside Ω. Applying Hölder's inequality and the continuity of the fractional integral (see, e.g., [17]), this expression can be bounded by

$$\big\|\nabla f(x)d(x)^\alpha\big\|_{L^p(\Omega)}\big\|I_{1-\alpha}g(x)\big\|_{L^{p'}(\mathbb{R}^n)}\leqslant C\big\|\nabla f(x)d(x)^\alpha\big\|_{L^p(\Omega)}\big\|g(x)\big\|_{L^{q'}(\Omega)}$$

thus proving (3.1) in the case $\alpha\in[0,1)$.

In the case $\alpha=1$ (that is, $p=q$), a standard calculation (see, e.g., [21, Lemma 2.8.3]) shows that (3.2) can be bounded by

$$C\int_\Omega Mg(x)d(x)\big|\nabla f(x)\big|\,dx\leqslant C\big\|Mg(x)\big\|_{L^{q'}(\Omega)}\big\|\nabla f(x)d(x)\big\|_{L^q(\Omega)},$$

and the desired result follows by continuity of the Hardy–Littlewood maximal function in $L^{q'}(\Omega)$ (see, e.g., [17]). □

Theorem 3.2. *If $\Omega \subseteq \mathbb{R}^n$ is a bounded John domain,*

$$\inf_{a\in\mathbb{R}} \|f-a\|_{L^{n/(n-1+\alpha)}(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha\right\|_{L^1(\Omega)} \tag{3.4}$$

whenever $f \in L^{n/(n-1+\alpha)}(\Omega)$ is a locally Lipschitz function, $\nabla f(x)d(x)^\alpha \in L^1(\Omega)$, $1-\alpha < n$, and $\alpha \in [0,1]$.

Proof. In the case $\alpha = 1$, inequality (3.4) can be proved as in the previous theorem, using the continuity of the maximal function in $L^\infty(\Omega)$.

In the case $\alpha \in [0,1)$, we follow the approach used in [8] to prove the Sobolev–Poincaré inequality for John domains, modifying it to include the distance to the boundary in our estimates.

For $g \in L^1(\Omega)$, let

$$E = \left\{x \in \Omega:\ \int_\Omega \frac{g(y)}{|x-y|^{n-1+\alpha}}\,dy > t\right\}.$$

Then,

$$|E| \leqslant \int_E\int_\Omega \frac{g(y)}{t|x-y|^{n-1+\alpha}}\,dy\,dx.$$

But,

$$\int_E \frac{1}{|x-y|^{n-1+\alpha}}\,dx \leqslant C|E|^{(1-\alpha)/n}$$

(see, e.g., [12, inequality (7.2.6)]). Therefore,

$$|E|t^{n/(n-1+\alpha)} \leqslant C\left(\int_\Omega \left|g(y)\right|dy\right)^{n/(n-1+\alpha)}.$$

Since, as in the proof of (3.1),

$$|f-f_\varphi| \leqslant C\int_\Omega \frac{|\nabla f(y)|d(y)^\alpha}{|x-y|^{n-1+\alpha}}\,dy,$$

we conclude that

$$\sup_{t>0}\left|\left\{x\in\Omega:\ |f-f_\varphi|>t\right\}\right|t^{n/(n-1+\alpha)} \leqslant C\left(\int_\Omega \left|\nabla f(y)\right|d(y)^\alpha\,dy\right)^{n/(n-1+\alpha)}.$$

This in turn implies, by [8, Theorem 4], that

$$\inf_{a\in\mathbb{R}}\left\|f(x)-a\right\|_{L^{n/(n-1+\alpha)}(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha\right\|_{L^1(\Omega)} \qquad \square$$

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded John domain. If w is a nonnegative function such that there exists a constant $K < \infty$ such that*

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,dx\right)^{1/p'} \leqslant K \tag{3.5}$$

where Q is any n-dimensional cube, and K is independent of Q, then

$$\inf_{a\in\mathbb{R}}\left\|\big(f(x)-a\big)w(x)\right\|_{L^q(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha w(x)\right\|_{L^p(\Omega)}$$

for all locally Lipschitz f such that $f(x)w(x) \in L^q(\Omega)$, where $0 < \alpha \leqslant 1$, $1 < p < \frac{n}{1-\alpha}$ and $q \leqslant \frac{np}{n-p(1-\alpha)}$.

Proof. By duality, it suffices to bound $\int_\Omega (f-f_\varphi)(y)g(y)\,dy$ for any g such that $\|g(x)w^{-1}(x)\|_{L^{q'}(\Omega)} < \infty$.

In the case $\alpha \in [0,1)$, using, as before, the bound (3.3) and Hölder's inequality, we obtain

$$\int_\Omega \left|\big(f-f_\varphi\big)(y)g(y)\right|dy \leqslant C\left\|\nabla f(x)d(x)^\alpha w(x)\right\|_{L^p(\Omega)}\left\|I_{1-\alpha}g(x)w(x)^{-1}\right\|_{L^{p'}(\Omega)}.$$

But, by condition (3.5), [16, Theorem 4] and the fact that $I_{1-\alpha}$ is self-adjoint,

$$\left\|I_{1-\alpha}g(x)w^{-1}(x)\right\|_{L^{p'}(\Omega)} \leqslant C\left\|g(x)w^{-1}(x)\right\|_{L^{q'}(\Omega)}$$

and the theorem follows.

In the case $\alpha = 1$, bound (3.3), as before, by

$$C\int_\Omega Mg(x)d(x)\left|\nabla f(x)\right|dx \leqslant C\left\|Mg(x)w^{-1}(x)\right\|_{L^{p'}(\Omega)}\left\|\nabla f(x)d(x)w(x)\right\|_{L^p(\Omega)}$$

and the result follows, since by condition (3.5) and [7, Theorem 1.2] (see also references therein for previously known results),

$$\left\|Mg(x)w^{-1}(x)\right\|_{L^{p'}(\Omega)} \leqslant C\left\|g(x)w^{-1}(x)\right\|_{L^{q'}(\Omega)}. \qquad \square$$

Remark 3.1. Notice that if w satisfies condition (3.5), then w^q belongs to Muckenhoupt's class A_r with $r = \frac{q}{p'} + 1$, and therefore it is a doubling weight (which in turn implies that it satisfies the weaker 'reverse doubling condition' required for [7, Theorem 1.2]).

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded John domain. If w is a nonnegative function such that there exists a constant $K < \infty$ such that*

$$\left(\frac{1}{|Q|}\int_Q w(x)^{\frac{n}{n-1+\alpha}}\,dx\right)^{\frac{n-1+\alpha}{n}}\left(\operatorname*{ess\,sup}_{x\in Q}\frac{1}{w(x)}\right) < K \tag{3.6}$$

where Q is any n-dimensional cube and K is independent of Q, then

$$\inf_{a\in\mathbb{R}}\left\|\big(f(x)-a\big)w(x)\right\|_{L^{n/(n-1+\alpha)}(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha w(x)\right\|_{L^1(\Omega)}$$

for all locally Lipschitz f such that $f(x)w(x) \in L^{n/(n-1+\alpha)}(\Omega)$ and $\alpha \in [0,1)$.

When $\alpha = 1$, condition (3.6) *should be replaced by*

$$M\big(w(x)\big) \leqslant Cw(x)$$

for almost every $x \in \Omega$ (that is, $w \in A_1$).

Proof. In the case $\alpha \in [0,1)$, for each $t > 0$ let $E_t = \{|I_{1-\alpha}g(x)| > t\}$. By [16, Theorem 5], if w satisfies condition (3.6),

$$\int_{E_t} w(x)^{\frac{n}{n-1+\alpha}}\,dx \leqslant Ct^{-\frac{n}{n-1+\alpha}}\left(\int_{\mathbb{R}^n}\left|g(x)\right|w(x)\,dx\right)^{\frac{n}{n-1+\alpha}}.$$

But, as before,

$$|f - f_\varphi| \leqslant C\int_\Omega \frac{|\nabla f(y)|d(y)^\alpha}{|x-y|^{n-1+\alpha}}\,dy = CI_{1-\alpha}\big(|\nabla f|d(x)^\alpha\big).$$

Therefore, setting $d\mu = w(x)^{n/(n-1+\alpha)}\,dx$, we obtain that

$$\mu\big\{|f-f_\varphi| > t\big\}t^{n/(n-1+\alpha)} \leqslant C\mu\big\{I_{1-\alpha}\big(|\nabla f|d(x)^\alpha\big) > t\big\}t^{n/(n-1+\alpha)} \leqslant C\left(\int_\Omega\left|\nabla f(x)\right|d(x)^\alpha w(x)\,dx\right)^{n/(n-1+\alpha)}$$

which, by [8, Lemma 4], implies

$$\inf_{a\in\mathbb{R}}\left(\int_\Omega |f-a|^{n/(n-1+\alpha)}\,d\mu\right)^{(n-1+\alpha)/n} \leqslant C\int_\Omega |\nabla f|\,d\nu$$

where $d\nu = d(x)^\alpha w(x)\,dx$, that is,

$$\inf_{a\in\mathbb{R}}\left\|(f-a)(x)w(x)\right\|_{L^{n/(n-1+\alpha)}(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha w(x)\right\|_{L^1(\Omega)}.$$

In the case $\alpha = 1$, bound (3.3), as before, by

$$C\int_\Omega Mg(x)d(x)\left|\nabla f(x)\right|dx \leqslant C\left\|Mg(x)w^{-1}(x)\right\|_{L^\infty(\Omega)}\left\|\nabla f(x)d(x)w(x)\right\|_{L^1(\Omega)}$$

and the result follows, since by [15, Theorem 4], if $w \in A_1$,

$$\left\| Mg(x)w^{-1}(x)\right\|_{L^\infty(\Omega)} \leqslant C\left\| g(x)w^{-1}(x)\right\|_{L^\infty(\Omega)}. \qquad \square$$

Remark 3.2. If a weight w satisfies condition (3.6), then $w^{n/(n-1+\alpha)}$ belongs to the class A_1.

4. The two-weighted case

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded John domain. If w_1 and w_2 are nonnegative functions such that there exists a constant $K < \infty$ such that*

$$|Q|^{\frac{1}{n}-1}\left(\int_Q w_1(x)\,dx\right)^{1/q}\left(\int_Q w_2(x)^{1-p'}\,dx\right)^{1/p'} \leqslant K \tag{4.1}$$

and w_1, $w_2^{1-p'}$ satisfy the following 'reverse doubling' condition:

$$\text{for any } \epsilon \in (0,1) \text{ there exists } \delta \in (0,1) \text{ such that } \int_{\epsilon Q} w(x)\,dx \leqslant \delta \int_Q w(x)\,dx \tag{4.2}$$

where Q is any n-dimensional cube, and K is independent of Q, then

$$\inf_{a\in\mathbb{R}}\left\|\big(f(x)-a\big)w_1^{1/q}(x)\right\|_{L^q(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha w_2(x)^{1/p}\right\|_{L^p(\Omega)}$$

for all locally Lipschitz f such that $f(x)w_1(x)^{1/q} \in L^q(\Omega)$, whenever $1 < p < q < \infty$ and $\alpha \in [0,1]$. If $p = q$, condition (4.1), should be replaced by requiring that there exists $r > 1$ such that

$$|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q w_1(x)^r\,dx\right)^{1/qr}\left(\frac{1}{|Q|}\int_Q w_2(x)^{(1-p')r}\,dx\right)^{1/p'r} \leqslant K(r). \tag{4.3}$$

Proof. As in the previous theorems, by duality it suffices to bound $\int_\Omega (f-f_\varphi)(y)g(y)\,dy$ for any g such that $\|g(x)w(x)^{-1/q}\|_{L^{q'}} < \infty$.

We begin by the case $\alpha \in [0,1)$. Using the bound (3.3) and Hölder's inequality, we obtain

$$\int_\Omega \left|(f-f_\varphi)(y)g(y)\right|dy \leqslant C\left\|\nabla f(x)d(x)^\alpha w_2(x)^{1/p}\right\|_{L^p(\Omega)}\left\|I_{1-\alpha}g(x)w_2(x)^{-1/p}\right\|_{L^{p'}(\Omega)}.$$

But, by condition (4.1) (respectively, condition (4.3)) and [19, Theorem 1],

$$\left\|I_{1-\alpha}g(x)w_2^{-1/p}\right\|_{L^{p'}} \leqslant C\left\|g(x)w_1(x)^{-1/q}\right\|_{L^{q'}}$$

as we wanted to show.

In the case $\alpha = 1$, bound (3.3), as before, by

$$C\int_\Omega Mg(x)d(x)\left|\nabla f(x)\right|dx \leqslant C\left\|Mg(x)w_2^{-1}(x)\right\|_{L^{p'}(\Omega)}\left\|\nabla f(x)d(x)w_2(x)\right\|_{L^p(\Omega)}$$

and the result follows, since by condition (3.5) and [7, Theorem 1.2],

$$\left\|Mg(x)w_2^{-1}(x)\right\|_{L^{p'}(\Omega)} \leqslant C\left\|g(x)w_1^{-1}(x)\right\|_{L^{q'}(\Omega)}. \qquad \square$$

Remark 4.1. In the previous theorem we may assume that $q \leqslant \frac{np}{n-p(1-\alpha)}$ (and thus $n > p(1-\alpha)$), since otherwise w_1 equals zero almost everywhere on $\{w_2 < \infty\}$. This was observed in [18, Remark b].

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded John domain. If w_1 and w_2 are nonnegative functions such that there exists a constant $K < \infty$ such that*

$$M\big(w_2(x)\big) \leqslant K w_1(x) \tag{4.4}$$

for almost all x, then

$$\inf_{a\in\mathbb{R}}\left\|\big(f(x)-a\big)w_1(x)\right\|_{L^1(\Omega)} \leqslant C\left\|\nabla f(x)d(x)w_2(x)\right\|_{L^1(\Omega)}$$

for all locally Lipschitz f such that $f(x)w_1(x) \in L^1(\Omega)$.

Proof. By duality, it suffices to bound $\int_\Omega (f - f_\varphi)(y)g(y)\,dy$ for any g such that $\|g(x)w_1^{-1}(x)\|_{L^\infty(\Omega)} < \infty$.

As before, bound (3.3) by

$$C\int_\Omega Mg(x)d(x)\left|\nabla f(x)\right|dx \leqslant C\left\|Mg(x)w_2^{-1}(x)\right\|_{L^\infty(\Omega)}\left\|\nabla f(x)d(x)w_2(x)\right\|_{L^1(\Omega)}$$

and the result follows, since by condition (3.6) and [15, Theorem 4],

$$\left\|Mg(x)w_2^{-1}(x)\right\|_{L^\infty(\Omega)} \leqslant C\left\|g(x)w_1^{-1}(x)\right\|_{L^\infty(\Omega)}. \qquad \square$$

Remark 4.2. Notice that if one wanted to prove the more general inequality

$$\inf_{a\in\mathbb{R}}\left\|\big(f(x)-a\big)w_1^{\frac{n-1+\alpha}{n}}(x)\right\|_{L^{\frac{n}{n-1+\alpha}}(\Omega)} \leqslant C\left\|\nabla f(x)d(x)^\alpha w_2(x)\right\|_{L^1(\Omega)}$$

following the proof of the one-weighted case, one would need to know that, if $E_t = \{|I_{1-\alpha}g(x)| > t\}$, then

$$\int_{E_t} w_1(x)\,dx \leqslant Ct^{-\frac{n}{n-1+\alpha}}\left(\int\left|g(x)\right|w_2(x)\,dx\right)^{\frac{n}{n-1+\alpha}}.$$

Unfortunately, we were unable to find neither proof of this inequality under the conditions of the previous theorem (or any other sufficient conditions on the weights w_1, w_2) nor any counterexample to the required weak inequality. Such a result is beyond the scope of this paper, but it is worth noticing that it would immediately imply the above two-weighted Sobolev–Poincaré inequality which would complete Theorem 4.1 in the case $p = 1$.

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